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CMMSE Geometric and topological properties of the complementary prism networks

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Abstract

In this paper we prove several optimal inequalities involving the hyperbolicity constant of complementary prisms networks. Moreover, we obtain bounds and closed formulas for the general topological indices $A(G) = \sum_{uv\in E(G)} a(d_u, d_v)$ and $B(G) = \sum_{u\in V(G)} b(d_u)$ of complementary prisms networks.

KEYWORDS:
Complementary prisms; generalized prism network; Gromov hyperbolicity; Geodesics.

1 | INTRODUCTION

The different kinds of products of graphs are an important research topic in graph theory, applied mathematics and computer science. Complementary products were introduced in [1] as a generalization of the Cartesian product. The complementary prism of a graph is a particular and interesting case. Let G be a graph and $\overline{G}$ the complement graph of G. The *complementary prism* of G, denoted by $G\overline{G}$, is the network obtained from the disjoint union of G and $\overline{G}$ by adding edges between the corresponding vertices of G and $\overline{G}$. In what follows, if $v \in V(G)$, we will denote by v' the corresponding vertex of v in $\overline{G}$.

Two well-known examples of complementary prisms are the Petersen graph and the corona product of K_n and K_1, $K_n \circ K_1$. In particular, the Petersen graph is the complementary prism $C_5\overline{C_5}$ and $K_n \circ K_1$ is the complementary prism $K_n\overline{K_n}$. From a structural geometrical point of view, complementary prism networks have been studied through their properties and indices (see chromatic index [2], domination number [3], cycle structure [4], complexity properties [5], spectral properties [6], convexity number [7], chromatic number [8], etc.).

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [9, 10, 11, 12]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [9, 10, 11, 12, 13, 14]). As observed in [11], the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. Moreover this conceptualization has multiple practical applications such as networks and algorithms ([15]), random graphs ([16, 17, 18]), real networks ([19, 20, 21, 22, 23]). Other problems that have been addressed are secure transmission of information, sensor networks, distance estimation, traffic flow, congestion minimization, etc. (see [19, 24, 25, 26, 27]).

The hyperbolicity constant of a geodesic metric space can be viewed as a measure of how tree-like the space is, this implication has been successfully applied to the study of chemical structures [28] and DNA study [29]. In mathematical chemistry a topological descriptor is a single number that represents a chemical structure in graph-theoretical terms via the molecular graph, they play a significant role in mathematical chemistry, especially in the QSPR/QSAR investigations. A topological descriptor is called a topological index if it correlates with a molecular property, for more information regarding the study of its main properties see [30, 31, 32, 33, 34].

In this paper, $G = (V, E) = (V(G), E(G))$ denotes a (finite or infinite) simple graph (not necessarily connected) such that $V \neq \emptyset$ and every edge has length 1. In order to consider a connected graph G as a geodesic metric space, identify (by an isometry) any edge $uv \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge uv (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. We denote by d_G or d this distance. If x, y are in different connected components of G, we define $d_G(x, y) = \infty$. These properties guarantee that any connected component of any graph is a geodesic metric space.

The geometrical and topological properties of several products of graphs have been investigated in [35, 36, 37, 38, 39, 40, 41, 42, 43]. So, it is natural to study the hyperbolicity constant and topological indices of complementary prisms. In this paper we prove several inequalities involving the hyperbolicity constant of complementary prisms networks and in many cases, we obtain the sharp value of the hyperbolicity constant. In the same direction, we obtain optimal bounds and closed formulas for the general topological indices $A(G) = \sum_{uv\in E(G)} a(d_u, d_v)$ and $B(G) = \sum_{u\in V(G)} b(d_u)$ of complementary prisms networks. Many important topological indices can be obtained from A and B by choosing appropriate symmetric functions a and b.

2 | ON THE HYPERBOLICITY CONSTANT IN COMPLEMENTARY PRISM NETWORKS

We collect in this section some previous definitions and results which will be useful along the paper.

We say that the curve γ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by uv.

