Supporting Information for “Inverse methods for quantifying time-varying subglacial perturbations from altimetry”

A.G. Stubblefield[1]

[1]Lamont-Doherty Earth Observatory, Columbia University

Contents of this file

Additional Supporting Information (Files uploaded separately)

Introduction. Text S1 contains a detailed derivation of the small-perturbation model used in the main text. Text S2 contains a description of the nonlinear subglacial lake model used to produce the synthetic data in Figures 8-9. We also provide captions for Movies S1-S5 at the end of the document.

Text S1. Derivation of the small-perturbation model.

Governing Equations

Here, we derive the forward model that is used in the main text. We assume that the domain is an ice slab of finite thickness and infinite horizontal extent. The bed underlying the ice layer is assumed to be oriented at an angle α relative to horizontal in the x direction (see Figure 1 in the main text). To account for this slope, we rotate the gravitational body force vector by this angle α. Therefore, in (x, y, z) space we assume that the ice slab is defined by $|x| < \infty$, $|y| < \infty$, and $s \leq z \leq h$, where h and s are the upper and basal surfaces of the ice sheet, respectively. We assume that h and s are uniform in the background state.

Assuming Newtonian and incompressible Stokes flow, ice deforms according to

$$-p_x + \eta(u_{xx} + u_{yy} + u_{zz}) = -\rho g \sin(\alpha) \tag{1}$$

$$-p_y + \eta(v_{xx} + v_{yy} + v_{zz}) = 0 \tag{2}$$

$$-p_z + \eta(w_{xx} + w_{yy} + w_{zz}) = \rho g \cos(\alpha) \tag{3}$$

$$u_x + v_y + w_z = 0, \tag{4}$$

where $[u, v, w]^T$ is the velocity, p is the pressure, ρ is the ice density, and g is gravitational acceleration. We assume a stress-free condition at the upper surface ($z = h$), which is equivalent to

$$2\eta w_z - p = 0 \tag{5}$$

$$\eta(u_z + w_x) = 0 \tag{6}$$

$$\eta(v_z + w_y) = 0. \tag{7}$$

At the basal surface ($z = s$), we consider a prescribed vertical velocity

$$w = w_b \tag{8}$$

for a given function w_b, along with a linear sliding law

$$\eta(u_z + w_x) = \beta u \tag{9}$$

$$\eta(v_z + w_y) = \beta v, \tag{10}$$

where β is the basal drag coefficient.

The upper and basal surfaces of the ice sheet evolve according to kinematic equations that are coupled to the Stokes flow equations. The upper surface evolves according to

$$h_t + uh_x + vh_y = w + a, \tag{11}$$

where a denotes accumulation or ablation at the ice-sheet surface. Similarly, the basal surface evolves according to

$$s_t + us_x + vs_y = w + m \tag{12}$$

where m is the melt rate.

Background states

We consider background states corresponding to flow down an inclined plane with no variations in the x or y directions. Perturbations will be taken with respect to the background states below. We let $\bar{u}_s$ denote the basal sliding velocity, $\bar{u}_h$ the horizontal surface velocity, and H the ice thickness (see Figure 1 in the main text). We set the background upper surface elevation to $\bar{h} = H$, the background basal surface elevation to $\bar{s} = 0$, the background basal melting rate to $\bar{m} = 0$, the background accumulation rate to $\bar{a} = 0$, the background horizontal velocity in the y direction to $\bar{v} = 0$, and the background vertical velocity to $\bar{w} = 0$. The remainder of the background state variables are determined via

$$\bar{u} = \bar{u}_s + \frac{\rho g \sin(\alpha)}{2\eta}(H^2 - (H-z)^2), \quad \bar{p} = \rho g \cos(\alpha)(H-z), \quad \bar{\beta}\bar{u}_s = \rho g \sin(\alpha) H \tag{13}$$

where $\bar{\beta}$ is the background basal drag coefficient.

