Supporting Information for "Modeling the impact of moulin shape on englacial hydrology"

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Introduction The supporting information provides additional information about the simulations and methods. **Text S1** shows the derivation for the change in head with cylindrical and non-cylindrical moulins. **Text S2** describes our moulin shape parameterization for constant meltwater input. **Text S3** describe the 1D discretized version of the single-conduit model used to compare the 0D version of the model used throughout the paper. **Figure S1** and **Figure S2** provide visualizations of the damping and oscillation

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timescales. **Figure S3** is an additional figure showing how an abrupt change in radius impacts the equilibration timescales. **Figure S4** demonstrates how the equilibrium head changes across the ice sheet. **Figure S5** shows the parametrization of moulin shape for an oscillating meltwater input. **Figure S6** compares the single-conduit model with a discretized subglacial channel model. **Table S1 to S6** summarize the input and fitting parameters for all the simulations and figures in the paper.

Text S1. Here we derive the moulin radius as a function of elevation

Case for a cylinder:

The continuity equation says that for Δt, the change in storage, ΔV, equals the input meltwater, Q_{in}, minus the discharge out of the channel, Q_{out}, times Δt, or

$$\frac{\Delta V}{\Delta t} = Q_{in} - Q_{out} \tag{1}$$

For each time-step, the **storage of water** $\Delta V = A_{\mathrm{r}} \Delta h$. If we plug in this relationship to Equation 1, then we get:

$$\frac{\Delta(hA_{\mathrm{r}})}{\Delta t} = Q_{in} - Q_{out} \tag{2}$$

If we rearrange then we obtain:

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{A_{\mathrm{r}}} \tag{3}$$

If $\Delta t \longrightarrow 0$ then:

$$\lim_{\Delta t \to 0} [\frac{\Delta h}{\Delta t}] = \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{Q_{in} - Q_{out}}{A_{\mathrm{r}}} \tag{4}$$

Case for a conical frustum:

If we use Equation 1 and plug in the volume of a frustum $\Delta V = \frac{1}{3}\pi(r_{\text{top}}^2 + r_{\text{top}}r_{\text{base}} + r_{\text{base}}^2) * \Delta h$. We obtain

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\frac{1}{3}\pi(r_{\text{top}}^2 + r_{\text{top}}r_{\text{base}} + r_{\text{base}}^2)} \tag{5}$$

We then define $r_{\text{top}}, r_{\text{base}}$ w.r.t. Δh or Δt. The slope $m = \frac{\Delta h}{\Delta r}$ and the change between the radius $\Delta r = r_{\text{base}} - r_{\text{top}} = \frac{\Delta h}{m}$. Therefore, we can express $r_{\text{base}} = r_{\text{top}} + \frac{\Delta h}{m}$ and replace r_{base} in Equation 5, giving

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\frac{\pi}{3}r_{\text{top}}^2 + [r_{\text{top}}(r_{\text{top}} + \frac{\Delta h}{m})] + (r_{\text{top}} + \frac{\Delta h}{m})^2} \tag{6}$$

We distribute and reorganize the denominator and get

$$\frac{\Delta h}{\Delta t} = \frac{Q_{in} - Q_{out}}{\pi r_{\text{top}}^2 + \pi r_{\text{top}}\frac{\Delta h}{m} + \frac{\pi}{3}\frac{\Delta h^2}{m^2}} \tag{7}$$

If $\Delta t \longrightarrow 0$, $\Delta h \longrightarrow 0$, then $\frac{2r_{\text{top}}\Delta h}{m}$ and $\frac{\Delta h^2}{m^2} \longrightarrow 0$, we are left with πr^2 at the denominator and we recover the continuity equation (1) for a cylindrical moulin:

$$\frac{dh}{dt} = \frac{Q_{in} - Q_{out}}{\pi r^2} = \frac{Q_{in} - Q_{out}}{A_{\text{r}}} \tag{8}$$

Text S2. Here we describe our moulin shape parameterization for constant meltwater input. We use a cone-shaped moulin with various wall slopes and the radius fixed at a certain depth. To explore equilibration timescales, we use a conical frustum where r_{base}/r_{top} can be greater than or less than 0.

$$A_{\text{r}}(z) = \pi(mz + r_{base})^2 \tag{9}$$

To fix the radius at the middle of the ice thickness, we define the base radius to be

$$r_{base} = r_{h_{\text{eq}}} - m(H/2) \tag{10}$$

. To fix the radius at equilibrium head, we define the base radius to be

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$$r_{base} = r_{h_{\text{eq}}} - mh_{\text{eq}} \tag{11}$$

Text S3.

