

1 **Quantum Calculation of Classical Kinetic Equations:**
2 **A Novel Approach for Numerical Analysis of 6D**
3 **Boltzmann-Maxwell Equations in Collisionless Plasmas**
4 **Using Quantum Computing**

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10 **Key Points:**

- 11 • A future fault-tolerant large-scale quantum computer speeds up simulations of the
12 6D collisionless Boltzmann equation in nonlinear plasmas.
13 • Future first principles simulators will have a huge number of lattices, leading to
14 more advanced understanding and prediction of physics.
15 • To solve nonlinear PDEs using quantum computation, we used the method of am-
16 plitude embedding and quantum walk.

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Abstract

A novel quantum algorithm for solving the Boltzmann-Maxwell equations of the 6D collisionless plasma is proposed. The equation describes the kinetic behavior of plasma particles in electromagnetic fields and is known for the classical first-principles equations in various domains, from space to laboratory plasmas. We have constructed a quantum algorithm for a future fault-tolerant large-scale quantum computer to accelerate its costly computation. This algorithm consists mainly of two routines: the Boltzmann solver and the Maxwell solver. Quantum algorithms undertake these dual procedures, while classical algorithms facilitate their interplay. Each solver has a similar structure consisting of three steps: Encoding, Propagation, and Integration. We conducted a preliminary implementation of the quantum algorithm and performed a parallel validation against a comparable classical approach. IBM Qiskit was used to implement all quantum circuits.

1 Introduction

The space plasma environment, extending from the Sun to the magnetosphere-ionosphere-atmosphere, includes regions of frozen conditions, zones of anomalous resistance caused by electromagnetic turbulence, interconnected regions characterized by weakly ionized gas systems in strong magnetic fields, coupled neutral-atmosphere chemical processes, and pure neutral-atmosphere collision systems. Owing to their complex interactions, an inclusive understanding and forecasting of the space environment remains an elusive goal, even with the advancements in high-performance instrumentation and in-situ observation of satellites. Therefore, it is imperative to develop space plasma simulations capable of providing comprehensive insights, ranging from local spatial domains to the global schematic.

Historically, the development of space plasma simulations has been constrained by computational time, memory capacity, and data storage limitations, resolving complex phenomena with restricted physics at local space scales. In light of these constraints, space plasma simulations can be divided into two principal scale hierarchies. One approach endeavors to reproduce Macroscopic phenomena using a coarse approximation, whereas the other aims to recreate Microscopic phenomena derived from first principles. Examples of the former include magnetohydrodynamics (MHD), while the latter include techniques such as particle-in-cell (PIC) or the Vlasov equation (hereafter referred to as the collisionless Boltzmann equation). The choice between global simulation and comprehensive simulation of physical processes depends on the required space and time scales. However, several thematic concerns have emerged that require simulation via coupling between scale hierarchies. For example, we describe the plasma instability of the current sheet and the initiation mechanism of magnetic reconnection. The importance of kinetic effects resulting from ion-electron dynamics during the onset of magnetic reconnection has been demonstrated (Daughton, 2003; Moritaka & Horiuchi, 2008). To emulate this, a multi-hierarchical simulation with inter-domain coupling of MHD and PIC has been developed, which allows to analyze the influence of macroscopic dynamics on the microscopic physics of magnetic reconnection (Usami et al., 2009, 2014).

In contrast, the collisionless Boltzmann equation requires advanced numerical computations of the 6D distribution function in both space (3D) and velocity (3D) of the particles, and has traditionally been limited to the analysis of low-dimensional, low-resolution or microscopic phenomena. Given the susceptibility of direct methods to numerical diffusion, the more accurate electromagnetic Vlasov method has been designed and implemented (Umeda, 2008; Umeda et al., 2009; Minoshima et al., 2011; Umeda et al., 2012). The considerable progress in its research has allowed the elucidation of numerous authentic physical phenomena through the use of full electromagnetic Vlasov simulation, notwithstanding certain limitations regarding dimensionality and lattice number, which depend on the availability of computational resources (Umeda, Miwa, et al.,

2010; Umeda, Togano, & Ogino, 2010; Umeda et al., 2011; Umeda, 2012; Umeda et al., 2013, 2014). Theoretically, the integration of a collision term into the Boltzmann-Maxwell equations provides a comprehensive representation of the collision effects present in the complex coupled magnetosphere-ionosphere-atmosphere system of the Earth.

However, the current state of simulation technology is such that the fluid equations incorporating these collision effects have not yet been successfully modeled. The effects resulting from ionospheric collisions affect a variety of facets, ranging from auroras to magnetospheric dynamics (e.g. Yoshikawa et al. (2013)), and further lead to the manifestation of complex phenomena (e.g. Ohtani and Yoshikawa (2016)). Consequently, the collisionality Boltzmann-Maxwell equations encompass a plethora of significant phenomena within their domain of interest that are relevant to space-earth electromagnetics. In an idealized scenario, the entirety of these phenomena could be computed using the collisional Boltzmann-Maxwell equations, eliminating the need for scaling factorial coupled simulations and the reliance on a variety of assumptions. However, performing high-order numerical computations for the first-principles collisional Boltzmann-Maxwell equation requires the establishment of extremely precise numerical methods, coupled with an enormous computational burden $O(L^6)$ (where L is the number of lattices per spatial degree of freedom), which is currently unattainable even with the computational power of today's supercomputers.

In recent years, advances in quantum computing, both software and hardware, have demonstrated numerous advantages of quantum algorithms, such as those represented by (Shor, 1994). Following Google's achievement of quantum supremacy in 2019 (Arute et al., 2019), the pragmatic implementation of quantum computing in plasma simulation, weather forecasting, fluid simulation, and various fields is attracting interest. In numerical computation, the first paper on solving linear equations with quantum computer, the so-called the HHL algorithm (Harrow et al., 2009), was published. Subsequently, a quantum algorithm for linear ordinary differential equations (ODE)(Berry et al., 2017) and for partial differential equations(PDE)(Childs et al., 2021), and many for fluid simulations have been reported in recent years (Mezzacapo et al., 2015; Budinski, 2022; Steijl & Barakos, 2018; Steijl, 2019, 2023; Arrazola et al., 2019; Cao et al., 2013; Wang et al., 2020; Gaitan, 2020, 2021). The employed methodologies vary considerably. Some use quantum computational versions of the lattice gas model (Yepez, 1998, 2001) or the lattice Boltzmann method (Miller et al., 2001), some use quantum Fourier transforms to solve the Poisson equation, some use HHL algorithms and Hamiltonian simulations and Some combine it with the HHL algorithm and Hamiltonian simulations, others reduce from PDEs to ODEs to solve nonlinear ODEs, and so on. Among them, the quantum lattice Boltzmann method is constructed by considering the streaming operation as Quantum Walk (Aharonov et al., 1993)(Succi et al., 2015). Similarly, a quantum algorithm for the Dirac equation was proposed (Fillion-Gourdeau et al., 2017), using the similarity of a sequence of time-evolving operations to Quantum Walk. And Todorova et al. developed a quantum algorithm for the collisionless Boltzmann equation that performs discrete real and discrete velocity space propagation by Quantum Walk using a discrete-velocity method (Todorova & Steijl, 2020). We consider that this method has an advantage over other quantum differential equation solving methods in that it is easier to introduce first-principles collision terms.

- Collisionless Boltzmann-Maxwell equations with \mathbf{u} (:velocity) constant and the electromagnetic field \mathbf{E}, \mathbf{B} under vacuum conditions acting one way:

$$\frac{\partial f}{\partial t} + \mathbf{u}_{const} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E} + \mathbf{u}_{const} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,$$

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$

115 We developed a quantum algorithm for the 6D Boltzmann-Maxwell equations for
 116 collisionless plasmas under the above conditions based on the efficient quantum walk cir-
 117 cuit(Douglas & Wang, 2009). In this process, we calculated the time evolution problem
 118 of the 6D distribution function with the addition of velocity space, referring to the quan-
 119 tum algorithms for the the discrete velocity method in the the Boltzmann equation(Todorova
 120 & Steijl, 2020) and the Macro step in the Navier-Stokes equations(Budinski, 2022). Thus,
 121 the implementation of the collision term, which is the final goal of our project, is much
 122 easier and can be developed step by step. Furthermore, according to our quantum al-
 123 gorithm, it is simpler and computationally less expensive to solve all regions with the
 124 collisionless Boltzmann-Maxwell equations than with Macro-Micro’s hierarchically cou-
 125 pled simulators. The quantum computer’s most important advantage, the lattice infor-
 126 mation in the spatial direction, is parallelized into a single state function by encoding
 127 amplitude embedding. The results show that the order of the Quantum Volume as the
 128 scale of the quantum circuit is $O(N_t (\log_2(L))^2)$, which is an improvement over the or-
 129 der of the computational volume $O(N_t L^6)$ of a similar classical algorithm.