Perturbation equations

We introduce perturbations (denoted by "1" superscripts) to the background states via

$$
\begin{aligned}
&u = \bar{u} + u^1, \quad v = \bar{v} + v^1, \quad \beta = \bar{\beta} + \beta^1 \\
&w = \bar{w} + w^1 \ \ (w_b = \bar{w}_b + w_b^1), \quad p = \bar{p} + p^1, \\
&s = \bar{s} + s^1, \quad h = \bar{h} + h^1, \quad m = \bar{m} + m^1, \quad a = \bar{a} + a^1
\end{aligned} \tag{14}
$$

where the perturbations are small (i.e., $O(\epsilon)$ where $\epsilon \ll 1$). We obtain equations for the perturbed fields by inserting the perturbations (14) into (1)-(12) and discarding the product terms (i.e., $f^1 g^1 = O(\epsilon^2)$).

In this way, we obtain a homogeneous Stokes system for the perturbed fields

$$-p_x^1 + \eta(u_{xx}^1 + u_{yy}^1 + u_{zz}^1) = 0 \tag{15}$$

$$-p_y^1 + \eta(v_{xx}^1 + v_{yy}^1 + v_{zz}^1) = 0 \tag{16}$$

$$-p_z^1 + \eta(w_{xx}^1 + w_{yy}^1 + w_{zz}^1) = 0 \tag{17}$$

$$u_x^1 + v_y^1 + w_z^1 = 0, \tag{18}$$

and the surface kinematic equations become

$$h_t^1 + \bar{u}_h h_x^1 = w^1 + a^1 \tag{19}$$

$$s_t^1 + \bar{u}_s s_x^1 = w^1 + m^1. \tag{20}$$

To account for changes in ice geometry, we linearize the upper and lower surface boundary conditions at $z = H + h^1$ and $z = s^1$ onto $z = H$ and $z = 0$, respectively. To do this, we use a 1$^{\text{st}}$-order Taylor expansion in depth for a function $f(z)$: $f(z^0 + z^1) \approx f(z^0) + f_z(z^0) z^1$. The stress-free condition at $z = H + h^1$ is approximated at $z = H$ by

$$2\eta w_z^1 - p^1 = -\rho g \cos(\alpha) h^1 \tag{21}$$

$$\eta(u_z^1 + w_x^1) = \rho g \sin(\alpha) h^1 \tag{22}$$

$$\eta(v_z^1 + w_y^1) = 0 \tag{23}$$

Equation (21) states that the perturbed normal stress is balanced by the perturbed cryostatic stress from the elevation anomaly. Similarly, boundary conditions at the base become

$$w^1 = w_b^1 \tag{24}$$

$$\eta(u_z^1 + w_x^1) = \bar{\beta} u^1 + \bar{u}_s \beta^1 - \bar{\tau} s^1 \tag{25}$$

$$\eta(v_z^1 + w_y^1) = \bar{\beta} v^1 \tag{26}$$

where the stress-gradient parameter

$$\bar{\tau} = \bar{\beta}\bar{u}_z - \eta \bar{u}_{zz}|_{z=0} \tag{27}$$

depends on the background state. We drop the "1" superscripts below.

Fourier transform solution

We use the Fourier transform (equation 2 in main text) to solve the system (15)-(26). The Stokes flow equations (15)-(18) become

$$-ik_x\widehat{p} + \eta(-k^2\widehat{u} + \widehat{u}_{zz}) = 0 \tag{28}$$

$$-ik_y\widehat{p} + \eta(-k^2\widehat{v} + \widehat{v}_{zz}) = 0 \tag{29}$$

$$-\widehat{p}_z + \eta(-k^2\widehat{w} + \widehat{w}_{zz}) = 0 \tag{30}$$

$$ik_x\widehat{u} + ik_y\widehat{v} + \widehat{w}_z = 0 \tag{31}$$

Equations (28)-(31) can be reduced to a fourth-order equation for the transformed vertical velocity