In [12] appear several different definitions of hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{x_1, x_2, x_3\}$. We say that T is *δ-thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T, i.e., $\delta(T) := \inf\{\delta \geq 0 \mid T \text{ is } \delta\text{-thin}\}$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) \mid T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If X has connected components $\{X_i\}_{i\in I}$, then we define $\delta(X) := \sup_{i\in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

For any connected graph G, we define, as usual,

$$\operatorname{diam} V(G) := \sup\left\{d_G(v, w) \mid v, w \in V(G)\right\},$$
$$\operatorname{diam} G := \sup\left\{d_G(x, y) \mid x, y \in G\right\},$$

i.e, $\operatorname{diam} V(G)$ is the diameter of the set of vertices of G, and $\operatorname{diam} G$ is the diameter of the whole graph G (recall that in order to have a geodesic metric space, G must contain both the vertices and the points in the interior of any edge of G).

Lemma 1. If G is a graph, then

$$\delta(G) \leq \frac{1}{2} \operatorname{diam} G \leq \frac{1}{2}(\operatorname{diam} V(G) + 1).$$

A subgraph H of G is said *isometric* if $d_H(x, y) = d_G(x, y)$ for every $x, y \in H$. Note that this condition is equivalent to $d_H(u, v) = d_G(u, v)$ for every vertices $u, v \in V(H)$.

The following result appears in [44, Lemma 9].

Lemma 2. If H is an isometric subgraph of G, then $\delta(H) \leq \delta(G)$.

Let G be a graph. A *pendant edge* uv of G is an edge such that or the only neighbor of u is v or vice verse.

Let G be a connected graph. If G has pendant edges and G_0 is the induced graph by removing all pendant edges of G, we define

$$\operatorname{diam}^* V(G) = \operatorname{diam} V(G_0), \qquad \operatorname{diam}^* G = \operatorname{diam} G_0.$$

Note that the graph G_0 defined above is an isometric subgraph of G.

One can check that the following result holds.

Proposition 1. Let G be a graph with $v_1, \dots, v_k \in V(G)$ and $w_1, \dots, w_k \notin V(G)$. Let Γ be the graph with

$$V(\Gamma) = V(G) \cup \{w_1, \dots, w_k\}, \qquad E(\Gamma) = E(G) \cup \{v_1 w_1, \dots, v_k w_k\}.$$

Then $\delta(\Gamma) = \delta(G)$.

Lemma 1 and Proposition 1 have the following consequence.

Corollary 1. If G is a graph, then

$$\delta(G) \le \frac{1}{2} \operatorname{diam}^* G \le \frac{1}{2}\big(\operatorname{diam}^* V(G) + 1\big).$$

From [45, Theorem 11] we have the following result.

Theorem 1. The following graphs have these precise values of δ.
- If P_n is a *path graph*, then $\delta(P_n) = 0$ for all $n \ge 1$.
- If C_n is a *cycle graph*, then $\delta(C_n) = \frac{1}{4} L(C_n) = \frac{n}{4}$ for all $n \ge 3$.
- If K_n is a *complete graph*, then $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$ and $\delta(K_n) = 1$ for all $n \ge 4$.

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Given a graph G, we denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following result, which appears in [46, Theorems 2.5, 2.6 and 2.7], will be used throughout the paper.

Theorem 2. For every graph G we have $\delta_1(G) = \delta(G)$. Furthermore, if G is hyperbolic, then $\delta(G)$ is an integer multiple of $1/4$ and there exists $T \in \mathbb{T}_1$ with $\delta(T) = \delta(G)$.

Since $\operatorname{diam} V(G\overline{G}\,) \le 3$, Lemma 1 gives the following result.

Theorem 3. If G is a graph, then

$$\delta(G\overline{G}\,) \le 2.$$

Theorem 4. Let G be a graph. If $\operatorname{diam} V(G) \le 3$, then

$$\delta(G) \le \delta(G\overline{G}\,).$$

If $\operatorname{diam} V(G) \ge 3$, then

$$\delta(\overline{G}) \le \delta(G\overline{G}\,).$$

Proof. If $\operatorname{diam} V(G) \le 3$, then G is an isometric subgraph in $G\overline{G}$, and so, Lemma 2 gives $\delta(G) \le \delta(G\overline{G}\,)$.