130 In the future, we will develop a quantum algorithm for the collisional Boltzmann-
 131 Maxwell equations and apply it to the plasma region from the sun to the Earth’s mag-
 132 netosphere-ionosphere-atmosphere. Thus, this will provide a framework in order to un-
 133 derstand and fully predict the space plasma environment. At that time, we expect the
 134 device to be used is a future fault-tolerant large-scale quantum computer. This paper
 135 develops the first quantum algorithm for this purpose and summarizes the methodology
 136 and verification results.

137 This paper is organized as follows: Section 1.1 and 1.2 describe the model of nu-
 138 merical computation, Section 2 describes our Quantum Algorithm of Boltzmann solver,
 139 and Section 3 compares and verifies the results of the quantum algorithm with similar
 140 classical algorithms. In Section 4, we discuss current issues and future solutions.

141 1.1 Governing equations

142 We employ the collisionless plasma Boltzmann and Maxwell equations within an
 143 electromagnetic field as governing equations. Specifically, these equations are given by

- The collisionless plasma Boltzmann equation with an electromagnetic field:

$$\frac{\partial f}{\partial t} + \mathbf{u}_{const} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} (\mathbf{E} + \mathbf{u}_{const} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

- Wave equation for the electric field \mathbf{E} in vacuum:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad (2)$$

- Wave equation for the magnetic field \mathbf{B} in vacuum:

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \quad (3)$$

144 Where f is the distribution function of the plasma particles, \mathbf{u} is the fluid velocity of the
 145 plasma, which we assume to be constant, q/m is the charge to mass ratio of the parti-
 146 cles and \mathbf{E} and \mathbf{B} are the electromagnetic fields. The Maxwell equations can be rewrit-
 147 ten in the form of wave equations for the electric and magnetic fields respectively, as above,
 148 to implement the quantum algorithms more efficiently.

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1.2 Numerical simulation method

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For the execution of nonlinear partial differential equations (1,2,3) on quantum computers, these equations require discretization by methods such as the finite difference technique or the finite element method. In the following discourse, the finite difference approach is adopted for the Boltzmann-Maxwell equation, resulting in difference equations that are implementable on quantum circuits.

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Proceeding with the application of the Forward Time Centered Space(FTCS) scheme, we differentiate the Boltzmann equations for collisionless plasma and derive a discretized representation. The differencing equation for the governing equation (1) is given by

$$\begin{aligned}
 f(x, y, z, v_x, v_y, v_z; t + \Delta t) = & f - \frac{u_x \Delta t}{2\Delta x} (f_{x+\Delta x} - f_{x-\Delta x}) - \frac{u_y \Delta t}{2\Delta y} (f_{y+\Delta y} - f_{y-\Delta y}) \\
 & - \frac{u_z \Delta t}{2\Delta z} (f_{z+\Delta z} - f_{z-\Delta z}) \\
 & - \frac{q(\mathbf{E} + \mathbf{u}_{const} \times \mathbf{B})_x \Delta t}{2m\Delta v_x} (f_{v_x+\Delta v_x} - f_{v_x-\Delta v_x}) \\
 & - \frac{q(\mathbf{E} + \mathbf{u}_{const} \times \mathbf{B})_y \Delta t}{2m\Delta v_y} (f_{v_y+\Delta v_y} - f_{v_y-\Delta v_y}) \\
 & - \frac{q(\mathbf{E} + \mathbf{u}_{const} \times \mathbf{B})_z \Delta t}{2m\Delta v_z} (f_{v_z+\Delta v_z} - f_{v_z-\Delta v_z}), \quad (4)
 \end{aligned}$$

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where the value of $f(x, y, z, v_x, v_y, v_z; t)$, namely the distribution function at the reference point x, y, z, v_x, v_y, v_z and time t , is simply denoted as f , and the same at the point deviating by one unit distance in each direction is denoted with subscripts:

$$(e.g.) f_{x+\Delta x} := f(x + \Delta x, y, z, v_x, v_y, v_z; t).$$

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We simplify the difference Boltzmann equation with the following assumption:

$$\frac{u_x \Delta t}{2\Delta x} = \frac{u_y \Delta t}{2\Delta y} = \frac{u_z \Delta t}{2\Delta z} = 1. \quad (5)$$

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Similarly, the difference equations for the electric and magnetic fields are given as

$$\begin{aligned}
 \mathbf{E}(x, y, z; t + \Delta t) = & \left(2 - 2 \frac{\Delta t^2}{\mu_0 \epsilon_0} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right) \right) \mathbf{E} - \mathbf{E}_{t-\Delta t} \\
 & + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta x^2} (\mathbf{E}_{x+\Delta x} + \mathbf{E}_{x-\Delta x}) + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta y^2} (\mathbf{E}_{y+\Delta y} + \mathbf{E}_{y-\Delta y}) \\
 & + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta z^2} (\mathbf{E}_{z+\Delta z} + \mathbf{E}_{z-\Delta z}), \quad (6)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{B}(x, y, z; t + \Delta t) = & \left(2 - 2 \frac{\Delta t^2}{\mu_0 \epsilon_0} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right) \right) \mathbf{B} - \mathbf{B}_{t-\Delta t} \\
 & + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta x^2} (\mathbf{B}_{x+\Delta x} + \mathbf{B}_{x-\Delta x}) + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta y^2} (\mathbf{B}_{y+\Delta y} + \mathbf{B}_{y-\Delta y}) \\
 & + \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta z^2} (\mathbf{B}_{z+\Delta z} + \mathbf{B}_{z-\Delta z}), \quad (7)
 \end{aligned}$$

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where quantities such as \mathbf{E} and \mathbf{B} are defined in the same manner as f above.

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Furthermore, for simplicity of notation, we set hereafter as the Lorentz force term as

$$\mathbf{F} := \mathbf{u} \times \mathbf{B}. \quad (8)$$

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Also, the speed of light c in equation (2,3) is rewritten here using the permittivity and the permeability (ϵ_0 and μ_0) in the vacuum. Similar to the Boltzmann equation example, we make the following assumption:

$$\frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta x^2} = \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta y^2} = \frac{\Delta t^2}{\mu_0 \epsilon_0 \Delta z^2} = 1. \quad (9)$$

170 Under the postulates of this manuscript, no velocity is obtained from the first-order ve-
 171 locity moment of the distribution function. Given the use of uniform velocities in both
 172 the temporal and spatial domains, the discretized magnetic field equation transforms into
 173 the propagation equation of the Lorentz force term.

174 As a result, we obtain the discretized Boltzmann-Maxwell equation to be implemented
 175 as follows:

$$\begin{aligned}
 f(x, y, z, v_x, v_y, v_z; t + \Delta t) &= f - (f_{x+\Delta x} - f_{x-\Delta x}) - (f_{y+\Delta y} - f_{y-\Delta y}) \\
 &\quad - (f_{z+\Delta z} - f_{z-\Delta z}) \\
 &\quad - \frac{q(\mathbf{E} + \mathbf{F})_x \Delta t}{2m\Delta v_x} (f_{v_x+\Delta v_x} - f_{v_x-\Delta v_x}) \\
 &\quad - \frac{q(\mathbf{E} + \mathbf{F})_y \Delta t}{2m\Delta v_y} (f_{v_y+\Delta v_y} - f_{v_y-\Delta v_y}) \\
 &\quad - \frac{q(\mathbf{E} + \mathbf{F})_z \Delta t}{2m\Delta v_z} (f_{v_z+\Delta v_z} - f_{v_z-\Delta v_z}), \quad (10)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{E}(x, y, z; t + \Delta t) &= -4\mathbf{E} - \mathbf{E}_{t-\Delta t} + (\mathbf{E}_{x+\Delta x} + \mathbf{E}_{x-\Delta x}) \\
 &\quad + (\mathbf{E}_{y+\Delta y} + \mathbf{E}_{y-\Delta y}) + (\mathbf{E}_{z+\Delta z} + \mathbf{E}_{z-\Delta z}), \quad (11)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{F}(x, y, z; t + \Delta t) &= -4\mathbf{F} - \mathbf{F}_{t-\Delta t} + (\mathbf{F}_{x+\Delta x} + \mathbf{F}_{x-\Delta x}) \\
 &\quad + (\mathbf{F}_{y+\Delta y} + \mathbf{F}_{y-\Delta y}) + (\mathbf{F}_{z+\Delta z} + \mathbf{F}_{z-\Delta z}). \quad (12)
 \end{aligned}$$

176 This allows us to evolve the values of f and (\mathbf{E}, \mathbf{B}) independently. We call the quan-
 177 tum routines that perform this evolution the Boltzmann solver and the Maxwell solver,
 178 respectively. For the evolution of f (Boltzmann solver), we need the values of \mathbf{E} and \mathbf{F}
 179 at each time step as they appear in the right-hand side of the equation (10), so we use
 180 the values obtained by the Maxwell solver.