$$\widehat{w}_{zzzz} - 2k^2\widehat{w}_{zz} + k^4\widehat{w} = 0. \tag{32}$$

The general solution to (32) is

$$\widehat{w} = \frac{A}{k}e^{kz} + \frac{B}{k}e^{-kz} + Cze^{kz} + Dze^{-kz}, \tag{33}$$

where the constants A, B, C, and D depend on k. To determine these coefficients, we will rewrite all of the boundary conditions in terms of $\widehat{w}$ and its z derivatives. The z derivatives of $\widehat{w}$ are

$$\widehat{w}_z = Ae^{kz} - Be^{-kz} + Ce^{kz} + Ckze^{kz} - Dkze^{-kz} + De^{-kz} \tag{34}$$

$$\widehat{w}_{zz} = Ake^{kz} + Bke^{-kz} + 2Cke^{kz} + Ck^2ze^{kz} + Dk^2ze^{-kz} - 2Dke^{-kz} \tag{35}$$

$$\widehat{w}_{zzz} = Ak^2e^{kz} - Bk^2e^{-kz} + 3Ck^2e^{kz} + Ck^3ze^{kz} - Dk^3ze^{-kz} + 3Dk^2e^{-kz} \tag{36}$$

For later reference, the surface kinematic equations (19)-(20) transform to

$$\widehat{h}_t + \bar{u}_h ik_x\widehat{h} = \widehat{w} + \widehat{a}, \tag{37}$$

$$\widehat{s}_t + \bar{u}_s ik_x\widehat{s} = \widehat{w} + \widehat{m}. \tag{38}$$

The sliding law (25)-(26) becomes

$$\eta(\widehat{u}_z + ik_x\widehat{w}) = \bar{\beta}\widehat{u} + \bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s} \tag{39}$$

$$\eta(\widehat{v}_z + ik_y\widehat{w}) = \bar{\beta}\widehat{v}. \tag{40}$$

Multiplying (39)-(40) by $-ik_x$ and $-ik_y$, respectively, summing the equations, and using the transformed incompressibility condition (31), we obtain

$$\eta(\widehat{w}_{zz} + k^2\widehat{w}) = \bar{\beta}\widehat{w}_z - ik_x(\bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s}). \tag{41}$$

Similarly, the shear stress condition (22)-(23) at the upper surface becomes

$$\eta(\widehat{w}_{zz} + k^2\widehat{w}) = -ik_x\rho g\sin(\alpha)\widehat{h}. \tag{42}$$

The normal-stress condition at the upper surface (21) transforms to $2\eta\widehat{w}_z - \widehat{p} = -\rho g\cos(\alpha)\widehat{h}$. The transformed Stokes equations (28)-(31) imply that $-k^2\widehat{p} = \eta(k^2\widehat{w}_z - \widehat{w}_{zzz})$, which reduces this expression for the normal-stress condition to

$$\eta(3k^2\widehat{w}_z - \widehat{w}_{zzz}) = -k^2\rho g\cos(\alpha)\widehat{h}. \tag{43}$$

To simplify notation, we define the aspect ratio

$$k' \equiv kH.$$

Using the formulas (33)-(36) evaluated at $z = H$, the normal stress condition (43) reduces to

$$Ae^{k'} - Be^{-k'} + Ck'e^{k'} - Dk'e^{-k'} = -\frac{\rho g\cos(\alpha)}{2\eta}\widehat{h} \equiv b_1, \tag{44}$$

and the upper-surface shear-stress condition (42) becomes

$$Ae^{k'} + Be^{-k'} + C(k'+1)e^{k'} + D(k'-1)e^{-k'} = -\frac{ik_x\rho g\sin(\alpha)}{2\eta k}\widehat{h} \equiv c_\alpha b_1, \tag{45}$$
$$c_\alpha \equiv \frac{ik_x}{k}\tan(\alpha).$$