If $\operatorname{diam} V(G) \ge 3$, then it is well-known that $\operatorname{diam} V(\overline{G}) \le 3$. Therefore, $\overline{G}$ is an isometric subgraph in $G\overline{G}$, and Lemma 2 gives $\delta(\overline{G}) \le \delta(G\overline{G}\,)$. □

Corollary 2. If G is a graph, then

$$\min\big\{\delta(G),\, \delta(\overline{G})\big\} \le \delta(G\overline{G}\,).$$

Theorem 5. If G is a graph with more than four vertices, then

$$\delta(G\overline{G}\,) \le 2\delta(G) + 2\delta(\overline{G}\,).$$

Proof. Since G has at least five vertices, [47, Theorem 4.3] gives that $\delta(G) + \delta(\overline{G}\,) \ge 1$. Hence, Theorem 3 gives the result. □

Remark 1. The inequality $\delta(G\overline{G}) \le \delta(G) + \delta(\overline{G})$ does not hold for every graph with more than four vertices: If $\overline{G}$ is P_5, note that $\delta(G) \le 3/2$ since $\operatorname{diam} V(G) = 2$. Then

$$\delta(G) + \delta(\overline{G}) \le 3/2 + 0 < 2 = \delta(G\overline{G}).$$

Note that if $\operatorname{diam} V(G) = 1$, then G is a complete graph.

Theorem 6. If G is a graph with $\operatorname{diam} V(G) = 1$, then

$$\delta(G) = \delta(G\overline{G}).$$

Proof. Since $\operatorname{diam} V(G) = 1$, G is isomorphic to a complete graph. Thus, $G\overline{G}$ is obtained from G by attaching a pendant edge to each vertex in $V(G)$ and so, Proposition 1 gives $\delta(G) = \delta(G\overline{G})$. □

Theorems 1 and 6 have the following consequence.

Corollary 3. $\delta(K_2\overline{K_2}) = 0$, $\delta(K_3\overline{K_3}) = 3/4$ and $\delta(K_n\overline{K_n}) = 1$ for every $n \ge 4$.

An *empty graph* G is a graph with $E(G) = \emptyset$. We denote by E_n the empty graph with n vertices.

Since the complementary prisms of G and $\overline{G}$ are isomorphic graphs, Theorem 6 has the following consequence.

Corollary 4. If G is an empty graph, then

$$\delta(\overline{G}) = \delta(G\overline{G}).$$

Theorem 7. If G is a graph with $\operatorname{diam} V(G) = 2$, then

$$5/4 \le \delta(G\overline{G}) \le 2.$$

Proof. Theorem 3 gives the upper bound.

Note that by hypothesis there exists a path P, such that $P \cap V(G) = \{u_0, u_1, u_2\}$ with $d_G(u_0, u_1) = d_G(u_1, u_2) = 1$ and $d_G(u_0, u_2) = 2$. Then the cycle C in $G\overline{G}$ with vertices $\{u_0, u_1, u_2, u_2', u_0'\}$ is an isometric subgraph of $G\overline{G}$. Thus, Lemma 2 and Theorem 1 give the lower bound. □

The lower bound in the previous theorem is attained by P_3 and the upper bound is attained by the path graph in Remark 1.

Next, we prove a kind of converse of Theorem 5.

Theorem 8. If G is a graph with $\delta(G) = 0$ or $\delta(\overline{G}) = 0$, then

$$\delta(G) + \delta(\overline{G}) \le \delta(G\overline{G}).$$

Proof. By symmetry, we can assume that $\delta(G) = 0$. Thus, G is a tree or a forest. So, if $\operatorname{diam} V(G) \ge 3$, then Theorem 4 gives the result. If $\operatorname{diam} V(G) = 2$, then G is the star graph S_n, $n \ge 3$, and $\overline{G}$ is the union of complete graph K_{n-1} and an isolated vertex; thus, $\delta(\overline{G}) = \delta(K_{n-1})$ and, by Theorem 1, we have $\delta(\overline{G}) \le 1$. Then Theorem 7 gives

$$\delta(G) + \delta(\overline{G}) \le 1 < 5/4 \le \delta(G\overline{G}).$$

If $\operatorname{diam} V(G) = 1$, then G is the path graph P_2 and $\delta(G) = \delta(\overline{G}) = 0$. □

Theorem 9. If G is a graph with $\operatorname{diam} V(G) = 3$, then

$$3/2 \le \delta(G\overline{G}) \le 2.$$

Proof. Theorem 3 gives the upper bound.