181 2 Quantum Algorithm

182 In this section, a quantum algorithm based on the discretized Boltzmann-Maxwell
 183 equations (4,6,7) is constructed and implemented on quantum circuits. This quantum
 184 algorithm can be divided into two independent routines: the Boltzmann solver and the
 185 Maxwell solver. They take an initial function of f and (\mathbf{E}, \mathbf{B}) as input, respectively. Both
 186 routines fix time and output physical quantities that evolve in one time step according
 187 to difference equations (11,12). By iterating this one-step evolution many times, we can
 188 obtain the value of a physical quantity that has evolved for an arbitrary time step.

189 The electric and magnetic fields derived by Maxwell solver are incorporated into
 190 the Propagation circuit of the Boltzmann solver as shown in the FIG. 1, thereby cou-
 191 pling each routine. The quantum calculations in this paper are carried out exactly in a
 192 way that deals with state vectors using a classical simulator provided by IBM Qiskit. It
 193 is straightforward to construct an authentic quantum algorithm based on measurements.
 194

195 2.1 Boltzmann

196 Our Boltzmann solver can be segmented into three principal steps: Encoding, Prop-
 197 agation and Integration.

198 2.1.1 Encoding

199 First of all, it is necessary to encode the classical information of the physical quan-
 200 tities into the amplitudes of quantum states. Fixing the number of lattice sites in all spa-
 201 tial and velocity directions to be L , f will have $V := L^6$ degrees of freedom. In the en-
 202 coding step, we associate each of these degrees of freedom with one computational ba-

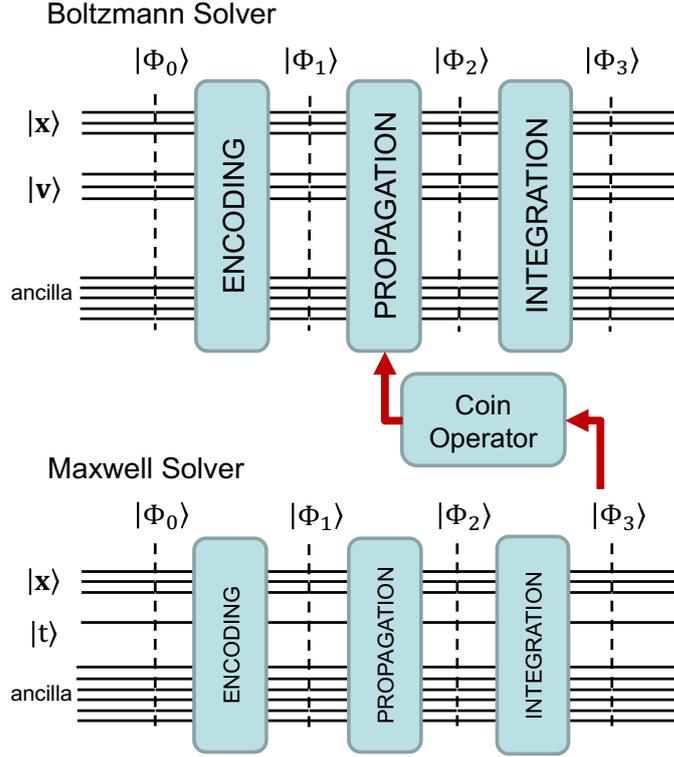


Figure 1. A Schematic of the quantum circuit of our algorithm for solving the Boltzmann-Maxwell equations. They consist of two routines that operate on the coin operator.

203 sis and encode the value of f in the amplitude of the corresponding quantum state. Thus,
 204 a total of V bases must be prepared in total, requiring $\lceil \log_2 V \rceil$ qubits. This method of
 205 encoding classical information into quantum information amplitudes is commonly referred
 206 to as the amplitude embedding technique.

207 To elucidate the relationship between physical quantities and probability ampli-
 208 tudes, the following conversion from a function $f(\mathbf{x}, \mathbf{v}; t)$ to a vector f_i , ($0 \leq i \leq V-1$)
 209 is implemented. The subscripts i specify a point in the 6D lattice space. For exam-
 210 ple, $i = 0$ corresponds to the origin point $(\mathbf{x}, \mathbf{v}) = (0, 0, 0, 0, 0, 0)$, and $i = 1$ represents
 211 the value of the distribution function moved by one lattice point in the x direction:

212 $(\mathbf{x}, \mathbf{v}) = (\Delta x, 0, 0, 0, 0, 0)$. Namely, the amount of f_i follows

$$(e.g.) \quad f_0 = f(0, 0, 0, 0, 0, 0; t = t_r), \quad (13)$$

$$f_1 = f(\Delta x, 0, 0, 0, 0, 0; t = t_r). \quad (14)$$

213 Note that the quantum state does not contain any information about time, since the prop-
 214 agation takes place with fixed time. We will assume $L = 2^{N_L}$ in the following. As ev-
 215 idenced in Section 3, our actual numerical calculations are executed with $N_L = 3$ ($L =$
 216 8).

217 The first important algorithm in the Encoding step is with a given distribution func-
 218 tion at a fixed $t = t_r$ to prepare a quantum state, which we name $|\phi_0\rangle_{\text{phys}}$, with these
 219 values in its amplitudes:

$$|\phi_0\rangle_{\text{phys}} = \sum_{i=0}^{V-1} \tilde{f}_i |i\rangle, \quad (15)$$

220 where \tilde{f} is the normalized distribution function as follows:

$$\tilde{f}_i = C f_i \quad , \quad C = \left(\sum_{i=0}^{V-1} |f_i|^2 \right)^{-1/2} . \quad (16)$$

221 At the initial time step of $t = 0$, an arbitrary distribution can be designated as an ini-
 222 tial function. Post the second step, the distribution function generated by the Boltzmann
 223 solver in the prior step ought to be provided as input. This iterative process allows for
 224 the computation of the distribution function at any desired time step. This procedure
 225 of state preparation can be executed in alignment with Appendix B.

226 It should be noted that, within the context of this manuscript, we have formulated
 227 the algorithm in a manner that measures f post each step and re-encodes it in the sub-
 228 sequent step, in order to circumvent excessive enlargement of the quantum circuit's depth.
 229 This design necessitates $O(V)$ measurements at every time step, failing the advantage
 230 of the quantum algorithm. However, it is straightforward to connect each time step seam-
 231 lessly. Namely, any measurements are required between each time step, implying that
 232 such a design will be beneficial when managing large-scale quantum apparatuses in the
 233 future. Further discussion on quantum advantage will be given in later sections.

234 The qubits prepared within this context are termed as the physical qubits, denoted
 235 as $|\text{phys}\rangle$. Looking more closely, $|\text{phys}\rangle$ is prepared by a total of 6 closed Hilbert spaces
 236 corresponding to spatial and velocity degrees of freedom, each having $N_L (= \log_2 L)$ qubits.
 237 Namely, we write it as

$$|\text{phys}\rangle = |\text{phys};x\rangle |\text{phys};y\rangle |\text{phys};z\rangle |\text{phys};v_x\rangle |\text{phys};v_y\rangle |\text{phys};v_z\rangle \quad (17)$$

238 Subsequent to the Propagation step, the ensuing quantum algorithms necessitate an ad-
 239 ditional qubit, which depending on their role, is identified as either subnode qubits $|\text{sub}\rangle$
 240 or ancilla qubits $|\text{ancilla}\rangle$. As will explained later the number of subnode and ancilla qubits
 241 are fixed to 4 and 1, respectively, regardless of the parameters and physical setup. Thus,
 242 the numbers of qubits required by the Boltzmann solver are

$$N_{\text{phis}} = 6N_L \quad , \quad N_{\text{sub}} = 4 \quad , \quad N_{\text{anc}} = 1, \quad (18)$$

243 and the following quantum state is prepared and output in after this Encoding step:

$$|\phi_1\rangle = |\phi_0\rangle_{\text{phys}} \otimes |0\rangle_{\text{sub}} \otimes |0\rangle_{\text{ancilla}}, \quad (19)$$

$$= \sum_{i=0}^{V-1} \tilde{f}_i |i\rangle_{\text{phys}} |0\rangle_{\text{sub}} |0\rangle_{\text{ancilla}}. \quad (20)$$

244 **2.1.2 Propagation**

245 In the Propagation step, we partially utilize the techniques of quantum algorithm
 246 method (Douglas & Wang, 2009) and implement an algorithm that multiplies each prob-
 247 ability amplitude of ϕ_1 by the coefficient of each term in the discretized equation.

248 The subnode bases and their corresponding physical quantities $f, \epsilon, \text{ and } \sigma$ denote
 249 the normalized distribution encoded as the amplitude of the associated state, the coef-
 250 ficients to be incorporated via the coin operator, and the sign to be multiplied during
 251 the integration step, respectively. To solve the evolution equation (10), we need to pre-
 252 pare and add up all the terms that arise in the equation such as:

$$f, \mp f_{x \pm \Delta x}, \dots, \mp \frac{q(\mathbf{E} + \mathbf{F})_x \Delta t}{2m \Delta v_x} f_{v_x \pm \Delta v_x}, \dots$$

253 After passing through the encoding step, we are now in possession of a quantum state
 254 $|\phi_1\rangle$, within which the data of the distribution function are encoded in the amplitude.