Using the formulas (33)-(36) evaluated at $z = 0$, the sliding law (41) reduces to

$$A(1-\gamma) + B(1+\gamma) + C(1-\gamma) - D(1+\gamma) = -\frac{ik_x}{2\eta k}(\bar{u}_s\widehat{\beta} + \bar{\tau}\widehat{s}) \equiv b_2 \tag{46}$$

where we have defined

$$\gamma = \bar{\beta}/(2\eta k).$$

The basal velocity anomaly boundary condition becomes

$$A + B = k\widehat{w}_b \equiv b_3. \tag{47}$$

Equations (44)-(47) lead to a linear system for the coefficients $\{A, B, C, D\}$. The vertical velocity at the upper surface takes the form

$$\begin{aligned}\widehat{w}|_{z=H} &= \frac{1}{k}\left(e^{k'}A + e^{-k'}B + k'e^{k'}C + k'e^{-k'}D\right) \\ &= -\mathsf{R}_g\widehat{h} + \mathsf{T}_w\widehat{w}_b + ik_x\mathsf{T}_\beta(\bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s}). \end{aligned} \tag{48}$$

The relaxation frequency R_g is given by

$$\mathsf{R}_g = \left(\frac{\rho g\cos(\alpha)}{2\eta k}\right)\frac{(1+\gamma)e^{4k'} - (2+4\gamma k' - 4c_\alpha k'(1+\gamma k'))e^{2k'} + 1 - \gamma}{(1+\gamma)e^{4k'} + (2\gamma + 4k' + 4\gamma k'^2)e^{2k'} - 1 + \gamma}, \tag{49}$$

the basal vertical velocity transfer function T_w is given by

$$\mathsf{T}_w = \frac{2(1+\gamma)(k'+1)e^{3k'} + 2(1-\gamma)(k'-1)e^{k'}}{(1+\gamma)e^{4k'} + (2\gamma + 4k' + 4\gamma k'^2)e^{2k'} - 1 + \gamma}, \tag{50}$$

and the basal drag transfer function is given by

$$\mathsf{T}_\beta = \left(\frac{k'}{\eta k^2}\right)\frac{e^{3k'} + e^{k'}}{(1+\gamma)e^{4k'} + (2\gamma + 4k' + 4\gamma k'^2)e^{2k'} - 1 + \gamma}. \tag{51}$$

In frequency space, equations (37) and (48) lead to the evolution equation

$$\frac{\partial\widehat{h}}{\partial t} + \left[ik_x\bar{u}_h + \mathsf{R}_g\right]\widehat{h} = \mathsf{T}_w\widehat{w}_b + ik_x\mathsf{T}_\beta(\bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s}) + \widehat{a}, \tag{52}$$

which is equation (6) from the main text.

Velocity solutions

Here, we derive the horizontal surface velocity anomaly solutions. We rearrange the transformed Stokes equations (28)-(29) to

$$\widehat{u}_{zz} - k^2\widehat{u} = \frac{ik_x}{\eta}\widehat{p} \tag{53}$$

$$\widehat{v}_{zz} - k^2\widehat{v} = \frac{ik_y}{\eta}\widehat{p}. \tag{54}$$

Equations (53) and (54) have the general solutions

$$\widehat{u}(z) = \frac{ik_x}{2\eta k}\left(e^{kz}\int_0^z \widehat{p}(z')e^{-kz'}\,\mathrm{d}z' - e^{-kz}\int_0^z \widehat{p}(z')e^{kz'}\,\mathrm{d}z'\right) + Ee^{kz} + Fe^{-kz} \tag{55}$$

$$\widehat{v}(z) = \frac{ik_y}{2\eta k}\left(e^{kz}\int_0^z \widehat{p}(z')e^{-kz'}\,\mathrm{d}z' - e^{-kz}\int_0^z \widehat{p}(z')e^{kz'}\,\mathrm{d}z'\right) + Ge^{kz} + Ie^{-kz} \tag{56}$$

where $\{E, F, G, I\}$ depend on the boundary conditions. As noted before, the body equations (28)-(31) imply that

$$\widehat{p} = \eta\left(\frac{1}{k^2}\widehat{w}_{zzz} - \widehat{w}_z\right). \tag{57}$$