Consider $u, v \in V(G)$ such that $d_G(u, v) = 3$ and let P be a geodesic in G joining u and v. Let x be the midpoint of P and y the midpoint of $u'v'$. Note that $d_{G\overline{G}}(x, y) = 3$. Consider two geodesics P' and P'' joining x to y such that $P' \cap P'' = \{x, y\}$, $uu' \in P'$ and $vv' \in P''$. Let us consider the geodesic triangle $T = \{P', [xv], [vy]\}$. We have

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(u, [xv] \cup [vy]) = 3/2.$$

□

Remark 2. The argument in the proof of Theorem 9 gives that if G is a connected graph with $\operatorname{diam} V(G) \ge 3$ then $\delta(G\overline{G}) \ge 3/2$.

Theorem 10. If G is a connected graph which is neither a complete graph nor a complete graph without an edge, then

$$3/2 \le \delta(G\overline{G}) \le 2.$$

Proof. Theorem 3 gives the upper bound.

If diam $V(G) \ge 3$, then Remark 2 gives $\delta(G\overline{G}) \ge 3/2$. Thus, we can assume that diam $V(G) = 2$, since G is not a complete graph. Since diam $V(G) = 2$ and $|V(G)| \ge 4$, there exist $u, v_0, v_1, w \in V(G)$ such that $d_G(u, v_0) = d_G(v_0, w) = 1$, $d_G(u, w) = 2$. Since G is neither a complete graph nor a complete graph without an edge, we can choose $u, v_0, v_1, w \in V(G)$ such that the induced subgraph of G by u, v_0, v_1, w is neither a complete graph nor a complete graph without an edge.

Case A. Suppose that $v_0v_1 \notin E(G)$. Since G is a connected graph without loss of generality we can assume that $v_1w \in E(G)$. We have the following cases:

Case A.1. Suppose that $uv_1 \in E(G)$. We have that $u'w', v'_0v'_1 \in E(\overline{G})$. Let x, y be the midpoints of $u'w'$ and $v'_0v'_1$ respectively. We have $d_{G\overline{G}}(x, y) \ge 3$.

Case A.1.1. Assume that there exists $s' \in V(\overline{G})$ such that $d_{G\overline{G}}(x, s') = d_{G\overline{G}}(y, s') = 3/2$. Without loss of generality assume that $s'u', s'v'_0 \in E(\overline{G})$. Let P be the geodesic joining x and y such that $P \cap V(\overline{G}) = \{u', s', v'_0\}$. Let z be the midpoint of v_1w. Let us consider the geodesic triangle $T = \{[xz], [yz], P\}$. We have

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(s', [xz] \cup [yz]) = 3/2.$$

Case A.1.2. Does not exist $s' \in V(\overline{G})$ with $d_{G\overline{G}}(x, s') = d_{G\overline{G}}(y, s') = 3/2$. Then there exist two geodesics P, P' joining x and y such that $P \cap V(G\overline{G}) = \{v'_0, v_0, u, u'\}$ and $P' \cap V(G\overline{G}) = \{w', w, v_1, v'_1\}$. Let z, p be the midpoints of P and P', respectively, and consider the geodesic triangle $T = \{[xz], [yz], P'\}$. We have that

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(p, [xz] \cup [yz]) = 3/2.$$

Case A.2. Suppose that $uv_1 \notin E(G)$. Let x, y be the midpoints of v_0w and $u'v'_1$, respectively. We have that $d_{G\overline{G}}(x, y) = 3$ and there exist two geodesics P, P' joining x and y such that $P \cap V(G\overline{G}) = \{u', u, v_0\}$ and $P' \cap V(G\overline{G}) = \{v'_1, v_1, w\}$. Consider the geodesic triangle $T = \{[xu], [yu], P'\}$. We have

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(v_1, [xu] \cup [yu]) = 3/2.$$

Case B. Suppose that $v_0v_1 \in E(G)$. Since the induced subgraph of G by u, v_0, v_1, w is neither a complete graph nor a complete graph without an edge, it is not possible to have $uv_1, v_1w \in E(G)$. We have the following cases:

Case B.1. $uv_1, v_1w \notin E(G)$. Thus, $u'w', u'v'_1, v'_1w' \in E(\overline{G})$ and let x, y be the midpoints of uv_0 and v'_1w', respectively. Then $d_{G\overline{G}}(x, y) = 3$ and there exist two geodesics P and P' in $G\overline{G}$ joining x and y such that $P \cap V(G\overline{G}) = \{u, u', v'_1\}$ and $P' \cap V(G\overline{G}) = \{w', w, v_0\}$. Consider the geodesic triangle $T = \{[xu'], [yu'], P'\}$. We have

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(w, [xu'] \cup [yu']) = 3/2.$$

Case B.2. $uv_1 \in E(G)$ or $v_1w \in E(G)$. We can assume that $v_1w \in E(G)$, and so $uv_1 \notin E(G)$. Let x, y be the midpoints of uu' and wv_1, respectively. There exist two geodesics P, P' in $G\overline{G}$ joining x and y such that $P \cap V(G\overline{G}) = \{u, v_0, v_1\}$ and $P' \cap V(G\overline{G}) = \{u', w', w\}$. Consider the geodesic triangle $T = \{[xv_0], [yv_0], P'\}$. We have

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(w', [xv_0] \cup [yv_0]) = 3/2.$$

□

Theorem 11. If $n \ge 3$ and G is the complete graph K_n without an edge, then

$$\delta(G\overline{G}) = 5/4.$$

Proof. Let $u, v, w \in V(G)$ such that $uw \notin E(G)$. Let x be the midpoint of $u'w'$, so, there exists two geodesics P and P' joining x and v. Let z, p be the midpoints of P and P', respectively. If we consider the geodesic triangle $T = \{[xz], [vz], P'\}$, then

$$\delta(G\overline{G}) \ge \delta(T) \ge d_{G\overline{G}}(p, [xz] \cup [vz]) = 5/4.$$

Since diam* $V(G\overline{G}) = 2$, Corollary 1 gives that $\delta(G\overline{G}) \le 3/2$. Since Theorem 2 gives that the hyperbolicity constant is an integer multiple of $1/4$, we have that $\delta(G\overline{G}) = 5/4$ or $\delta(G\overline{G}) = 3/2$. Seeking for a contradiction assume that $\delta(G\overline{G}) = 3/2$.

By Theorem 2, there exist $x, y, z \in J(G\overline{G})$, a geodesic triangle $T = \{[xy], [yz], [xz]\}$ which is a cycle with $\delta(G\overline{G}) = \delta(T) = 3/2$, and $p \in [xy]$ with $d_{G\overline{G}}(p, [xz] \cup [yz]) = 3/2$. We have that $d_{G\overline{G}}(x, y) = 3$, since $\operatorname{diam}^* G\overline{G} \leq 3$ and

$$3 \geq d_{G\overline{G}}(x, y) = d_{G\overline{G}}(x, p) + d_{G\overline{G}}(p, y) \geq 3.$$

Since $\operatorname{diam}^* V(G\overline{G}) = 2$, thus x and y must be midpoints of edges in $E(G\overline{G})$.

If $x \in u'_x v'_x$ and $y \in u_y v_y$, with $u'_x v'_x \in E(\overline{G})$ and $u_y v_y \in E(G)$, then we have that $\{u_x, v_x\} \cap \{u_y, v_y\} = \emptyset$. If P is a geodesic in $G\overline{G}$ joining x and y, then $u_x u'_x \subset P$ or $v_x v'_x \subset P$; by symmetry we can assume that $u_x u'_x \subset P$. Since T is a cycle and $x \in u'_x v'_x$, we have that $v_x v'_x \subset [xz] \cup [yz]$. We can assume that $u_y \in P$ and so, v_y belongs to $[xz] \cup [yz]$, since T is a cycle. Since p is the midpoint of $[xy]$, we have $p = u_x$ and

$$3/2 = d_{G\overline{G}}(u_x, [xz] \cup [yz]) \leq d_{G\overline{G}}(u_x, v_y) = 1,$$

a contradiction.