Table 1. The subnode bases and their corresponding physical quantities. f , ϵ , and σ respectively represent the (unnormalized) distribution function associated with each basis state, the coefficients to be incorporated via the coin operator, and the sign to be multiplied during the integration step. These are the quantities that appear on the right-hand side of the difference equation (10).

j	$ j\rangle_{\text{sub}}$	f_j	ϵ_j	σ_j
0	$ 0000\rangle$	$f(x, y, z, v_x, v_y, v_z)$	1	+1
1	$ 0001\rangle$	$f(x + \Delta x, y, z, v_x, v_y, v_z)$	1	-1
2	$ 0010\rangle$	$f(x - \Delta x, y, z, v_x, v_y, v_z)$	1	+1
3	$ 0011\rangle$	$f(x, y + \Delta y, z, v_x, v_y, v_z)$	1	-1
4	$ 0100\rangle$	$f(x, y - \Delta y, z, v_x, v_y, v_z)$	1	+1
5	$ 0101\rangle$	$f(x, y, z + \Delta z, v_x, v_y, v_z)$	1	-1
6	$ 0110\rangle$	$f(x, y, z - \Delta z, v_x, v_y, v_z)$	1	+1
7	$ 0111\rangle$	$f(x, y, z, v_x + \Delta v_x, v_y, v_z)$	$q \frac{E_x(x, y, z) + F_x(x, y, z) \Delta t}{2m \Delta v_x}$	-1
8	$ 1000\rangle$	$f(x, y, z, v_x - \Delta v_x, v_y, v_z)$	$q \frac{E_x(x, y, z) + F_x(x, y, z) \Delta t}{2m \Delta v_x}$	+1
9	$ 1001\rangle$	$f(x, y, z, v_x, v_y + \Delta v_y, v_z)$	$q \frac{E_y(x, y, z) + F_y(x, y, z) \Delta t}{2m \Delta v_y}$	-1
10	$ 1010\rangle$	$f(x, y, z, v_x, v_y - \Delta v_y, v_z)$	$q \frac{E_y(x, y, z) + F_y(x, y, z) \Delta t}{2m \Delta v_y}$	+1
11	$ 1011\rangle$	$f(x, y, z, v_x, v_y, v_z + \Delta v_z)$	$q \frac{E_z(x, y, z) + F_z(x, y, z) \Delta t}{2m \Delta v_z}$	-1
12	$ 1100\rangle$	$f(x, y, z, v_x, v_y, v_z - \Delta v_z)$	$q \frac{E_z(x, y, z) + F_z(x, y, z) \Delta t}{2m \Delta v_z}$	+1
13	$ 1101\rangle$	0	0	-1
14	$ 1110\rangle$	0	0	+1
15	$ 1111\rangle$	0	0	-1

255 Therefore, by considering an algorithm that multiplies each coefficient such as $\frac{q(\mathbf{E}+\mathbf{F})_x \Delta t}{2m \Delta v_x}$
 256 by the corresponding state, the amplitudes of all states are updated to the state with
 257 the appropriate coefficient appearing in equation (10). We will deal with the explicit sign
 258 in the equation later. The values of \mathbf{E} and \mathbf{F} at the certain time step are obtained from
 259 Maxwell solver.

260 Subnodes serve to identify the terms that arise at a specific time step, namely
 261 $f, f_{x \pm \Delta x}, \dots, f_{v_x \pm \Delta v_x} \dots$. In total, there are 13 ($= 1+2 \times 6$) terms: one term f ,
 262 which precedes propagation, and terms propagated by each ± 1 unit for each of the six
 263 directions in space and velocity. Hence, 4 ($= \lceil 13 \rceil$) qubits are necessitated as a subnode.
 264 It should be noted that this number remains uninfluenced by physical quantities
 265 like volume. For simplicity, we have associated them as depicted in TABLE 1. Here, ϵ_j
 266 is the coefficient applied to each term, and σ_j is the sign explicitly attributed to each term
 267 in TABLE 1. In fact, both ϵ_j and σ_j are coefficients in the difference equation (10), so
 268 it is possible to define epsilon to include the sign of σ_j . However, we choose to distinguish
 269 between them because ϵ_j represents a quantity that depends on a specific assumption
 270 as indicated by the assumption (5,9), while σ_j is a universally determined quantity.
 271 By making this distinction, we think we can minimize the part that we need to be modified
 272 based on different assumptions.

273 As elucidated below, the coin operator is accountable for the multiplication of these
 274 coefficients, and the shift operator assumes responsibility for correlating each term with
 275 the basis of the subnode.

276 We can create the appropriate coefficients by first make the subnodes in superpoti-
 277 tion using the H-gate. Then apply the diagonal matrix with $\{\epsilon\}$ as components:

$$\Lambda := \text{diag}(\epsilon_0, \epsilon_1, \dots, \epsilon_{15}). \quad (21)$$

278 The operation with this diagonal matrix is not a unitary and thus it must be embedded
 279 in a unitary matrix of larger size. Since the coefficients are real, this procedure can be
 280 done easily as explained in the Appendix Appendix B. Here, we use the ancilla qubit
 281 $|a_0\rangle$ to create a unitary matrix of larger size. We call this whole operator acting on the
 282 subnode (and the ancilla qubit) the “coin operator” according to the terminology of quan-
 283 tum walk. As a result, we obtain the state after operating the coin operator as follows:

$$\begin{aligned}
 U_{\text{Coin}}|\phi_1\rangle &= \sum_{i=0}^{V-1} U_{\text{Coin}}\tilde{f}_i|i\rangle_{\text{phys}}|0\rangle_{\text{sub}}|0\rangle_{\text{ancilla}}, \\
 &= \sum_{i=0}^{V-1} \sum_{j=0}^{15} \tilde{f}_i\tilde{\epsilon}_j|i\rangle_{\text{phys}}|j\rangle_{\text{sub}}|0\rangle_{\text{ancilla}} + |*\rangle|1\rangle_{\text{ancilla}}, \quad (22)
 \end{aligned}$$

284 where $\tilde{\epsilon}$ represents a normalized quantity. $|*\rangle$ represents the computationally unneces-
 285 sary states, which are identified by the ancilla qubit being $|1\rangle_{\text{ancilla}}$.

286 Next, so-called increment/decrement gates are applied on both subnode and phys-
 287 ical qubits to associate the basis of subnode and physical amount at different points. The
 288 increment/decrement gates are operators that shift one computational basis, respectively.
 289 Specifically, those operator satisfy

$$\begin{aligned}
 U_{\text{Incr.}}|i\rangle &= |i+1\rangle, \\
 U_{\text{Decr.}}|i\rangle &= |i-1\rangle. \quad (23)
 \end{aligned}$$

290 Suppose the periodic boundary condition on the N -qubits system:

$$\begin{aligned}
 U_{\text{Incr.}}|2^N-1\rangle &= |0\rangle, \\
 U_{\text{Decr.}}|0\rangle &= |2^N-1\rangle, \quad (24)
 \end{aligned}$$

291 those operator follow the relation: $U_{\text{Incr.}}^\dagger = U_{\text{Decr.}}$. The increment circuit can be specif-
 292 ically configured as follows.

$$U_{\text{Incr.}} = \text{Circuit with 4 qubits, 3 CNOTs, and X gate}, \quad U_{\text{Decr.}} = \text{Circuit with 4 qubits, 3 CNOTs, and X gate} \quad (25)$$

293 By performing controlled-Increment/Decrement gates on the subnode as control
 294 registers and the physical qubits as target registers, we can map the subnode to a phys-
 295 ical quantity on each lattice point. We call this sequential operations as the “shift op-
 296 erator”. The circuit of the shift operator is shown in FIG.2.

