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CMMSE: State-space realization of periodic input/output quaternionic linear systems

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The investigation of linear periodic systems is a prominent subject of research within the domain of linear systems theory. In this context, the state-space realization of such systems is of particular significance. In fact, after successfully treating the realization problem for time-invariant systems, various researchers have directed their attention toward investigating the case of linear periodic systems. Meanwhile, in the late 1980s, Jan C. Willems proposed an approach that broadened the range of systems studied, now referred to as the behavioral approach. This approach views the behavior of a system as its fundamental element, including all signals that adhere to the system laws (also called system trajectories). More recently, the behavioral framework has also been extended separately to periodic and quaternionic behavioral systems.

Our work consists of considering linear periodic input/output quaternionic behavioral systems and using recent developments in order to obtain quasi-minimal and uniform state-space realizations.

KEYWORDS:
Discrete-time systems, input/output systems, quaternionic systems, lifted systems, periodic systems, system realization

1 | INTRODUCTION

In the realm of linear systems theory, the state-space realization problem is a well-established fundamental topic for time-invariant systems. However, in the case of periodic systems, this area remains relatively less investigated. Over the past few decades, there has been some research yielding results on fast realization algorithms as, for instance, the algorithm proposed by Aleixo and Rocha[1,2]. Another area that has received little attention is that of quaternionic linear systems, especially in the periodic context. Our goal was to merge these two contributions, in order to explore the feasibility of applying the realization algorithm proposed by Aleixo and Rocha to periodic quaternionic linear systems.

The paper is organized as follows. Section 2 contains the essential background on quaternions necessary to support our approach along with an overview of the targeted periodic quaternionic linear system intended for realization. Section 3 and Section 4 are devoted to the introduction of the algorithm proposed by Aleixo and Rocha[1,2], as well as its practical implementation using Python™. Section 5 is devoted to the presentation of a numerical example and the conclusions are left to Section 6.

2 | QUATERNIONS AND QUATERNIONIC SYSTEMS

2.1 | Quaternions

Let $\mathbb{H} = \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} : a, b, c, d \in \mathbb{R}\}$ be the quaternion field, where $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$, $\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}$, and $\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$. For $\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}$, the conjugate of $\boldsymbol{q}$ is defined as $\overline{\boldsymbol{q}} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$. Thus, a, the real part of $\boldsymbol{q}$, denoted by $\mathrm{Re}(\boldsymbol{q})$, is given by $a = (\boldsymbol{q} + \overline{\boldsymbol{q}})/2$ and $\boldsymbol{q}\overline{\boldsymbol{q}} = \overline{\boldsymbol{q}}\boldsymbol{q} = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}$. The norm of $\boldsymbol{q}$, denoted by $|\boldsymbol{q}|$, is defined to be $\sqrt{\boldsymbol{q}\overline{\boldsymbol{q}}}$. By routine computation, we can show the following basic property.

Theorem 1. For any $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H}$, $\overline{\boldsymbol{q}_1 + \boldsymbol{q}_2} = \overline{\boldsymbol{q}_1} + \overline{\boldsymbol{q}_2}$, and $\overline{\boldsymbol{q}_1\boldsymbol{q}_2} = \overline{\boldsymbol{q}_2}\,\overline{\boldsymbol{q}_1}$.

As is well known, see Tian[3], the real quaternion algebra $\mathbb{H}$ is algebraically isomorphic to the matrix algebra through the bijective map that associates to each quaternion $\boldsymbol{q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ the matrix

$$\omega(\boldsymbol{q}) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

Furthermore, Tian[3] proved the following result.

Theorem 2. Let $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{H}$ and $\lambda \in \mathbb{R}$. Then

(a) $\boldsymbol{q}_1 = \boldsymbol{q}_2 \Longleftrightarrow \omega(\boldsymbol{q}_1) = \omega(\boldsymbol{q}_2)$.

(b) $\omega(\boldsymbol{q}_1 + \boldsymbol{q}_2) = \omega(\boldsymbol{q}_1) + \omega(\boldsymbol{q}_2)$.