It is not possible to have $x \in u_x v_x$ and $y \in u_y v_y$, or $x \in u_x u'_x$ and $y \in u_y v_y$, or $x \in u_x u'_x$ and $y \in u_y u'_y$, or $x \in u_x u'_x$ and $y \in u'_x v'_y$, since thus $d_{G\overline{G}}(x, y) \leq 2$. Hence, $\delta(G\overline{G}) \neq 3/2$ and the proof is finished. □

Theorem 12. If G is a graph with $\operatorname{diam} V(G) = 4$, then

$$7/4 \leq \delta(G\overline{G}) \leq 2.$$

Proof. Theorem 3 gives the upper bound.

Consider $u, v \in V(G)$ such that $d_G(u, v) = 4$. Let P be a geodesic in G joining u and v, and let $w \in P \cap V(G)$ such that $d_G(u, w) = 1$. Let x be the midpoint of uw. Note that $d_{G\overline{G}}(x, v) = 7/2$ and there exist geodesics P', P'' joining x and v such that $P' \subset P$ and $\{u, u', v, v'\} = P'' \cap V(G\overline{G})$. Let z, p be the midpoints of P' and P'', respectively, and consider the geodesic triangle $T = \{[xz], [vz], P''\}$. Therefore,

$$d(G\overline{G}) \geq \delta(T) \geq d_{G\overline{G}}(p, [xz] \cup [vz]) = 7/4.$$

□

Theorem 13. If G is a connected graph with $\operatorname{diam} V(G) \geq 5$, then

$$\delta(G\overline{G}) = 2.$$

Proof. Theorem 3 gives $\delta(G\overline{G}) \leq 2$. Let us prove the converse inequality. Since $\operatorname{diam} V(G) \geq 5$, there exists $v_0, v_5 \in V(G)$ such that $d_G(v_0, v_5) = 5$. Let P be a geodesic joining v_0 to v_5 in G, with $P \cap V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and $v_{i-1}v_i \in E(G)$ for $1 \leq i \leq 5$. Let x, y, z be the midpoints of $v'_0 v'_5$, $v_0 v_1$, $v_4 v_5$, respectively. We consider the geodesic $P' = [xy]$, $P'' = [yz]$, $P''' = [xz]$ such that $P' \cap V(G\overline{G}) = \{v_0, v'_0\}$, $P'' \cap V(G\overline{G}) = \{v_1, v_2, v_3, v_4\}$ and $P''' \cap V(G\overline{G}) = \{v_5, v'_5\}$. Let p be the midpoint of P'', and consider the geodesic triangle $T = \{[xy], [xz], [yz]\}$. Hence,

$$d(G\overline{G}) \geq \delta(T) \geq d_{G\overline{G}}(p, [xy] \cup [xz]) = 2.$$

□

The results in this section allow to obtain the following result, which characterizes the connected graphs with $\delta(G\overline{G}) \leq 5/4$.

Theorem 14. If G is a connected graph, then:

- $\delta(G\overline{G}) = 0$ if and only if G is K_2.
- $\delta(G\overline{G}) = 3/4$ if and only if G is K_3.
- $\delta(G\overline{G}) = 1$ if and only if G is K_n, $n \geq 4$.
- $\delta(G\overline{G}) = 5/4$ if and only if G is K_n without an edge, $n \geq 3$.

We consider now the complementary prism of non-connected graphs.

Theorem 15. If G is a graph with at least two non-empty connected components, then

$$3/2 \leq \delta(G\overline{G}) \leq 2.$$

Proof. Theorem 3 gives the upper bound.

Denote by G_1 and G_2 two non-empty connected components. Let $u_0, u_1, v_0, v_1 \in V(G)$ such that $u_0u_1 \in E(G_1)$ and $v_0v_1 \in E(G_2)$. Let x, y be the midpoints of u_0u_1 and v_0v_1, respectively. Let P, P' be the two geodesics joining x and y such that $P \cap V(G\overline{G}) = \{u_1, u_1', v_1', v_1\}$ and $P' \cap V(G\overline{G}) = \{u_0, u_0', v_0', v_0\}$, and choose $z \in P'$. Let p be the midpoint of $u_1'v_1'$. Consider the geodesic triangle $T = \{[xz], [yz], P\}$. We have

$$\delta(G\overline{G}) \geq \delta(T) \geq d_{G\overline{G}}(p, [xz] \cup [yz]) = d_{G\overline{G}}(p, \{u_2', v_2'\}) = d_{G\overline{G}}(p, \{u_1', v_1'\}) + 1 = 3/2.$$

□

Theorem 16. If G is a non-empty disconnected graph, then

$$5/4 \leq \delta(G\overline{G}) \leq 2.$$

Proof. Theorem 3 gives the upper bound.