297 As a result, after applying both the coin operator and the shift operator, we ob-
 298 tain the following state as a final output of this propagation step:

$$|\phi_2\rangle = \sum_{i=0}^{V-1} \sum_{j=0}^{15} \tilde{\epsilon}_j\tilde{f}_{i,j}|i\rangle_{\text{phys}}|j\rangle_{\text{sub}}|0\rangle_{\text{ancilla}} + |*\rangle|1\rangle_{\text{ancilla}}. \quad (26)$$

299 We can articulate the exact correlation between \tilde{f}_i and $\tilde{f}_{i,j}$ as outlined herein. Initially,
 300 we had the capacity to signify the index i as $i = sL + t$, ($0 \leq s < 6$, $0 \leq t < L$),
 301 which, for instance, correlates with the direction x when $s = 0$, y when $s = 1$, and so
 302 forth, and the coordinates of the corresponding directions are symbolized by t . The shift

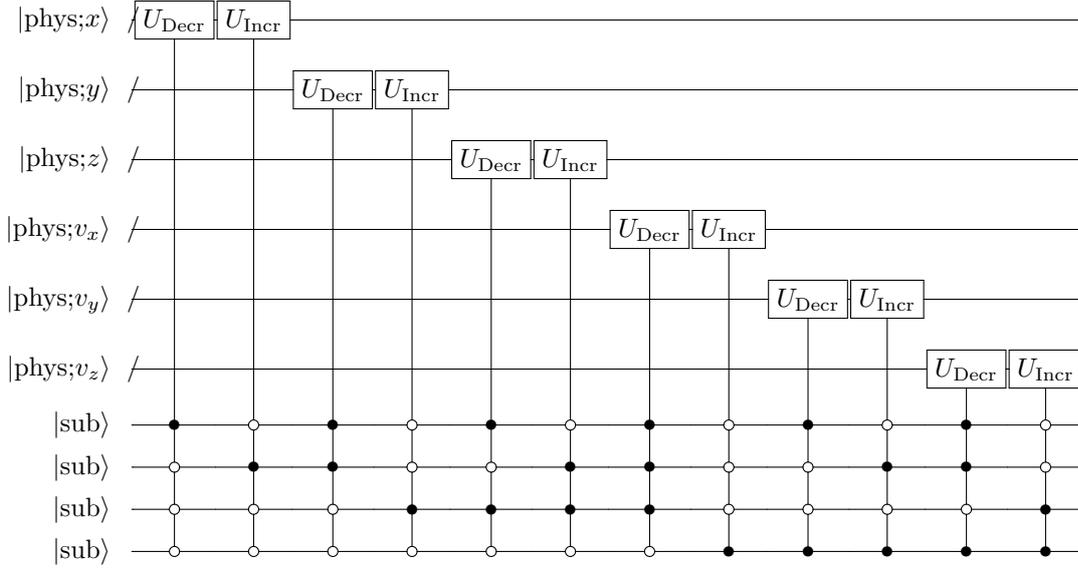


Figure 2. A Quantum circuit for the shift operators. Increment and Decrement operators controlled by subnodes are aligned according to the order of TABLE 1.

303 operator moves computational bases in each subspace by ± 1 , respecting periodic bound-
 304 ary conditions in each orientation. This ± 1 direction is specified by the index j as shown
 305 in TABLE 1. Therefore, $\tilde{f}_{i,j}$ can be represented as follows:

$$\tilde{f}_{i,j} = \tilde{f}_{sL+t+(-1)^j \text{mod} L}, \quad (27)$$

306 when $i = sL + t$, ($0 \leq s < 6$, $0 \leq t < L$).

307 2.1.3 Integration

308 Passing through the encoding and propagation steps so far, we obtain a state in
 309 which the all 13 terms arising in the right-hand side of the equation (10) for a fixed time
 310 step under are encoded in the amplitude of each basis state. In this step, we perform a
 311 superposition of subnode states to compute the sum of all terms and collect them into
 312 the amplitude of a single state $|0000\rangle_{\text{sub}}$. However, As a preprocessing step, we need to
 313 invert the phases of certain states as explained below.

314 The amplitude of each basis are multiplied by the coefficients in the difference equa-
 315 tion 10, excluding the explicit sign, which is denoted by sigma in TABLE 1. Therefore,
 316 we need to inverse the phase of corresponding state for the terms with a minus sign. This
 317 process is also very simple and only requires one application of Z gate as shown in cir-
 318 cuit 28 before applying H gates.

Finally, we superimpose all sunode states by applying H as shown in circuit(28).

$$\begin{array}{c}
 |\text{sub}\rangle \text{ --- } [Z] \text{ --- } [H] \text{ ---} \\
 |\text{sub}\rangle \text{ --- } [H] \text{ ---} \\
 |\text{sub}\rangle \text{ --- } [H] \text{ ---} \\
 |\text{sub}\rangle \text{ --- } [H] \text{ ---}
 \end{array} \quad (28)$$

319 As a result, the amplitudes of the states from $|0000\rangle_{\text{sub}}$ to $|1111\rangle_{\text{sub}}$ are summed and gath-
 320 ered as the amplitude of $|0000\rangle_{\text{sub}}$ state with equal weighting of $1/4$. Therefore, we fi-
 321 nally obtain the following state

$$|\phi_3\rangle = \frac{1}{4} \sum_{i=0}^V \sum_{j=0}^{12} \sigma_j \tilde{\epsilon}_j \tilde{f}_{i,j} |i\rangle_{\text{phys}} |0000\rangle_{\text{sub}} |0\rangle_{\text{ancilla}} + |*\rangle. \quad (29)$$

322 With more clear form, we can write:

$$\begin{aligned} \sum_{j=0}^{12} \sigma_j \tilde{\epsilon}_j \tilde{f}_{i,j} &\sim f - (f_{x+\Delta x} - f_{x-\Delta x}) - (f_{y+\Delta y} - f_{y-\Delta y}) - (f_{z+\Delta z} - f_{z-\Delta z}) \\ &\quad - \frac{q(\mathbf{E} + \mathbf{F})_x \Delta t}{2m\Delta v_x} (f_{v_x+\Delta v_x} - f_{v_x-\Delta v_x}) - \frac{q(\mathbf{E} + \mathbf{F})_y \Delta t}{2m\Delta v_y} (f_{v_y+\Delta v_y} - f_{v_y-\Delta v_y}), \\ &= f(x, y, z, v_x, v_y, v_z; t + \Delta t), \end{aligned} \quad (30)$$

323 where the distribution function is at the corresponding point of (x, y, z, v_x, v_y, v_z) to the
 324 index i . Since the normalizing factors of f and ϵ are involved here, the relation is denoted
 325 as “ \sim ”.

326 According to the resultant state $|\phi_3\rangle$, we can measure the physical and subnode
 327 qubits and focus on the $|0\rangle_{\text{sub}}$ to obtain a distribution function that is one time step evo-
 328 lved according to the Boltzmann-Maxwell equation. For further time steps, we can use
 329 this distribution function as an initial value to input to the first encoding step, and fur-
 330 ther time evolution can be implemented by performing similar steps.

331 Here are remarks on this algorithm, most of what is touched on here will be dis-
 332 cussed more comprehensively in the Section 4. First, we asserted that the measurement
 333 of the state delivers the value of the distribution function; however, what is specifically
 334 attained is the square of the absolute value of the distribution function. Nevertheless,
 335 given that the value of the distribution function f is consistently real and non-negative,
 336 the precise value of f can be accurately recovered from the measurements. On the other
 337 hand, \mathbf{E} and \mathbf{B} handled by Maxwell solver in Appendix Appendix A are real but also
 338 have negative values, so not exactly the same algorithm can be used. However, during
 339 computation with real quantum algorithms, there isn't a genuine necessity to measure
 340 the values of \mathbf{E} and \mathbf{B} . The primary function of the Maxwell solver is simply to convey
 341 these values to the Boltzmann solver within the quantum circuit, hence this does not pre-
 342 sent a significant issue. If one want to measure \mathbf{E} and \mathbf{B} values as well, a further an-
 343 cilla node that identifies the sign must be prepared, and an additional quantum oracle
 344 is also needed.

345 Next, Actually measuring f does not lead to quantum advantage. This is because
 346 f still has $O(V = L^6)$ degrees of freedom, and it is inevitable to measure it $O(V)$ times
 347 in order to obtain full information. However, this problem can be avoided because what
 348 we are physically interested in is not f itself, but the velocity moment quantity obtained
 349 by integrating f with respect to velocity v . If we could implement this integral, i.e., just
 350 a sum in the discrete system, in an efficient quantum algorithm, the computational com-
 351 plexity would be superior to that of a naive classical algorithm. Furthermore, we believe
 352 that it is possible to reduce the Hilbert space to be measured based on physical condi-
 353 tions such as uniformity with respect to a certain spatial direction, limiting the measure-
 354 ment to the physical space of interest, etc.

355 3 Comparison

356 In this paper, all quantum circuits were exactly simulated by dealing directly with
 357 statevectors. Thus it is expected that the results will be in exact agreement with numer-
 358 ical calculations using conventional classical algorithms. We prepared $L = 8$ lattice sites

359 in each spatial and velocity direction and calculated with the volume $V = 8^6$. As for
 360 the quantum algorithm $6 \times \lceil \log_2 L \rceil = 18$ qubits were used as $|\text{phys}\rangle$.

361 And we set $\Delta x = \Delta y = \Delta z = 30\text{m}$, $\Delta t = 10^{-7}\text{s}$, satisfying the assumption (9).
 362 Thereby, $v_x = v_y = v_z = 3 \times 10^8\text{m/s}$ is constant at the speed of light. The plasma
 363 particles are assumed to be positrons and set $e = 1.6 \times 10^{-19}\text{C}$, $m_e = 9.1 \times 10^{-31}\text{kg}$, so
 364 we put $\Delta v_x = \Delta v_y = \Delta v_z = 10^5\text{m/s}$. In this section, for simplicity, we re-scale vari-
 365 ables x, y, \dots dividing by the unit $\Delta x, \Delta y, \dots$ and denote them as coordinates on a lat-
 366 tice space. That is, $x = n$ denotes the point where $x = n\Delta x$ physically.