Here we describe the discretized subglacial channel model used as a comparison in Supplemental Figure S6. The model uses Equations 1-3 from the main text, which are the same as for the 0D model described in the model description Section 2 in the main text. However, instead of calculating the effective pressure at the moulin only, we use a one dimensional grid set up from Landlab (Hobley et al., 2017) with nodes every 400m along the conduits. At each node calculate the hydraulic gradient as well as the conduit cross-section evolution via melt and creep for every node along the conduit. The ice thickness is calculated with the square-root glacier function from equation 5 in Section 2. The code for the 1D model is in `https://github.com/speleophysics/landlab/tree/add-pressurized-flow-network-solver/landlab/components/conduit_networks`.

References

Hobley, D. E. J., Adams, J. M., Nudurupati, S. S., Hutton, E. W. H., Gasparini, N. M., Istanbulluoglu, E., & Tucker, G. E. (2017). Creative computing with Landlab: an open-source toolkit for building, coupling, and exploring two-dimensional numerical models of Earth-surface dynamics. *Earth Surface Dynamics*, *5*(1), 21–46. doi: 10.5194/esurf-5-21-2017

Meierbachtol, T., Harper, J., & Humphrey, N. (2013). Basal Drainage System Response to Increasing Surface Melt on the Greenland Ice Sheet. *Science*, *341*(6147), 777–779. doi: 10.1126/science.1235905

Röthlisberger, H. (1972). Water Pressure in Intra- and Subglacial Channels. *Journal of Glaciology*, *11*(62), 177–203. doi: 10.1017/S0022143000022188

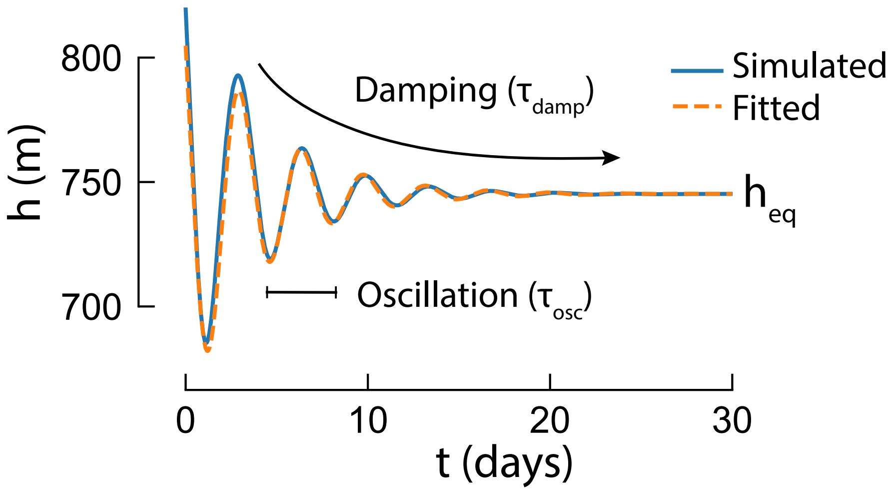

Figure S1. Oscillation of head during equilibration. The solid line shows the full numerical result, and the dashed line shows the fit of an idealized solution for a damped harmonic oscillator. The simulation is for a cylindrical moulin, with $Q_{\mathrm{in}} = 3\,\mathrm{m}^3/\mathrm{s}$ and $r = 10\,\mathrm{m}$.