(c) $\omega(\boldsymbol{q}_1\boldsymbol{q}_2) = \omega(\boldsymbol{q}_1)\omega(\boldsymbol{q}_2)$.

(d) $\omega(\lambda\boldsymbol{q}_1) = \lambda\omega(\boldsymbol{q}_1)$.

(e) $\omega(1) = I_4$.

(f) $\omega(\overline{\boldsymbol{q}_1}) = \omega^T(\boldsymbol{q}_1)$.

(g) $\omega\left(\boldsymbol{q} - 1^{-1}\right) = \omega^{-1}(\boldsymbol{q}_1)$, if $\boldsymbol{q}_1 \neq 0$.

For $A = \left[\boldsymbol{q}_{ij}\right] \in \mathbb{H}^{m\times n}$ the real matrix representation of the quaternion matrix A is defined to be

$$\Omega(A) = \left[\omega(\boldsymbol{q}_{ij})\right] \in \mathbb{R}^{4m\times 4n}. \tag{1}$$

From (1) and Theorem 2 it is easy to prove the following result.

Theorem 3. Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{n\times p}$, and $\lambda \in \mathbb{R}$. Then

(a) $\Omega(I_n) = I_{4n}$.

(b) $\Omega(AB) = \Omega(A)\Omega(B)$.

(c) $\Omega(AB) = \Omega(A)\Omega(B)$.

(d) $\Omega(AB) = \Omega(A)\Omega(B)$.

(e) $\Omega(A + B) = \Omega(A) + \Omega(B)$.

(f) $\Omega(\lambda A) = \lambda\Omega(A)$.

Due to the non-commutative nature of the quaternion multiplication, some Linear Algebra concepts suffer some modifications. One such concept that has relevance in this work is the following.

Definition 1 (Wei et al.[4]). Let $v_1, v_2, \dots, v_r \in \mathbb{H}^n$ be vectors with quaternion entries. The vectors $v_1, v_2, \dots, v_r$ are left linearly independent, if for $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{H}$,

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r = 0$$

implies $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. Otherwise we say that $v_1, v_2, \dots, v_r$ are left linearly dependent. Similarly, the vectors $v_1, v_2, \dots, v_r$ are right linearly independent, if for $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{H}$,

$$v_1 \lambda_1 + v_2 \lambda_2 + \cdots + v_r \lambda_r = 0$$

implies $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. Otherwise, we say that $v_1, v_2, \dots, v_r$ are right linearly dependent.

Another concept is the **rank** of a quaternion matrix.

Definition 2. Let $A \in \mathbb{H}^{m\times n}$. The rank of A is the maximum number of columns of A which are right linearly independent and is denoted by $\mathrm{rank}(A)$.

It can be proved, see Wan[5], that the maximum number of right linearly independent columns is equal to the maximum number of left independent rows.

Definition 3 (Wan[5]). Let $A, B \in \mathbb{H}^{m\times n}$. These matrices are said to be equivalent if exists $P \in \mathbb{H}^{m\times m}$ and $Q \in \mathbb{H}^{n\times n}$ such that $A = PBQ$.

The following results and respective proofs can be found in Wan[5].

Theorem 4. Equivalent matrices have the same rank.

Theorem 5. Let $A \in \mathbb{H}^{m\times n}$ and $\mathrm{rank}(A) = r$. Then A is equivalent to

$$\begin{bmatrix} I_r & * \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \in \mathbb{H}^{m\times n}.$$

Proposition 1. Let $A \in \mathbb{H}^{m\times n}$. Then $\mathrm{rank}(\Omega(A)) = 4\,\mathrm{rank}(A)$.

Proof. Let $A \in \mathbb{H}^{m\times n}$ and $\mathrm{rank}(A) = r$. Then, by Theorem 5, A is equivalent to

$$B = \begin{bmatrix} I_r & * \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix} \in \mathbb{H}^{m\times n}.$$

Hence, by Theorem 3

$$\Omega(B) = \begin{bmatrix} I_{4r} & * \\ 0_{4(m-r)\times 4r} & 0_{4(m-r)\times 4(n-r)} \end{bmatrix} \in \mathbb{R}^{4m\times 4n}.$$

By Theorem 4, we conclude that $\mathrm{rank}(\Omega(B)) = \mathrm{rank}(\Omega(A)) = 4r$. □

There is also a full rank matrix decomposition in quaternion matrix algebra. Wang[6] presents the next theorem.