367 3.1 Initial condition

368 As the initial distribution function, we employed a simple setup: we set 0 for $(x =$
 369 $1, y = 1)$ or $(v_x = 1, v_y = 1)$, and set 1 for the other spaces. Namely,

$$\begin{aligned} f(x, y, z, v_x, v_y, v_z; t = 0)|_{x=1 \cap y=1} &= 0, \\ f(x, y, z, v_x, v_y, v_z; t = 0)|_{v_x=1 \cap v_y=1} &= 0, \\ f(x, y, z, v_x, v_y, v_z; t = 0) &= 1 \text{ (otherwise)}. \end{aligned}$$

370 This is a simple setup to compare the agreement with the classical algorithm, and in prac-
 371 tice it is necessary to give a suitable initial condition corresponding to considering phys-
 ical phenomena such as plasma.

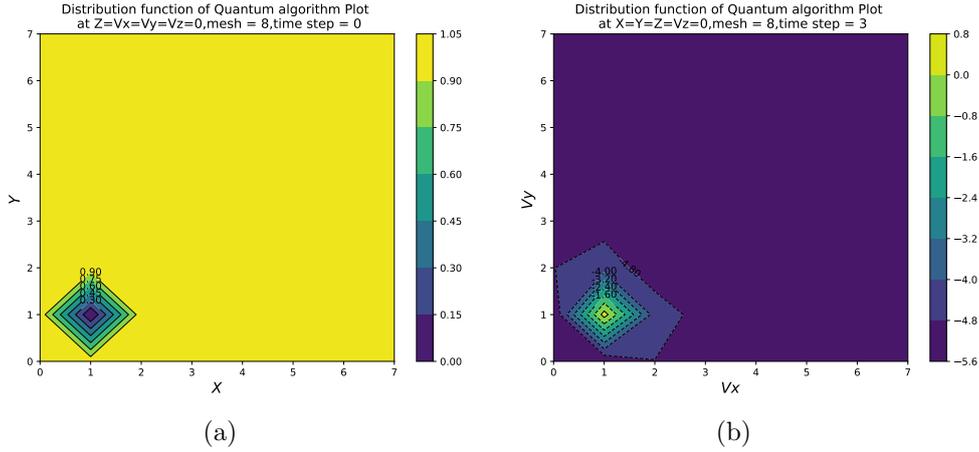


Figure 3. The initial distribution function in the space for (a) the $x - y$ subplane with $z = v_x = v_y = v_z = 0$, and (b) the $v_x - v_y$ subplane with $x = y = z = v_z = 0$. This makes it possible to check the influence of electromagnetic fields on propagation in velocity space as well as in real space.

372

373 Since we implemented the Increment/Decrement circuits periodic (24), the simu-
 374 lation results are also periodic so that the 0-th and L -th lattice points are identical for
 375 all directions.

376 3.2 Simulation result

377 We implemented our quantum algorithm with the input conditions and advanced
 378 time evolution from time step = 0 to time step = 3.

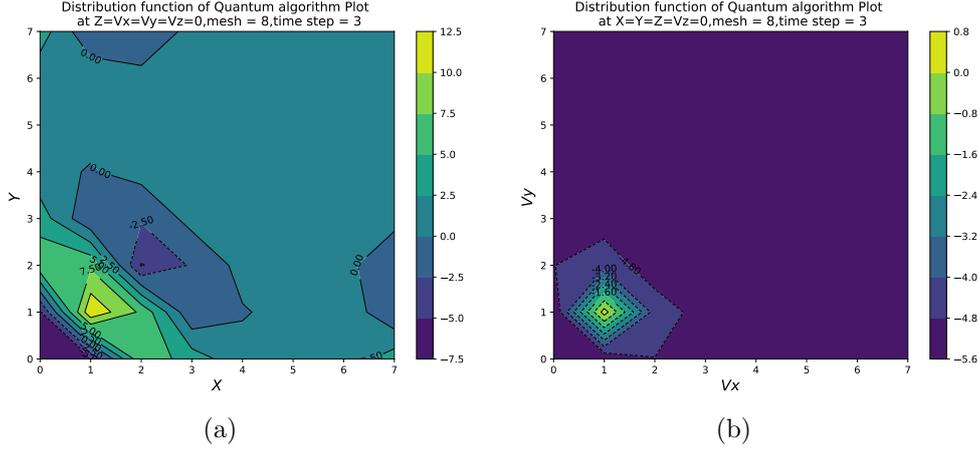


Figure 4. The results show (a) real space propagation at $z = v_x = v_y = v_z = 0$ and (b) velocity space propagation at $x = y = z = v_z = 0$ with time evolution to time step = 3 using our quantum algorithm.

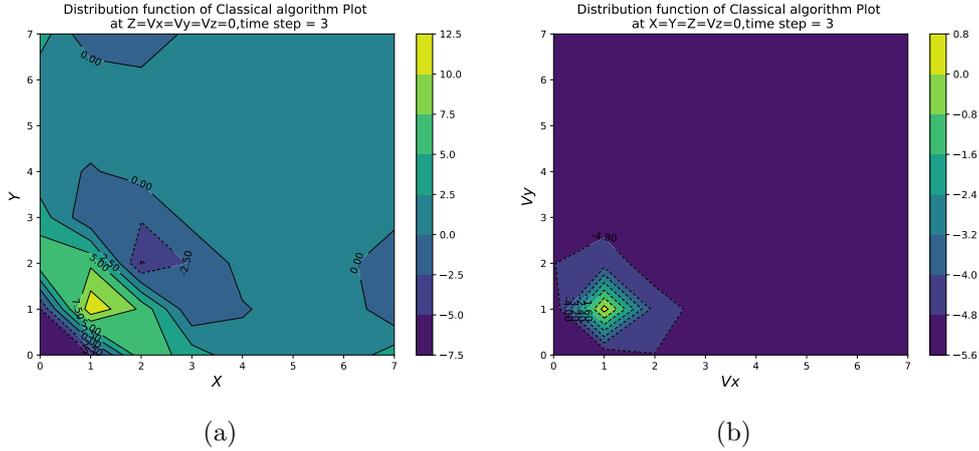


Figure 5. The results are based on a classical algorithm of the time evolution of the difference equations (4,6,7) using the same FTCS scheme as in this paper, with similar initial and boundary conditions. (a) shows real space propagation at $z = v_x = v_y = v_z = 0$ and (b) shows velocity space propagation at $x = y = z = v_z = 0$ with time evolution to time step = 3

379 Comparing FIG. 4 and FIG. 5, the simulation results of the quantum algorithm
 380 perfectly match those of the classical algorithm with similar conditions and methods. This
 381 is because we are simulating exactly with statevector in this case, and the actual results
 382 based on measurements will have statistical errors depending on the number of shots.

383 Although f should take values between 0 and 1, this is not the case in FIG. 4 and
 384 FIG. 5. This is a consequence of numerical diffusion due to discretization using the FTCS
 385 scheme, which occurs universally in classical algorithms. As noted in the discussion, the
 386 numerical diffusion is reduced by $O(\Delta t)$ in the time direction and $O((\Delta x)^2)$ in the space
 387 direction, so it is guaranteed to give correct results if the calculation is performed on
 388 a sufficiently large system.

389 The propagation in real space and velocity space is different, showing that it is acted
 390 upon by the electromagnetic field solved with the Maxwell solver. We achieved one of
 391 our goals in this paper, that is, the coupling of the Boltzmann equation and the Maxwell
 392 equation. However, note that this is a unilateral interaction from the Maxwell equation,
 393 since the assumption of uniform velocity and vacuum condition is used.

394 4 Discussion

395 Our plasma simulator is not yet able to cover generic phenomena according to the
 396 governing equations (1,2,3). This paper is in the middle stage of our project. This means
 397 that our plasma simulator does not yet account for velocity inhomogeneity in the con-
 398 vective term of the distribution function, the interaction between electromagnetic fields
 399 and plasma particles, and the collisional effects. To add these physical effects, new quan-
 400 tum algorithms must be developed.

- 401 • Self-consistent collisionless Boltzmann-Maxwell equations interacting with the elec-
 402 tromagnetic field by calculating ρ charge density, velocity, and \mathbf{j} current density
 403 in moment quantities of the distribution function:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} &= 0, \\ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \mathbf{j}}{\partial t}, \\ \nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 (\nabla \times \mathbf{j}). \end{aligned}$$

404 The next stage will be to improve the current quantum algorithm to the quantum al-
 405 gorithm for the collisionless Boltzmann-Maxwell equation described above. To do this,
 406 a quantum algorithm that calculates the amount of velocity moments in the distribu-
 407 tion function should be developed. Thereby, the electromagnetic field and plasma par-
 408 ticles can interact with each other via velocity inhomogeneity, charge density, and cur-
 409 rent density. This stage can simulate all the complex kinetic effects of collisionless plasma
 410 in an electromagnetic field; it simulates macroscopic MHD phenomena that reflect ki-
 411 netic effects as Micro phenomena. In other words, even macroscopic phenomena can fall
 412 back to microscopic phenomena, thus contributing to the complete understanding of the
 413 physical process and to the prediction. The domain covers space plasmas in space plan-
 414 etary science, such as the solar surface, and the earth's magnetosphere and astrophysics,
 415 such as black hole accretion disks and interstellar winds.