We substitute (57) into equations (55)-(56) and integrate the $\widehat{w}_{zzz}$ term by parts twice. To this end, we use the identity

$$\int_0^z \left(\frac{1}{k^2}\widehat{w}_{zzz} - \widehat{w}_z\right)e^{\pm kz'}\,\mathrm{d}z' = \frac{1}{k^2}\left[e^{\pm kz'}\widehat{w}_{zz}\right]_0^z - \frac{\pm k}{k^2}\left[\widehat{w}_z e^{\pm kz'}\right]_0^z \tag{58}$$

and find that the pressure integrals reduce to

$$\begin{aligned} &e^{kz}\int_0^z \widehat{p}(z')e^{-kz'}\,\mathrm{d}z' - e^{-kz}\int_0^z \widehat{p}(z')e^{kz'}\,\mathrm{d}z' \\ &= \frac{2\eta}{k}\left(\widehat{w}_z - \widehat{w}_z|_{z=0}\cosh(kz) - \frac{1}{k}\widehat{w}_{zz}|_{z=0}\sinh(kz)\right). \end{aligned} \tag{59}$$

Therefore, we obtain

$$\widehat{u}(z) = \frac{ik_x}{k^2}P(z) + Ee^{kz} + Fe^{-kz} \tag{60}$$

$$\widehat{v}(z) = \frac{ik_y}{k^2}P(z) + Ge^{kz} + Ie^{-kz} \tag{61}$$

$$P(z) = \widehat{w}_z - \widehat{w}_z|_{z=0}\cosh(kz) - \frac{1}{k}\widehat{w}_{zz}|_{z=0}\sinh(kz). \tag{62}$$

We can determine the constants $\{E, F, G, I\}$ from the sliding law and stress-free upper-surface condition. First, we note that

$$\widehat{u}_z(z) = \frac{ik_x}{k^2}P_z(z) + Eke^{kz} - Fke^{-kz} \tag{63}$$

$$\widehat{v}_z(z) = \frac{ik_y}{k^2} P_z(z) + Gke^{kz} - Ike^{-kz} \tag{64}$$

$$P_z(z) = \widehat{w}_{zz} - \widehat{w}_z|_{z=0} k \sinh(kz) - \widehat{w}_{zz}|_{z=0} \cosh(kz) \tag{65}$$

$$P_z(0) = 0 = P(0). \tag{66}$$

The stress-free conditions at $z = H$ (i.e., $\widehat{u}_z = -ik_x\widehat{w} + \frac{\rho g \sin(\alpha)}{\eta}\widehat{h}$, etc.) imply

$$Ee^{k'} - Fe^{-k'} = -\frac{ik_x}{k^2}\left(k\widehat{w}_h + \frac{1}{k}P_z(H) - 2\kappa c_\alpha b_1\right) \equiv b_4 \tag{67}$$

$$Ge^{k'} - Ie^{-k'} = -\frac{ik_y}{k^2}\left(k\widehat{w}_h + \frac{1}{k}P_z(H)\right) \equiv b_4', \tag{68}$$

where we have defined

$$\kappa = k^2/k_x^2.$$

For later convenience in deriving the response functions, we note that b_4' is analogous to b_4 but with $\kappa = 0$. We rearrange the sliding law as $\widehat{u}_z - \frac{\bar{\beta}}{\eta}\widehat{u} = \frac{1}{\eta}(\bar{u}_s\widehat{\beta} + \bar{\tau}\widehat{s}) - ik_x\widehat{w}_b$ to obtain

$$E(1-2\gamma) - F(1+2\gamma) = -\frac{ik_x}{k^2}\left(k\widehat{w}_b - 2\kappa b_2\right) \equiv b_5, \tag{69}$$

and, similarly,

$$G(1-2\gamma) - I(1+2\gamma) = -\frac{ik_y}{k^2}(k\widehat{w}_b) \equiv b_5'. \tag{70}$$