Theorem 6. Let $A \in \mathbb{H}^{m\times n}$ and $\mathrm{rank}(A) = r$. Then there exist $B \in \mathbb{H}^{m\times r}$, $C \in \mathbb{H}^{r\times n}$ with $\mathrm{rank}(B) = \mathrm{rank}(C) = r$, such that $A = BC$.

2.2 | Quaternionic linear systems

The theory of quaternionic systems in the classic state-space approach began to take shape in the early 1980s, thanks to some significant contributions of Hazewinkel et al.[7,8]. Subsequently, quaternionic systems underwent a reexamination within the framework of the behavioral approach, marked by contributions from Pereira et al.[9,10].

The periodic case as described in Aleixo and Rocha[1,2] for real systems is taken into consideration throughout this work even though we deal with quaternionic systems since it is possible to view every quaternionic system as a real (or a complex) system of higher dimension with a special structure. That is, we assume that for a given periodic quaternionic linear system with period P, m inputs, and p outputs, a minimal n-dimensional time-invariant state-space realization of a lifted system is known and given by the quadruplet:

$$F \in \mathbb{H}^{n\times n}, \qquad G \in \mathbb{H}^{n\times m\mathrm{P}}, \qquad H \in \mathbb{H}^{p\mathrm{P}\times n}, \qquad J \in \mathbb{H}^{p\mathrm{P}\times m\mathrm{P}}$$

with matrices G, H, and J being partitioned as

$$G = \left[\, G_1 \quad G_2 \quad \cdots \quad G_{\mathrm{P}} \,\right], \qquad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_{\mathrm{P}} \end{bmatrix}, \qquad J = \begin{bmatrix} J_{1,1} & 0 & \cdots & 0 \\ J_{2,1} & J_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_{\mathrm{P},1} & J_{\mathrm{P},2} & \cdots & J_{\mathrm{P,P}} \end{bmatrix}$$

and where the lower block-triangularity of matrix J is inherited by the necessary condition for the existence of a periodic realization.

3 | ALEIXO AND ROCHA ALGORITHM

The algorithm proposed by Aleixo and Rocha introduces the concept of an n-chain (of size s) of matrices ${}^{1}M, \dots, {}^{s}M$, each one holding

$$\operatorname{rank} {}^{\ell}M \leqslant n, \text{ with } \ell = 1, \dots, s$$

such that:

$${}^{1}M := \left[\, {}^{1}M_1 \,\middle|\, {}^{1}M_2 \,\right] = \left[\, {}^{0}Q \,\middle|\, {}^{1}M_2 \,\right] = \left[\begin{array}{c|c} F & G_1 \\ H_{\mathrm{P}} & J_{\mathrm{P},1} \\ \vdots & \vdots \\ H_2 & J_{2,1} \end{array}\right] \equiv K_1$$

and

$${}^{\ell+1}M := \left[\, {}^{\ell+1}M_1 \,\middle|\, {}^{\ell+1}M_2 \,\right] = \left[\, {}^{\ell}Q \,\middle|\, {}^{\ell+1}M_2 \,\right],$$

where ${}^{\ell}Q$ is a $(n + (\mathrm{P} - (\ell + 1))\, p) \times n$ matrix such that $\exists\, {}^{\ell}R, {}^{\ell}S$, and ${}^{\ell}T$ satisfying:

$${}^{\ell}M = \begin{bmatrix} {}^{\ell}Q \\ {}^{\ell}R \end{bmatrix} \begin{bmatrix} {}^{\ell}S & {}^{\ell}T \end{bmatrix}$$

and

$${}^{\ell+1}M_2 := \begin{bmatrix} G_{\ell+1} \\ J_{\mathrm{P},\ell+1} \\ \vdots \\ J_{\ell+2,\ell+1} \end{bmatrix}, \quad \ell = 1, \dots, s-1.$$

It follows that each matrix ${}^{\ell}M$, with $\ell = 1, \dots, s$, has dimension $(n + (\mathrm{P} - \ell)\, p) \times (n + m)$, while the size of the chain, s, has maximum value $\mathrm{P} - 1$.