- 416 • Self-consistent collisional Boltzmann-Maxwell equations interacting with an elec-
 417 tromagnetic field, with the addition of a first-principles collision term:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} &= Col(f, f'), \\ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \mathbf{j}}{\partial t}, \\ \nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\mu_0 (\nabla \times \mathbf{j}). \end{aligned}$$

418 Furthermore, in the final stage, this quantum algorithm will be improved to a quantum
 419 algorithm for computing the collision term from the distribution function. By adding a
 420 first-principles collision term, the domain of coverage is further extended. It covers the
 421 highly complex collisional effects of space plasma versus neutral atmospheres, simulat-
 422 ing the ionospheric dynamics of various planetary systems; except for Maxwell solver,
 423 it calculates non-equilibrium states of rarefied gases first principles; apply Boltzmann so-
 424 lver and it solves problems of neutrinos and bubble structure in the universe.

425 We used a finite difference FTCS scheme as our numerical model; the FTCS scheme
 426 has numerical errors on the order of $O(\Delta t)$, $O(\Delta x_i^2)$ and $O(\Delta v_{x_i}^2)$ per time evolution.
 427 Previously, 6D Vlasov simulation research using classical computers has been able to al-
 428 locate only $L \sim 100$ (L : lattices per spatial degree of freedom), even using supercom-
 429 puters. Therefore, simple numerical methods such as the FTCS scheme are not very ap-
 430 propriate for classical algorithms because of the large numerical errors. However, in the
 431 case of quantum computation with a large-scale quantum computer in a domain that is
 432 impossible with a classical computer, the number of lattices per spatial degree of free-
 433 dom ($\gg 100$ lattices) is a very large quantity, and thus the numerical error is inevitably
 434 very small. For example, we estimate that $L > 10^6$ is needed to simulate the auroral elec-
 435 tron acceleration problem in the magnetosphere-ionosphere. For that very large L , the
 436 numerical error from the FTCS scheme is small enough. Moreover, since L increases ex-
 437 ponentially with the line increase in hardware logical qubits, the speed of expansion and
 438 growth of the computational domain and the speed of improvement in accuracy become
 439 exponential.

440 The greatest advantage of quantum algorithms over classical algorithms is massively
 441 parallelization. We estimate the Quantum Volume of our quantum algorithm and de-
 442 scribe the quantum advantage of the Boltzmann-Maxwell equation. Simply, we will call
 443 Quantum Volume=width(number of qubits) \times depth(number of gates) in our quantum
 444 algorithm. The width of this quantum algorithm is $6 \log_2(L) + 6$ where L denotes the
 445 number of lattice points in each direction. Comparing to the classical algorithm $O(L)$
 446 computational complexity of the classical algorithm, the fact that it can be expressed
 447 in $\log_2(L)$ qubits is a quantum advantage. On the other hand, the measured quantum
 448 circuits for $L = 2$, $L = 4$, and $L = 8$, were found to be approximately $600 \times \log_2(L)$
 449 per time evolution. In case of time evolution to Time step = N_t , the approximated Quan-
 450 tum Volume would be $3600 \times N_t \log_2(L) (\log_2(L) + 1)$. This is of the order of of the scale:
 451 $O(N_t (\log_2(L))^2)$. Compared to the computational volume of a similar classical algo-
 452 rithm $O(N_t L^6)$, the order is improved by compression of 6D spatial information. Thus,
 453 the larger L is, the higher the quantum superiority.

454 Our quantum algorithms are intended for a future large-scale quantum computer,
 455 but there remain several issues in terms of efficient algorithms. There is a problem of
 456 the efficient preparation of the initial distribution function on quantum circuits. The En-
 457 coding step Appendix B method has the exponential complexity $O(2^N)$ of preparing ar-
 458bitrary quantum states in a 2^N -dimensional Hilbert space with an N qubit(Zalka, 1998;
 459 Georgescu et al., 2014). This problem is an important topic in quantum computation,
 460 and various efficient methods have been proposed. For example, Georgescu et al. devel-
 461 oped an efficient method to prepare quantum states with polynomial complexity in a num-
 462 ber of qubits(Georgescu et al., 2014), and other efficient quantum state initialization meth-
 463 ods such as log-concave. Other efficient methods for specific cases, such as log-concave
 464 probability distribution functions, have been reported as well(Grover & Rudolph, 2002).
 465 Although the initial distribution function varies depending on the physical phenomenon
 466 to be simulated, the Maxwell velocity distribution function, for example, is a log-concave
 467 probability distribution function and may be efficiently prepared(Todorova & Steijl, 2020).

468 Our quantum algorithm is more efficient than the classical algorithm in spatial in-
 469 formation, but not in the time direction. The reason for this is that the finite difference
 470 method of a numerical computation does not allow time information to enter the width
 471 of quantum circuits. The finite difference method is a time-marching-based method for
 472 classical numerical calculations using the forward term on the left side of the difference
 473 equation. Due to its nature, one of the degrees of freedom must always be in the depth
 474 when implemented in a quantum computer. Variables that are not set to width are not
 475 accelerated, so there are restrictions on the number of lattices with respect to the num-
 476 ber of degrees of freedom that can be set to depth, even for large-scale quantum com-
 477 putation. One simple way to improve this is to rewrite the difference equation of the fi-

478 nite difference method so that the smallest number of lattice degrees of freedom is the
 479 evolution parameter instead of time. Although only one degree of freedom is restricted,
 480 this method can keep the depth relatively small.

481 A common problem in quantum differential equation solving is the problem of van-
 482 ishing time-marching-based measurement probabilities. In general terms, quantum lin-
 483 ear system algorithms have an exponentially decreasing measurement probability with
 484 respect to the time step, depending on the number of time steps. The quantum algorithm
 485 in this study suffers from the same problem. The first possible solution to this problem
 486 is the application of the compression gadget proposed by Fang et al (Fang et al., 2023).
 487 This is a time-marching-based quantum differential equation solving method that is in-
 488 dependent of time steps by repeating uniform singular value amplification. They verified
 489 their implementation on linear ODEs, but it may be applicable to our PDEs. Next, we
 490 also consider the use of different quantum differential equation solving methods as a so-
 491 lution. Hamiltonian simulations are a common method for solving quantum differential
 492 equations, and the Vlasov-poisson and Vlasov-Maxwell equations have already been used
 493 (Toyoizumi et al., 2023; Engel et al., 2019). While it is easy to implement the compres-
 494 sion gadget (Fang et al., 2023) within a Hamiltonian simulation, we consider that it is
 495 difficult to implement the nonlinear Boltzmann-Maxwell equations with first-principles
 496 collision terms in a Hamiltonian simulation.

497 5 Summary

498 In this paper, a novel quantum algorithm for solving the Boltzmann-Maxwell equa-
 499 tion for collisionless plasmas has been formulated; both the Boltzmann and Maxwell equa-
 500 tion solvers were structured with a similar quantum circuit. To confirm the validity of
 501 our quantum algorithm, we performed simulations of the distribution function propa-
 502 gation process under the background electromagnetic field propagation using the Qiskit
 503 platform. We compared the results of the quantum calculation with the results of the
 504 parallel classical calculation and found perfect agreement between them. This completes
 505 the framework for efficiently solving nonlinear problems in various plasmas, such as space
 506 plasmas. Prospective endeavors may cultivate the development of a more generalized quan-
 507 tum algorithm for the Boltzmann-Maxwell equation for collisional plasmas, wherein the
 508 vacuum condition is eliminated and first-principles collision terms are incorporated.

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517 Appendix A Maxwell solver

518 The basic structure of the Maxwell solver is almost identical to that of the Boltz-
 519 mann solver. Similar to the Boltzmann solver, the Maxwell solver consists of three steps:
 520 encoding, propagation, and integration. The algorithm is briefly described, with special
 521 emphasis on the differences to the Boltzmann solver.