As before, we note that b_5' is analogous to b_5 but with $\kappa = 0$. At this point, we can solve equations (67)-(70) symbolically for $\{E, F, G, I\}$. Then, we can solve for the velocity solutions via

$$\widehat{u}|_{z=H} \equiv \widehat{u}_h = \frac{ik_x}{k^2}P(H) + Ee^{k'} + Fe^{-k'} \tag{71}$$

$$\widehat{v}|_{z=H} \equiv \widehat{v}_h = \frac{ik_y}{k^2}P(H) + Ge^{k'} + Ie^{-k'}. \tag{72}$$

The velocities can be written in terms of $\{\widehat{h}, \widehat{s}, \widehat{w}, \widehat{\beta}\}$ as

$$\widehat{u}_h = -\mathsf{U}_\beta(\bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s}) - ik_x\left(\mathsf{U}_h\widehat{h} + \mathsf{U}_w\widehat{w}_b\right) \tag{73}$$

$$\widehat{v}_h = -\mathsf{V}_\beta(\bar{u}_s\widehat{\beta} - \bar{\tau}\widehat{s}) - ik_y\left(\mathsf{V}_h\widehat{h} + \mathsf{V}_w\widehat{w}_b\right), \tag{74}$$

which are equations (A1) and (A2) in the main text. The response functions for the u component are given by

$$\begin{aligned}
\mathsf{U}_h &= \frac{\rho g\cos(\alpha)}{2\eta k^2}\Big(2k'(\gamma k'+1)(2\gamma+(2\gamma+1)e^{2k'}-1)e^{k'} + \mathsf{P}_\alpha\Big)\mathsf{D}^{-1}\\
\mathsf{U}_w &= \frac{1}{k}k'\Big(2\gamma^2-3\gamma+2(2\gamma^2-1)e^{2k'}+(2\gamma^2+3\gamma+1)e^{4k'}+1\Big)\mathsf{D}^{-1}\\
\mathsf{U}_\beta &= \frac{k_x^2}{2\eta k^3}\Big(2\kappa(\gamma-1)+k'(2\gamma-1)+2(2\gamma\kappa+4\gamma k'^2(\kappa-1)+k'(4\kappa-3))e^{2k'}\\
&+(2\kappa(\gamma+1)-k'(2\gamma+1)-1)e^{4k'}+1\Big)\mathsf{D}^{-1}\\
\mathsf{D} &= \Big((2\gamma^2+3\gamma+1)e^{6k'}+(6\gamma^2+4\gamma k'^2(2\gamma+1)+4k'(2\gamma+1)+3\gamma-1)e^{4k'}\\
&+(6\gamma^2+4\gamma k'^2(2\gamma-1)+4k'(2\gamma-1)-3\gamma-1)e^{2k'}+2\gamma^2-3\gamma+1\Big)/(2e^{k'})
\end{aligned}$$

and those for the v component are given by

$$\mathsf{V}_h = \mathsf{U}_h|_{\kappa=0} \tag{75}$$

$$\mathsf{V}_w = \mathsf{U}_w \tag{76}$$

$$\mathsf{V}_\beta = \frac{k_y}{k_x}\mathsf{U}_\beta|_{\kappa=0}. \tag{77}$$

The additional terms P_α entering the expression for U_h (74) when $\alpha > 0$ are given by

$$\begin{aligned}
\mathsf{P}_\alpha &= c_\alpha\Big(-2\gamma^2+3\gamma+2\kappa\left(2\gamma^2-3\gamma+1\right)+\Big[2\gamma^2+3\gamma-2\kappa\left(2\gamma^2+3\gamma+1\right)+1\Big]e^{6k'}\\
&+\Big[-16\gamma^2k'^2-8\gamma^2k'-2\gamma^2+8\gamma k'^2-12\gamma k'-3\gamma+2\kappa(8\gamma^2k'^2+2\gamma^2-4\gamma k'^2+8\gamma k'\\
&-\gamma-4k'+1)+8k'-1\Big]e^{2k'}+\Big[16\gamma^2k'^2-8\gamma^2k'+2\gamma^2+8\gamma k'^2+12\gamma k'-3\gamma
\end{aligned}$$