It has been shown by Aleixo and Rocha[1,2] that a linear time-invariant system $\Sigma = (F, G, H, J)$, which generates an (n, t)-chain of size $\mathrm{P} - 1$ (with $0 \leqslant t \leqslant \max\{(\mathrm{P} - 1)\, p, m\}$), is induced by a P-periodic state-space system $\Sigma(k) = (A(k), B(k), C(k), D(k))$, of state dimension $n + t$, such that:

$$B(\mathrm{P} - 1) = \left[\begin{array}{c} 0_{t\times m} \\ \hline G_{\mathrm{P}} \end{array}\right], \qquad C(0) = \left[\, 0_{p\times t} \,\middle|\, H_1 \,\right], \qquad D(t) = J_{t+1,t+1}, \quad t = 0, \dots \mathrm{P} - 1$$

and, consequently, the realization algorithm is completed by specifying

$$\begin{aligned} &A(t), \quad t = 0, \dots, \mathrm{P} - 1 \\ &B(t), \quad t = 0, \dots, \mathrm{P} - 2 \\ &C(t), \quad t = 1, \dots, \mathrm{P} - 1. \end{aligned} \tag{2}$$

Thus, for the computation of the unknowns (2), this chain of matrices must have size $\mathrm{P} - 1$ allowing each matrix ${}^{\ell}M$, for $\ell = 1, \dots, \mathrm{P} - 1$, to be factored as

$${}^{\ell}M = \begin{bmatrix} {}^{\ell}Q \\ C(\ell) \end{bmatrix} \left[\, A(\ell - 1) \quad B(\ell - 1) \,\right],$$

with the remaining matrix $A(\mathrm{P}-1)$ obtained from the last matrix (factorization) of the chain

$$ {}^{\mathrm{P}-1}M = \begin{bmatrix} A\left(\mathrm{P}-1\right) \\ C\left(\mathrm{P}-1\right) \end{bmatrix} \begin{bmatrix} A\left(\mathrm{P}-2\right) & B\left(\mathrm{P}-2\right) \end{bmatrix}. $$

4 | ALEIXO AND ROCHA ALGORITHM IMPLEMENTATION

In the sequel, we consider the algorithm presented in Aleixo and Rocha [1,2].

Aleixo and Rocha's Algorithm

procedure REALIZATION(F, G, H, J, P)
  $t = 0$ and $\ell = 1$
  while $\ell < P - 1$ **do**
    Construct ${}^{\ell}_{t}M$
    if rank $\left({}^{\ell}_{t}M\right) \leqslant n + t$ **then**
      Increment ${}^{\ell}_{t}M$ with appropriate number of zeros
    end if
    Factorize ${}^{\ell}_{t}M$
  end while
  return $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$
end procedure

If the software we are using, in this case Python™, natively incorporated the quaternion algebra, we could directly implement this technique within the computer. However, this is not the current scenario. To handle quaternion computations within the isomorphic real matrix space and switch back to quaternions, we must account for (1).

When working in the isomorphic real matrix space, three concerns may be seen by looking at the steps in the algorithm above. First, we must take Proposition 1 into account while calculating the rank. Furthermore, we must also make sure that the quaternionic structure is maintained while factorizing the ${}^{\ell}M$ matrices. For that, we take into consideration a Gaussian-based transformation of the quaternionic matrices similar to the algorithm proposed by Wang et al. [6]. Finally, when incrementing a matrix ${}^{\ell}M$ with zeros, we must remember that these zeros are quaternions, and so each zero is represented by a 4×4 nil matrix.

We can come back to the quaternions whenever needed.

In the following section, we present an example where the procedure was implemented as previously stated. However, we show the matrices that were produced in quaternion form instead.