By Theorem 15 we can assume that G has an isolated vertex. Let $u, v, w \in V(G)$ such that $uv \in E(G)$ and w is an isolated vertex. There exists an isometric cycle C such that $V(G\overline{G}) \cap C = \{u, v, v', w', u'\}$, and Lemma 2 and Theorem 1 give the desired result. □

Theorem 17. If G is a disconnected graph such that one of its connected components has at least two edges, then

$$3/2 \leq \delta(G\overline{G}) \leq 2.$$

Proof. Theorem 3 gives the upper bound.

By Theorem 15 we can assume that G has an isolated vertex $u \in V(G)$. Denote by G_1 the connected component of G with at least two edges $v_1v_2, v_2v_3 \in E(G)$. Let x, y be the midpoints of $u'v_1'$ and v_2v_3, respectively. Note that $d_{G\overline{G}}(x, y) = 3$. Let P, P' be geodesics joining x and y such that $P \cap V(G\overline{G}) = \{v_1', v_1, v_2\}$ and $P' \cap V(G\overline{G}) = \{u', v_3', v_3\}$.

Case A. $v_1v_3 \notin E(G)$. Consider the geodesic triangle $T = \{[xv_3'], [yv_3'], P\}$. Thus

$$\delta(G\overline{G}) \geq \delta(T) \geq d_{G\overline{G}}(v_1, [xv_3'] \cup [yv_3']) = 3/2.$$

Case B. $v_1v_3 \in E(G)$. If we consider the geodesic triangle $T = \{[xv_1], [yv_1], P'\}$, then

$$\delta(G\overline{G}) \geq \delta(T) \geq d_{G\overline{G}}(v_3', [xv_1] \cup [yv_1]) = 3/2.$$

□

Since the complementary prism of G and $G\overline{G}$ are isomorphic graphs, Theorem 11 has the following consequence.

Proposition 2. If G is the union of an edge and one or several isolated vertices, then

$$\delta(G\overline{G}) = 5/4.$$

The results in this section allow to characterize the disconnected graphs G with $\delta(G\overline{G}) \leq 5/4$.

Theorem 18. If G is a disconnected graph, then:

- $\delta(G\overline{G}) = 0$ if and only if G is E_2.
- $\delta(G\overline{G}) = 3/4$ if and only if G is E_3.
- $\delta(G\overline{G}) = 1$ if and only if G is E_n, $n \geq 4$.
- $\delta(G\overline{G}) = 5/4$ if and only if G is the union of an edge and one or several isolated vertices.

Let us define

$$\operatorname{diam}' V(G) = \max\{\operatorname{diam} V(H) : H \text{ is a connected component of } G\}.$$

Finally, the results in sections 3 and 4 allow to obtain the following result.

Theorem 19. If G is a disconnected graph, then:

(a) $\delta(G\overline{G}) \geq 5/4$ if $\operatorname{diam}' V(G) \geq 1$.

(*b*) $\delta(G\overline{G}\,) \geq 3/2$ if $\operatorname{diam}' V(G) \geq 2$.

(*c*) $\delta(G\overline{G}\,) \geq 7/4$ if $\operatorname{diam}' V(G) \geq 4$.

(*d*) $\delta(G\overline{G}\,) = 2$ if $\operatorname{diam}' V(G) \geq 5$.

Proof. Corollary 4 and Theorems 16 and 17 give the items (*a*) and (*b*). The arguments in the proofs of Theorems 12 and 13 give items (*c*) and (*d*), respectively. □

3 | ON GENERAL TOPOLOGICAL INDICES IN COMPLEMENTARY PRISM NETWORKS

Topological indices based on end-vertex degrees of edges have been used over 50 years. Among them, several indices are recognized to be useful tools in chemical researches (see, e.g., [48], [49], [50]).