$$-2\kappa(8\gamma^2 k'^2 + 2\gamma^2 + 4\gamma k'^2 + 8\gamma k' + \gamma + 4k' + 1) + 8k' + 1\Big]e^{4k'} - 1\Big). \tag{78}$$

Text S2. Nonlinear subglacial lake model.

Model description

Here, we outline the nonlinear subglacial lake model that is used to construct the synthetic data in main text Figures 8-9. The model setup is the same as in Stubblefied, Spiegelman, & Creyts (2021) and Stubblefied, Creyts, et. al (2021), so we only state the relevant portions here. The code is openly available (DOI: 10.5281/zenodo.5775182).

First, on the basal boundary we assume a linear sliding law of the form

$$\mathcal{T}\sigma\mathbf{n} = -\beta\mathcal{T}\mathbf{u}, \tag{79}$$

where σ is the stress tensor, $\mathbf{u}$ is the velocity, β is the basal drag coefficient, and $\mathcal{T} = \mathcal{I} - \mathbf{n}\mathbf{n}^T$ is a tangential projection operator with $\mathbf{n}$ being an outward-pointing unit normal to the domain boundary and $\mathcal{I}$ the identity tensor. For consistency with the small-perturbation model parameters in the main text (Table 1), we set $\beta = 5 \times 10^9$ Pa s/m on the ice-bed boundary. We note that $\beta = 0$ over the ice-water boundary in the nonlinear model.

Second, the ice viscosity in the nonlinear model follows Glen's law (Cuffey & Paterson, 2010; Glen, 1955), which is given by

$$\eta(\mathsf{D}) = \frac{1}{2}B(|\mathsf{D}|^2 + \epsilon_v)^{\frac{1-n}{2n}}. \tag{80}$$

In the flow law (eq. 80), $|\mathsf{D}| = \sqrt{\mathrm{tr}(\mathsf{D}^{\mathrm{T}}\mathsf{D})}$ denotes the Frobenius norm of the strain-rate tensor D, $n = 3$ is the stress exponent, $B = 8.6 \times 10^7$ Pa s$^{1/n}$ is the ice hardness, and ϵ_v is a regularization parameter that prevents infinite viscosity at zero strain rate. For consistency with the small-perturbation model, we assume that the zero-strain-rate (i.e., $|\mathsf{D}| = 0$) viscosity coincides with the Newtonian viscosity in the main text (10^{13} Pa s; Table 1). We accomplish this by setting $\epsilon_v = (2 \times 10^{13}/B)^{\frac{2n}{1-n}} \approx 7.89 \times 10^{-17}$ s^{-1}.

Third, instead of prescribing a basal vertical velocity w_b, we prescribe the subglacial lake water volume change V through the integral constraint

$$\dot{V} = \int_\Gamma \mathbf{u} \cdot (-\mathbf{n}) \, \mathrm{d}\mathbf{s}, \tag{81}$$

where $\dot{V}$ is the rate of water volume change and Γ is the lower boundary of the ice. The prescribed volume change is shown in Figure 8. The vertical velocity w_b in Figure 8 is extracted from the nonlinear Stokes solution $\mathbf{u}$.

Finally, we assume a cryostatic stress condition on the side-walls of the domain to ensure consistency with the (infinite-domain) small-perturbation model. This also limits the influence of the slippery spot over the lake on the elevation change—i.e., the basal drag anomaly doesn't appear at first-order when the background state is cryostatic—making an inversion for the basal vertical velocity feasible with altimetry data alone. With these specifications, the elevation solution from the nonlinear model is then used to produce synthetic data (Figures 8 and 9) as described in Section 4.2 of the main text.