5 | EXAMPLE

Consider the minimal 2-dimensional time-invariant realization of a quaternionic lifted model given by the quadruplet:

$$ F = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j}+\boldsymbol{k} \end{bmatrix}, \quad G = \left[\begin{array}{c|c|c} 1 & \boldsymbol{i} & \boldsymbol{j} \\ 2\boldsymbol{j} & 1 & \boldsymbol{i} \end{array}\right], \quad H = \left[\begin{array}{cc} 1 & \boldsymbol{i} \\ \hline \boldsymbol{j} & \boldsymbol{j} \\ \hline 7\boldsymbol{i} & \boldsymbol{k} \end{array}\right], \text{ and } J = \left[\begin{array}{c|c|c} \boldsymbol{i} & 0 & 0 \\ \hline 2\boldsymbol{j} & 2\boldsymbol{k} & 0 \\ \hline 3 & 3\boldsymbol{k} & 3\boldsymbol{j} \end{array}\right] $$

and assume, conformably the partitions made, that $P = 3$, $m = 1$, and $p = 1$.

Consider matrix ${}^{1}M$ given by

$$ {}^{1}M = \left[\begin{array}{c|c} F & G_1 \\ \hline H_3 & J_{31} \\ H_2 & J_{21} \end{array}\right] = \left[\begin{array}{cc|c} \boldsymbol{i} & \boldsymbol{j} & 1 \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j}+\boldsymbol{k} & 2\boldsymbol{j} \\ \hline 7\boldsymbol{i} & \boldsymbol{k} & 3 \\ \boldsymbol{j} & \boldsymbol{j} & 2\boldsymbol{j} \end{array}\right]. $$

It is easy to check that, $\operatorname{rank}\left({}^{1}M\right) = 3 > n = 2$.
Define a new time-invariant realization with increased dimension $\bar{n} = n + (3-2) = 3$ as follows:

$$ {}_{1}F := \left[\begin{array}{c|c} 0 & 0_{1\times 2} \\ \hline 0_{2\times 1} & F \end{array}\right] = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \boldsymbol{i} & \boldsymbol{j} \\ 0 & \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j}+\boldsymbol{k} \end{array}\right] $$

$$ {}_{1}G := \left[\, {}_{1}G_1 \middle| {}_{1}G_2 \middle| {}_{1}G_3 \,\right] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline G_1 & G_2 & G_3 \end{array}\right] = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 1 & \boldsymbol{i} & \boldsymbol{j} \\ 2\boldsymbol{j} & 1 & \boldsymbol{i} \end{array}\right] $$

$$ {}_{1}H := \left[\begin{array}{c} {}_{1}H_1 \\ \hline {}_{1}H_2 \\ \hline {}_{1}H_3 \end{array}\right] = \left[\begin{array}{c|c} 0 & H_1 \\ \hline 0 & H_2 \\ \hline 0 & H_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & \boldsymbol{i} \\ \hline 0 & \boldsymbol{j} & \boldsymbol{j} \\ \hline 0 & 7\boldsymbol{i} & \boldsymbol{k} \end{array}\right] \qquad \text{and} \qquad {}_{1}J := J. $$

Now, define a new matrix ${}_{1}^{1}M$ as:

$$ {}_{1}^{1}M = \left[\begin{array}{c|c} {}_{1}F & {}_{1}G_1 \\ \hline {}_{1}H_3 & J_{31} \\ {}_{1}H_2 & J_{21} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{i} & \boldsymbol{j} & 1 \\ 0 & \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j}+\boldsymbol{k} & 2\boldsymbol{j} \\ \hline 0 & 7\boldsymbol{i} & \boldsymbol{k} & 3 \\ 0 & \boldsymbol{j} & \boldsymbol{j} & 2\boldsymbol{j} \end{array}\right]. $$

Since $\operatorname{rank}\left({}_{1}^{1}M\right) = 3 \leqslant \bar{n}$, it is possible to factor this matrix as:

$$ {}_{1}^{1}M = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \boldsymbol{i} & \boldsymbol{j} & 1 \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j} & 2\boldsymbol{j} \\ \boldsymbol{i} & \boldsymbol{k} & 3 \\ \hline \boldsymbol{j} & \boldsymbol{j} & 2\boldsymbol{j} \end{array}\right] \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] =: \left[\begin{array}{c} {}_{1}^{1}Q \\ \hline {}_{1}^{1}R \end{array}\right] \left[\, {}_{1}^{1}S \;\middle|\; {}_{1}^{1}T \,\right]. $$