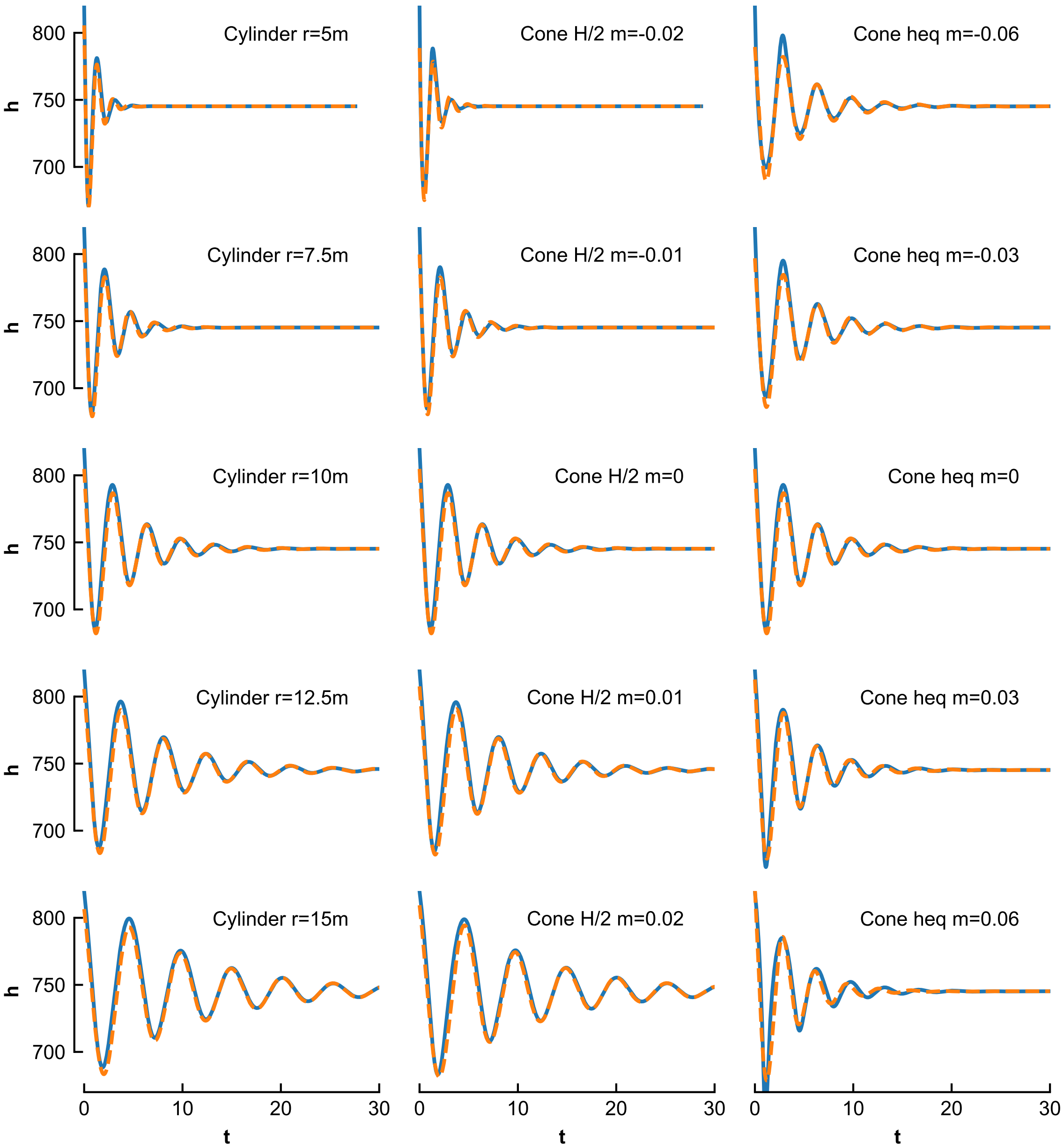

Figure S2. Comparison between simulated and fitted oscillations for many cases.