522

A1 Encoding

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In Maxwell solver, the physical quantities \mathbf{E} and \mathbf{B} are written together as \mathbf{g} , and develop them simultaneously according to the equations (11,12). Since there are no velocity degrees of freedom, only $N_{\text{phys}} = 3\lceil\log_2 L\rceil$ qubit are prepared for $|\text{phys}\rangle$, and one additional qubit representing time is also prepared. $|\text{sub}\rangle$ requires $N_{\text{sub}} = 6$ qubit in this case. This is because we need $N_{\text{species}} = 1$ qubit to distinguish the difference of the physical quantity, namely \mathbf{E} or \mathbf{B} , $N_{\text{direction}} = 2(= \lceil\log_2 3\rceil)$ qubits to specify the elements of the vector for them as they are vector, and $N_{\text{term}} = 3(= \lceil\log_2 8\rceil)$ qubits to indicate 8 terms appearing the equations (11,12). Collectively, these are called subnodes, but their roles are actually divided as follows:

$$|\text{sub}\rangle \rightarrow |\text{species}\rangle|\text{direction}\rangle|\text{sub}\rangle. \quad (\text{A1})$$

532

533

534

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These correspondences are shown in Table A1 where ϵ and σ represent the the coefficient and explicit sign of each term in the equations (11,12). Therefore, using exactly the same algorithm as the Boltzmann solver, we obtain the following state as the outcome of this encoding step:

$$|\phi_1\rangle = \sum_{i=0}^{V-1} \sum_{s=0}^1 \sum_{d=0}^2 \tilde{g}_{i,t,d} |i\rangle_{\text{phys}} |0\rangle_{\text{time}} |s\rangle_{\text{species}} |d\rangle_{\text{direction}} |0\rangle_{\text{term}} |0\rangle_{\text{ancilla}}, \quad (\text{A2})$$

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where the subscript i indicates a lattice point using the same rules as in the Boltzmann solver, $g_{i,t,d}$ are given in TABLE A1, and \tilde{g} is normalized g . At the first time step we need to specify the initial values for g .

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A2 Propagation

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The structure of the Propagation step in Maxwell solver is fundamentally a Quantum Walk, similar to the Propagation in Boltzmann solver. Thus we need to construct the coin operator and the shift operator. However, the elements of the Coin operator, the time qubits, and the type of subnodes are different. Furthermore, the time increment circuit is used only with respect to the state $|\text{111}\rangle_{\text{sub}}$ to use the physical quantity of one previous time. Therefore, in this section, Propagation step generate the states corresponding to the terms propagated in space-time by using the increment and decrement circuits.

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The coin operator acts on the subnodes.

$$U_{\text{coin}} |s\rangle_{\text{species}} |d\rangle_{\text{direction}} |j\rangle_{\text{sub}} = \tilde{\epsilon}_{s,d,j} |s\rangle_{\text{species}} |d\rangle_{\text{direction}} |j\rangle_{\text{sub}}, \quad (\text{A3})$$

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where you can also find $\epsilon_{s,d,j}$ in TABLE A1 and $\tilde{\epsilon}$ is normalized ϵ .

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One difference from the Boltzmann solver is that the right-hand side of the expression (11,12) contains a term $g_{i,t-1,s,d}$ that also evolves in the time direction. This effect can be easily implemented by treating time as part of the spatial direction and applying the shift operator in the same way, but note that only the increment circuit is operated since the direction is only negative. After operating the coin and the shift operator, we obtain the following state as the outcome of this propagation step:

$$|\phi_2\rangle = \sum_{i=0}^{V-1} \sum_{t=0}^1 \sum_{s=0}^1 \sum_{d=0}^2 \sum_{j=0}^7 \tilde{\epsilon}_{s,d,j} \tilde{g}_{i,t,s,d} |i\rangle_{\text{phys}} |t\rangle_{\text{time}} |s\rangle_{\text{species}} |d\rangle_{\text{direction}} |j\rangle_{\text{sub}} |0\rangle_{\text{ancilla}} + |*\rangle |1\rangle_{\text{ancilla}}, \quad (\text{A4})$$

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where $\tilde{g}_{i,t,s,d}$ represents the shift of ± 1 unit in each spatial and the temporal. As for the time direction, $|1\rangle_{\text{time}} |\text{111}\rangle_{\text{sub}}$ and the initial amplitude at $|0\rangle_{\text{time}} |000\rangle_{\text{sub}}$ are exchanged by the increment circuit (25). The reason for this exchange is because one previous time state is needed to generate a term that propagates in the time direction.

559

A3 Integration

In contrast to the Boltzmann equation, the Maxwell equation is a second-order differential equation. As a result, the signs σ_j that appear in the corresponding difference equation (10) differ from those in the Boltzmann equation (as shown in Table A1). In such cases, an controlled-inverse gate, which is shown as follows, should be applied prior to the superposition by the H gate:

$$U_{\text{Inv.}} = \begin{array}{c} | \text{sub} \rangle \\ | \text{sub} \rangle \\ | \text{sub} \rangle \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

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The rest of the integration step can use the same method as the Boltzmann solver, but this time we are dealing with different physical quantities, \mathbf{E} and \mathbf{B} , in the same circuit, so we need to sum each of them and not confuse them. As a result, we can specify the spatial lattice point (i) and the species, and obtain the time-evolved quantities \mathbf{E} , \mathbf{B} developed in the amplitude of $|000\rangle_{\text{sub}}$.

Table A1. The subnode bases and their corresponding physical quantities. g , ϵ , and σ respectively represent the (unnormalized) electromagnetic fields associated with each basis state, the coefficients to be incorporated via the coin operator, and the sign to be multiplied during the integration step. These are the quantities that appear on the right side of the difference equations (11,12). Here we write only for $|\text{direction}\rangle = |00\rangle_{\text{direction}}$ as an example; $|01\rangle_{\text{direction}}$ and $|10\rangle_{\text{direction}}$ correspond to the y - and z - components of \mathbf{E} and \mathbf{F} , respectively.

$ s\rangle_{\text{species}} d=0\rangle_{\text{direction}} j\rangle_{\text{sub}}$	$g_{s,d=0,j}$	$\epsilon_{s,d=0,j}$	σ_j
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 000\rangle_{\text{sub}}$	$E_x(x, y, z; t)$	-4	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 001\rangle_{\text{sub}}$	$E_x(x + \Delta x, y, z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 010\rangle_{\text{sub}}$	$E_x(x - \Delta x, y, z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 011\rangle_{\text{sub}}$	$E_x(x, y + \Delta y, z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 100\rangle_{\text{sub}}$	$E_x(x, y - \Delta y, z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 101\rangle_{\text{sub}}$	$E_x(x, y, z + \Delta z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 110\rangle_{\text{sub}}$	$E_x(x, y, z - \Delta z; t)$	1	+1
$ 0\rangle_{\text{species}} 00\rangle_{\text{direction}} 111\rangle_{\text{sub}}$	$E_x(x, y, z; t - \Delta t)$	1	-1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 000\rangle_{\text{sub}}$	$F_x(x, y, z; t)$	-4	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 001\rangle_{\text{sub}}$	$F_x(x + \Delta x, y, z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 010\rangle_{\text{sub}}$	$F_x(x - \Delta x, y, z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 011\rangle_{\text{sub}}$	$F_x(x, y + \Delta y, z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 100\rangle_{\text{sub}}$	$F_x(x, y - \Delta y, z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 101\rangle_{\text{sub}}$	$F_x(x, y, z + \Delta z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 110\rangle_{\text{sub}}$	$F_x(x, y, z - \Delta z; t)$	1	+1
$ 1\rangle_{\text{species}} 00\rangle_{\text{direction}} 111\rangle_{\text{sub}}$	$F_x(x, y, z; t - \Delta t)$	1	-1

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Appendix B Construction of our coin operator

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In this section we consider an algorithm to multiply a vector to each quantum basis. Let Λ denote the multiplying vector:

$$\Lambda = (\lambda_0, \lambda_2, \dots, \lambda_{M-1}), \quad (\text{B1})$$

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where we suppose that $\{\lambda\}$ take real values and Λ be normalized: $\sum_i \lambda_i^2 = 1$.

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To implement this algorithm, we need operate a diagonal matrix \mathcal{A} having entries corresponding to Λ but this cannot be done directly because it is not unitary operator

571 in general. Thus we realized this non-unitary operation by using one ancilla qubit and
 572 embedding the matrix \mathcal{A} in a unitary matrix with larger size, which is known as the block
 573 encoding method. As $\{\lambda\}$ are always real, this procedure can easily be implemented as
 574 follows:

$$U = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & -\mathcal{A} \end{pmatrix}, \quad (\text{B2})$$

575 with

$$\mathcal{A} = \text{diag}(\lambda_1, \lambda_2, \dots), \quad (\text{B3})$$

$$\mathcal{B} = \text{diag}\left(\sqrt{1 - \lambda_1^2}, \sqrt{1 - \lambda_2^2}, \dots\right). \quad (\text{B4})$$

576 After performing this unitary operation on an arbitrary state:

$$|\psi\rangle = \sum_i \alpha_i |i\rangle_{\text{phys}} |0\rangle_{\text{anc}}, \quad (\text{B5})$$

577 we obtain the following state:

$$|\psi'\rangle = U|\psi\rangle, \quad (\text{B6})$$

$$= \sum_i \lambda_i \alpha_i |i\rangle_{\text{phys}} |0\rangle_{\text{ancilla}} + |*\rangle |1\rangle_{\text{ancilla}}, \quad (\text{B7})$$

578 which we can distinguish desired/unnecessary states with $|0/1\rangle_{\text{ancilla}}$.

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