Now, define matrix ${}_{1}^{2}M$ as:

$$ {}_{1}^{2}M = \left[\begin{array}{c|c} {}_{1}^{1}Q & \begin{array}{c} {}_{1}G_2 \\ \hline J_{32} \end{array} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \boldsymbol{i} & \boldsymbol{j} & 1 & \boldsymbol{i} \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j} & 2\boldsymbol{j} & 1 \\ \cline{4-4} \boldsymbol{i} & \boldsymbol{k} & 3 & 3\boldsymbol{k} \end{array}\right]. $$

Since rank $\left({}_1^2M\right) = 3 \leqslant \bar{n}$, one can factor this matrix as:

$$
{}_1^2M = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \boldsymbol{i} & \boldsymbol{j} & 1 \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j} & 2\boldsymbol{j} \\ \hline \boldsymbol{i} & \boldsymbol{k} & 3 \end{array}\right]\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{5} - \frac{11}{10}\boldsymbol{i} + \frac{11}{5}\boldsymbol{j} + \frac{11}{10}\boldsymbol{k} \\ 0 & 1 & 0 & \frac{7}{5} + 2\boldsymbol{i} + \boldsymbol{j} + \frac{4}{5}\boldsymbol{k} \\ 0 & 0 & 1 & -\frac{1}{10} + \frac{2}{5}\boldsymbol{i} - \frac{3}{10}\boldsymbol{j} - \frac{1}{5}\boldsymbol{k} \end{array}\right]
$$

$$
=: \left[\begin{array}{c} {}_1^2Q \\ \hline {}_1^2R \end{array}\right]\left[\begin{array}{c|c} {}_1^2S & {}_1^2T \end{array}\right].
$$

Thus, we can deduce that the provided lifted model induces a 3-periodic system with a state-space realization of quasi-minimal (and uniform) dimension 3. The realization is:

$$
A(0) = {}_1^1S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A(1) = {}_1^2S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A(2) = {}_1^2Q = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{i} & \boldsymbol{j} & 1 \\ \boldsymbol{i}+\boldsymbol{k} & 1+\boldsymbol{j} & 2\boldsymbol{j} \end{bmatrix}
$$

$$
B(0) = {}_1^1T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad B(1) = {}_1^2T = \frac{1}{10}\begin{bmatrix} -2 - 11\boldsymbol{i} + 22\boldsymbol{j} + 11\boldsymbol{k} \\ 14 + 20\boldsymbol{i} + 10\boldsymbol{j} + 8\boldsymbol{k} \\ -1 + 4\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k} \end{bmatrix} \quad B(2) = {}_1G_3 = \begin{bmatrix} 0 \\ \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix}
$$

$$
C(0) = {}_1H_1 = \begin{bmatrix} 0 & 1 & \boldsymbol{i} \end{bmatrix} \quad C(1) = {}_1^1R = \begin{bmatrix} \boldsymbol{j} & \boldsymbol{j} & 2\boldsymbol{j} \end{bmatrix} \quad C(2) = {}_1^2R = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{k} & 3 \end{bmatrix}
$$

$$
D(0) = J_{11} = \begin{bmatrix} \boldsymbol{i} \end{bmatrix} \quad D(1) = J_{22} = \begin{bmatrix} 2\boldsymbol{k} \end{bmatrix} \quad D(2) = J_{33} = \begin{bmatrix} 3\boldsymbol{j} \end{bmatrix}.
$$

6 | CONCLUSION

As we have shown, quaternion algebra does not need to be computerized to examine the periodic quaternionic linear systems proposed in this work. Utilizing the isomorphic real matrix representation of quaternions suffices. However, before proceeding with the full rank decomposition and computing the rank, one must exercise caution. This is because the isomorphism results in a fourfold increase in the dimensions of all matrices while the algorithm must ensure the preservation of the quaternion structure of these matrices.

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Conflict of interest

The authors declare no potential conflict of interest.

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