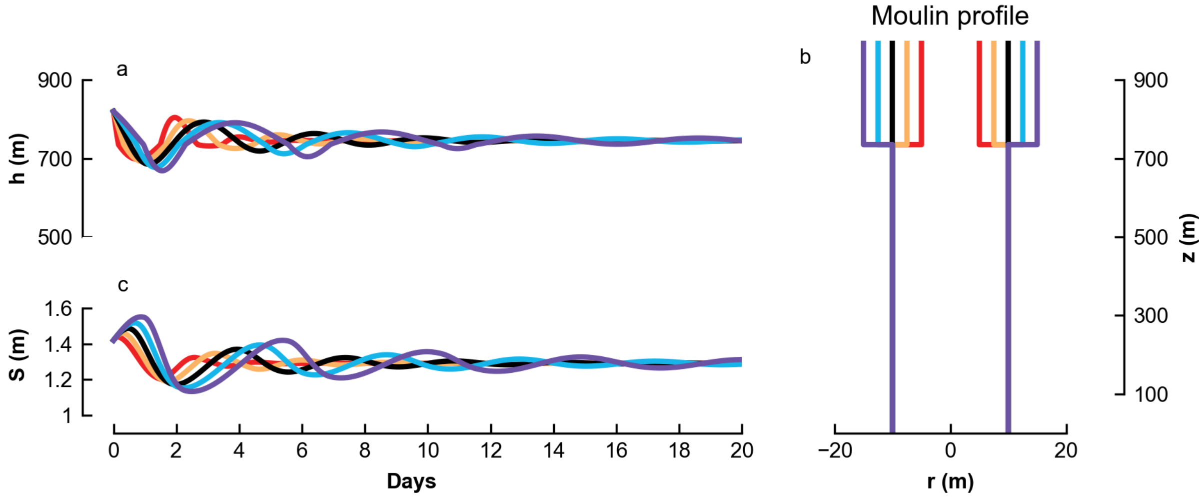

Figure S3. Timeseries of **head (h)** and **channel cross-sectional area (S)** for a fixed meltwater input Q_{in} for bottle-shaped moulins (red and yellow), a cylindrical moulin (black) and goblet-shaped moulins (blue and purple)

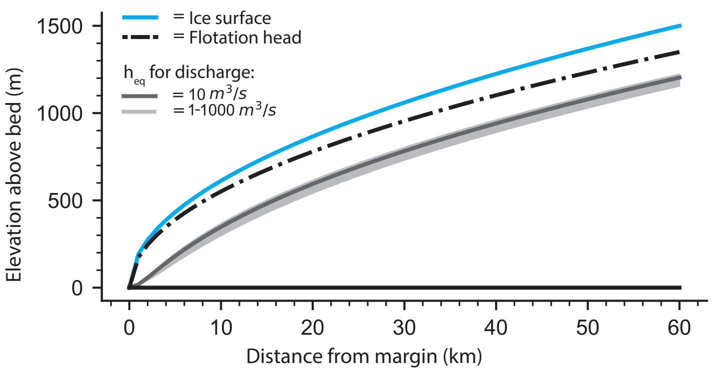

Figure S4. Equilibrium head (h_{eq}) along an ice sheet profile for a wide range of Q_{in}. Equilibrium head (h_{eq}) calculated with the model depends on channel length, ice thickness, and Q_{in}. Meierbachtol et al. (2013); Röthlisberger (1972) described similar profiles.

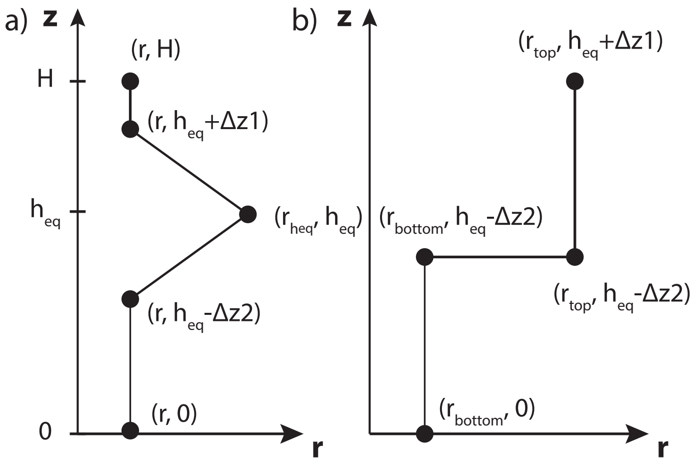

Figure S5. Cartesian coordinates of the moulin shape used for the simulation with oscillating meltwater input. Parameterization for hourglass, diamond and superposed-cylinder shaped moulin. To explore oscillating meltwater input, we define the shape by interpolating the radius defined in the cartesian coordinate system, with r in the x axis, and z in the y axis. Shape coordinates are displayed in Figure S6. The radius (r) is interpolated every meter along the axis z. (a) Hourglass and diamond shaped moulins are defined by five points. (b) Goblet and bottle-shaped moulins defined by four points.