Additional Supporting Information (Files uploaded separately)

Caption for Movie S1

Movie of the computational example in Figure 4 over the entire simulation time. h^{obs} is the synthetic elevation anomaly data, $h^{\mathrm{fwd}} = \mathcal{H}_{w_b}(w_b^{\mathrm{inv}})$ is the modelled elevation associated with the inversion w_b^{inv}, and w_b^{true} is the "true" solution used to create the synthetic data. The elevations have been normalized by $\|h^{\mathrm{obs}}\|_\infty \approx 4$ and the vertical velocities have been normalized by $\|w_b^{\mathrm{true}}\|_\infty = 5$. File name: ms01.mp4.

Caption for Movie S2

Movie of the computational example in Figure 5 over the entire simulation time. h^{obs} is the synthetic elevation anomaly data, $h^{\mathrm{fwd}} = \mathcal{H}_\beta(\beta^{\mathrm{inv}})$ is the modelled elevation associated with the inversion β^{inv}, and β^{true} is the "true" solution used to create the synthetic data. The elevations have been normalized by $\|h^{\mathrm{obs}}\|_\infty \approx 2.87$ and the basal drag anomalies have been normalized by $\|w_b^{\mathrm{true}}\|_\infty = 0.08$. File name: ms02.mp4.

Caption for Movie S3

Movie of the computational example in Figure 7 over the entire simulation time. h^{obs} is the synthetic elevation anomaly data, $\mathbf{u}^{\mathrm{obs}}$ is the synthetic horizontal surface velocity anomaly data, $h^{\mathrm{fwd}} = \mathcal{H}_c(w_b^{\mathrm{inv}}, \beta^{\mathrm{inv}})$ is the modelled elevation associated with the inversion $[w_b^{\mathrm{inv}}, \beta^{\mathrm{inv}}]^T$, $\mathbf{u}^{\mathrm{fwd}} = [\mathcal{U}_c(w_b^{\mathrm{inv}}, \beta^{\mathrm{inv}}), \mathcal{V}_c(w_b^{\mathrm{inv}}, \beta^{\mathrm{inv}})]^T$ is the modelled horizontal surface velocity associated with the inversion, and $[w_b^{\mathrm{true}}, \beta^{\mathrm{true}}]^T$ is the "true" solution used to create the synthetic data. The elevations have been normalized by $\|h^{\mathrm{obs}}\|_\infty \approx 5.6$, the horizontal velocities have been normalized by the maximum flow speed in the observed anomaly ($\sim$38.9), the vertical velocities have been normalized by $\|w_b^{\mathrm{true}}\|_\infty = 5$, and the basal drag anomalies have been normalized by $\|w_b^{\mathrm{true}}\|_\infty = 0.08$. File name: ms03.mp4.

Caption for Movie S4

Movie of the nonlinear subglacial lake simulation depicted in Figure 8. For reference, the maximum elevation anomaly is $\sim$1.4m and the bed trough is 8m deep in dimensional terms. See Movie S5 and Figure 9 for a more detailed view of the elevation anomaly h.

The basal vertical velocity w_b is normalized by its maximum absolute value $\|w_b\|_\infty \approx 6.58$. File name: ms04.mp4.

Caption for Movie S5

Movie of the inversion shown in Figure 9 over the entire simulation time. h^{obs} is the synthetic elevation anomaly data, $h^{\mathrm{fwd}} = \mathcal{H}_{w_b}(w_b^{\mathrm{inv}})$ is the modelled elevation associated with the inversion w_b^{inv}, and w_b^{true} is the "true" solution used to create the synthetic data. The elevations have been normalized by $\|h^{\mathrm{obs}}\|_\infty \approx 1.42$ and the vertical velocities have been normalized by $\|w_b^{\mathrm{true}}\|_\infty \approx 6.58$. File name: ms05.mp4.