

Simplest energy balance models for the global temperature anomaly are deterministic, c.f. those used in IAMs (DICE/PAGE/FUND).

$$\frac{d\Delta Q}{dt} = C \frac{d\Delta T}{dt} = -\lambda \Delta T + F(t)$$

Hasselmann's approach of adding white (delta-correlated) noise improved realism.

$$\left(\frac{d}{dt} + \frac{\lambda}{C}\right) \Delta T = \frac{F(t)}{C} + \frac{\xi}{C}$$

$$\langle \xi(t) \xi(t') \rangle = \sigma^2 \delta(t - t')$$

However as noted by [Leith, 1994](#) there is a case for making the noise itself in this model red [c.f. [Padilla et al, 2011](#)], or even long-range dependent ("1/f"), and/or exploring non-exponential long tailed response kernels to replace constant lambda [c.f. ongoing work of Tromso group starting with [Rypdal, JGR, 2012](#)].

In this poster we modify the kernel, and propose using the mapping between the Langevin equation of Brownian motion and the Hasselmann equation in climate to suggest other equations. Noise can be left white-we don't assume presence of a fluctuation dissipation theorem relating kernel to noise.

Heat capacity	\longleftrightarrow	Mass
Temperature anomaly		Velocity
Feedback parameter		Damping constant
Forcing		Drift Force

We show how this approach gives a "generalised Hasselmann model" for arbitrary forcing and noise, and a "fractional Hasselmann model" in the special 1/f case of long range memory. The LHS of the latter is the same as that of [Lovejoy et al's](#) FEBE model.

NWW thanks the Centre for Space Physics at Boston University, The Santa Fe Institute, the Kantz group at MPIPKS in Dresden, and the Metzler group in Potsdam for hospitality; and the MPI, London Mathematical Laboratory, ONR & NORKLIMA for financial support during the last 6 years.



CLIMATE

"Generalised Hasselmann Equation"

$$\left(\frac{d}{dt} \Delta T(t) + \int_0^t \gamma(t-t') \Delta T(t') dt'\right) = \frac{F(t)}{C} + \frac{\xi}{C}$$

Allows full choice of memory kernel γ
long range, short range, hybrid etc.

Specialise to long range memory:
power law kernel $\gamma \sim t^{-\alpha}$,

"Fractional Hasselmann Equation"

when integral operator $\int_0^t dt' \frac{v(t')}{(t-t')^\alpha}$

written (equivalently) as fractional derivative

$$\left(\frac{d}{dt} \Delta T(t) + \gamma_\alpha \frac{d^{\alpha-1}}{dt^{\alpha-1}} \Delta T(t)\right) = \frac{F(t)}{C} + \frac{\xi}{C}$$

Fractionally integrate w.r.t $t^{\alpha-1}$

(LHS) is [Lovejoy et al](#) FEBE
(Fractional Energy Balance Equation)

$$\left(\frac{d^{2-\alpha}}{dt^{2-\alpha}} \Delta T(t) + \gamma_\alpha \Delta T(t)\right) =$$

If [Lovejoy et al's](#) $H \equiv$ our $2-\alpha$
and their $\tau^{-H} \equiv$ our γ_α .

Meaning of RHS is more opaque ...

$$\frac{d^{1-\alpha}}{dt^{1-\alpha}} \frac{1}{C} F(t) + \frac{d^{1-\alpha}}{dt^{1-\alpha}} \frac{1}{C} \xi$$

but can see that even if ξ white,
noise will usually become fractional.

Infrared Climate [Leith, NCAR Conference, 1994](#)

Unfortunately, there is evidence that such a model would be unsatisfactory to capture some of the low-frequency phenomena observed in the atmosphere. This is referred to as the *infrared climate problem* and appears to be caused by non-linear interactions of the chaotic internal weather frequencies that potentially induce a "piling up" of extra variance at the low frequencies.

This infrared climate problem further obscures the predictability of climate change induced by slowly changing external influences, anthropogenic or not.

Specialise to absence
of memory $\gamma(t) = (\lambda / C) \delta(t)$

[Hasselmann, 1976] Equation

$$\left(\frac{d}{dt} + \frac{\lambda}{C}\right) \Delta T(t) = \frac{F(t)}{C} + \frac{\xi}{C}$$

Hasselmann solution for starting time of 0

$$\Delta T(t) = \Delta T(0) e^{-(\lambda/C)t} + \frac{1}{C} \int_0^t e^{-(\lambda/C)(t-\tau)} [F(\tau) + \xi] d\tau$$

(NB in some cases the anomaly $\Delta T(0) = 0$)
becomes

$$\Delta T(t) = \frac{1}{C} \int_{-\infty}^t e^{-(\lambda/C)(t-\tau)} [F(\tau) + \xi] d\tau$$

in the case when start time is $-\infty$.