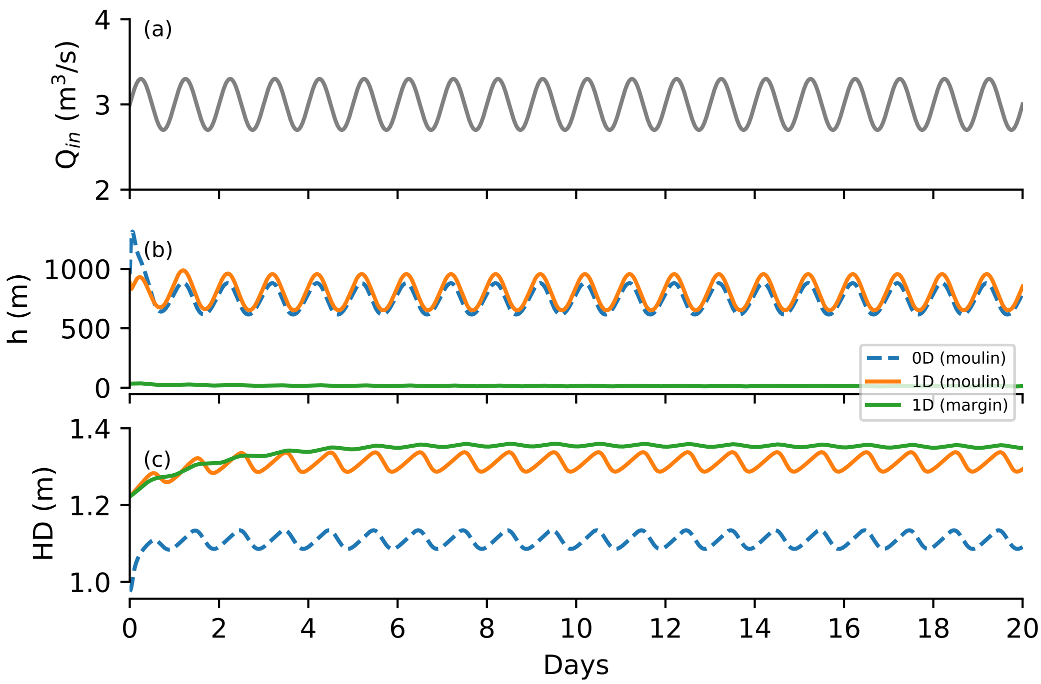

Figure S6. Comparison between the 0D single-conduit model used in the paper and a similar 1D single-conduit model, where each node along the conduit has a different effective pressure and cross-sectional area, using an idealized square-root glacier (see equation 5 in Section 2). (a) Meltwater input (Q_{in}): we use the same sinusoidal meltwater input as for the simulations in the paper: $Q_{\text{mean}} = 3\,\text{m}^3/\text{s}$, $Q_{\text{min}} = 2.6\,\text{m}^3/\text{s}$; (b) Hydraulic head and (c) Hydraulic diameter (HD) at the moulin for the 0D and the 1D model, and at the margin for the 1D model.

Table S1. Constants and model parameters used in the simulations.

Symbol	**Value**	**Description**
ρ_w	1000 kg/m^3	Water density
ρ_i	910 kg/m^3	Ice density
g	9.8 m/s^2	Gravitational acceleration
f	0.1	Darcy-Weissbach friction factor
L_f	3.32e5 J/kg	Latent heat of fusion
B	6e-24 $1/Pa^3 s$	Glen's law fluidity coefficient (Basal softness)
n	3	Glen's law exponent
C_1	$1/(\rho_\mathrm{i} * L_f)$	Melt opening parameter
C_2	$2Bn^{-n}$	Closure parameter
C_3	$2^{5/4}\sqrt{\pi}/(\pi^{1/4}\sqrt{\pi+2}\sqrt{\rho_\mathrm{w} f})$	Flux parameter

Table S2. Model parameters for simulations with constant Q_in in Figure 2 (main text). For the simulations with this parameters, the equilibrium head $h_\mathrm{eq} = 745$, and equilibrium subglacial channel cross-section area $S_\mathrm{eq} = 1.3$

Parameter	Value	Unit	Description
Q_in	3	m^3/s	Constant meltwater input
t_0	0	d	Initial time
t_f	100	d	Final time
H_0	6	m	Ice thickness
h_0	$1.1h_\mathrm{eq}$	m	Initial head
S_0	$1.1S_\mathrm{eq}$	m	Initial subglacial channel cross-section