In "Fractional Hasselmann" case exponential
will be replaced by Mittag-Leffler functions
c.f. [Lovejoy](#) FEBE model (e.g. [EGU, 2019](#))

[Mori, 1965; Kubo, 1966] Generalised Langevin Equation

$$\left(\frac{d}{dt} v(t) + \int_0^t \gamma(t-t') v(t') dt'\right) = \frac{F(t)}{M} + \frac{\xi}{M}$$

$$v = \dot{x}$$

Specialise to long range
memory: kernel $\gamma \sim t^{-\alpha}$,

Specialise to absence
of memory $\gamma(t) = (\eta / M) \delta(t)$

Fractional Langevin Equation [e.g. [Lutz, PRE, 2001](#)]

when integral operator $\int_0^t dt' \frac{v(t')}{(t-t')^\alpha}$

written (equivalently) as fractional derivative

$$\left(\frac{d}{dt} v(t) + \gamma_\alpha \frac{d^{\alpha-1}}{dt^{\alpha-1}} v(t)\right) = \frac{F(t)}{M} + \frac{\xi}{M}$$

If an FDT holds, ξ becomes fractional Gaussian noise,
 ξ_H whose Hurst exponent obeys $2H = 2 - \alpha$

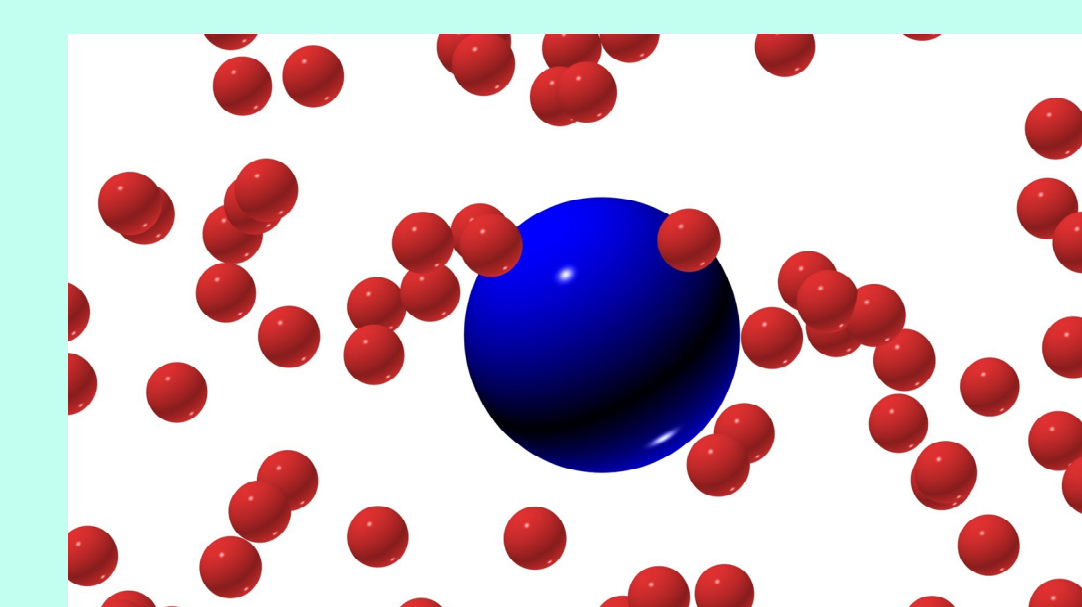
Fractionally integrate w.r.t $t^{\alpha-1}$

$$\left(\frac{d^{2-\alpha}}{dt^{2-\alpha}} v(t) + \gamma_\alpha v(t)\right) =$$

$$\frac{d^{1-\alpha}}{dt^{1-\alpha}} \left(\frac{F(t)}{M} + \frac{\xi}{M}\right)$$

[Langevin, 1908] Equation

$$\left(\frac{d}{dt} + \frac{\eta}{M}\right) v = \frac{F(t)}{M} + \frac{\xi}{M}$$



BROWNIAN MOTION

Standard text [[Gardiner, 2004](#)] points out
(in $F(t)=0$ case) that solution for starting time 0
 $v(t) = v(0) e^{-(\eta/M)t} + \frac{1}{M} \int_0^t e^{-(\eta/M)(t-\tau)} [F(\tau) + \xi] d\tau$
becomes
 $v(t) = \frac{1}{M} \int_{-\infty}^t e^{-(\eta/M)(t-\tau)} [F(\tau) + \xi] d\tau$
in the case when start time is $-\infty$.
In Fractional Langevin case exponential
replaced by Mittag-Leffler functions
e.g. [Lutz, 2001](#); [Kupferman, 2004](#).