Table S3. Moulin shape parameters for simulations with constant Q_{in} in Figure 2 (main text). The radius (r) is in meters and the slope (m) is given in percent (%) and degrees (°) from the vertical axis.

plot color	red	yellow	black	blue	purple
Cylinder					
m	0	0	0	0	0
r	5	7.5	10	12.5	15
Cone $H/2$					
$m\%$	-2	-1	0	1	2
$m°$	-1.15	-0.57	0	0.57	1.15
r_{middle}	10	10	10	10	10
r_{heq}	5	7.5	10	12.5	15
r_{base}	20	15	10	5	0
r_{top}	0	5	10	15	20
Cone h_{eq}					
$m\%$	-6	-3	0	3	6
$m°$	-3.43	-1.72	0	1.72	3.43
r_{heq}	10	10	10	10	10
r_{base}	25	17.5	10	2.5	-5
r_{top}	5	7.5	10	12.5	15
Diamond-Hourglass h_{eq}					
$m\%$	-6	-3	0	3	6
$m°$	-3.43	-1.72	0	1.72	3.43
r_{heq}	10	10	10	10	10

Table S4. Fitting parameters for simulations in Figure 2 (main text). The damping timescale (τ_{damp}), the period of oscillation (τ_{osc}), the amplitude (a) in meters, and the phase shift (ϕ) in days. A visual comparison between simulations and fits is provided in Figure S2.

Cylinder	radius	τ_{damp}	τ_{osc}	a	ϕ
red	5.0	0.94	1.64	0.14	2.65
yellow	7.5	2.23	2.53	0.11	2.49
black	10.0	4.08	3.42	0.09	2.37
blue	12.5	6.61	4.30	0.09	2.28
purple	15.0	10.00	5.18	0.08	2.20

Cone $H/2$	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.02	1.18	1.75	0.12	2.73
yellow	-0.01	2.34	2.56	0.10	2.52
black	0.00	4.08	3.42	0.09	2.37
blue	0.01	6.53	4.28	0.09	2.26
purple	0.02	9.87	5.14	0.09	2.18

Cone h_{eq}	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.06	4.22	3.44	0.08	2.49
yellow	-0.03	4.18	3.43	0.09	2.43
black	0.00	4.08	3.42	0.09	2.37
blue	0.03	3.86	3.37	0.10	2.31
purple	0.06	3.29	3.25	0.11	2.24

Diamond-Hourglass h_{eq}	slope	τ_{damp}	τ_{osc}	a	ϕ
red	-0.06	2.71	3.03	0.09	2.59
yellow	-0.03	3.40	3.23	0.09	2.48
black	0.00	4.08	3.42	0.09	2.37
blue	0.03	4.75	3.61	0.10	2.25
purple	0.06	5.39	3.81	0.10	2.14

Table S5. Model parameters from graphs for oscillating Q_{in}, Figure 4 and 5 (main text).

Parameter	Value	Unit	Description
Q_{mean}	3	$\mathrm{m^3/s}$	Mean meltwater input
Q_{a}	0.4	$\mathrm{m^3/s}$	Amplitude of oscillation of the meltwater input
Q_{period}	1	d	Period of oscillation of meltwater input
t_0	0	d	Initial time
t_{f}	50	d	Final time
H	1000	m	Ice thickness
L	30000	m	Subglacial channel length

Table S6. Moulin shape parameters from graphs for oscillating Q_{in}, Figure 4 and 5 (main text). The radius (r) is in meters.

cylinder	red	yellow	black	blue	purple
r (m)	1	3.5	5	8	15

Hourglass-Diamond 1	red	yellow	black	blue	purple
r	1	2	5	10	19
r_{heq}	5	5	5	5	5

Hourglass-Diamond 2	red	yellow	black	blue	purple
r	5	5	5	5	5
r_{heq}	1	3.5	5	8	15

Diamond	red	yellow	black	blue	purple
r	1	1.5	2	4	10
r_{heq}	5	5.5	6	8	14

Hourglass	red	yellow	black	blue	purple
r	5	6.5	8	10	18
r_{heq}	1	2.5	4	6	14

Bottle-Goblet 1	red	yellow	black	blue	purple
r_{top}	3	4	5	6	10
r_{base}	5	5	5	5	5

Bottle-Goblet 2	red	yellow	black	blue	purple
r_{top}	5	5	5	5	5
r_{base}	1	2	4	6	12