Given a symmetric function $a : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ and a graph G with n vertices, we consider the general topological index

$$A(G) = \sum_{uv\in E(G)} a(d_u, d_v), \qquad A_{red}(G) = \sum_{uv\in E(G)} a(d_u - 1, d_v - 1).$$

Hence,

$$A(\overline{G}) = \sum_{uv\in E(\overline{G})} a(n-1-d_u, n-1-d_v) = \sum_{uv\notin E(G)} a(n-1-d_u, n-1-d_v).$$

Given a function $b : \mathbb{Z}^+ \to \mathbb{R}$ and a graph G with n vertices, we consider the general topological index

$$B(G) = \sum_{u\in V(G)} b(d_u), \qquad B_{red}(G) = \sum_{u\in V(G)} b(d_u - 1).$$

Note that the topological index $B(G)$ also can be written as

$$B(G) = \sum_{uv\in E(G)} \left(\frac{b(d_u)}{d_u} + \frac{b(d_v)}{d_v} \right) = \sum_{u\in V(G)} b(d_u).$$

Furthermore, given a symmetric function $c : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$, the topological index $C(G)$ defined by

$$C(G) = \prod_{uv\in E(G)} c(d_u, d_v)$$

verifies that

$$\log C(G) = \sum_{uv\in E(G)} \log c(d_u, d_v),$$

i.e., $\log C(G) = A(G)$ with $a(d_u, d_v) = \log c(d_u, d_v)$. Hence, also these kind of topological indices is essentially contained in the class of indices A.

One can check that the following general results hold for any topological index of these kinds.

Theorem 20. Let G be any graph with n vertices and $a : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ a symmetric function. Then

$$A(G\overline{G}) = \sum_{uv\in E(G)} a(d_u + 1, d_v + 1) + \sum_{uv\notin E(G)} a(n - d_u, n - d_v) + \sum_{u\in V(G)} a(d_u + 1, n - d_u)$$

$$A_{red}(G\overline{G}) = A(G) + A(\overline{G}) + \sum_{u\in V(G)} a(d_u, n - 1 - d_u).$$

Theorem 21. Let G be any graph with n vertices and $b : \mathbb{Z}^+ \to \mathbb{R}$. Then

$$B(G\overline{G}) = \sum_{u\in V(G)} \big(b(d_u + 1) + b(n - d_u) \big)$$

$$B_{red}(G\overline{G}) = \sum_{u\in V(G)} \big(b(d_u) + b(n - 1 - d_u) \big) = B(G) + B(\overline{G}).$$

Corollary 5. Let G be any graph and $b : \mathbb{Z}^+ \to \mathbb{R}$.

(1) If b is an increasing function, then

$$B(G\overline{G}) \geq B(G) + B(\overline{G}).$$

(2) If b is an decreasing function, then

$$B(G\overline{G}) \leq B(G) + B(\overline{G}).$$

Recall that the study of topological indices starts with the seminal work by Wiener [51]. The *Wiener index* of G is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v),$$

where $\{u, v\}$ runs over every pair of vertices in G.

It is interesting to generalize the Wiener index in the following way

$$W^\lambda(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v)^\lambda,$$

with $\lambda \in \mathbb{R}$. Obviously, if $\lambda = 1$, then W^λ coincides with the ordinary Wiener index W. Note that W^{-2} is the Harary index; W^{-1} is the reciprocal Wiener index; the quantity W^2 is closely related to the hyper-Wiener index, since $WW = (W^1 + W^2)/2$. Another topological index, proposed in [52] is expressed in terms of W^1, W^2 and W^3 as $(2W^1 + 3W^2 + W^3)/6$. See [53] for more connections of the same kind.

Theorem 22. Let G be a graph with n vertices and $\lambda \in \mathbb{R}$.

(1) If $\lambda > 0$, then

$$\frac{3}{2}\, 2^\lambda (n^2 - n) + \frac{1}{2}(n^2 + n) \le W^\lambda(G\overline{G}) \le \Big(\frac{1}{2}\, 3^\lambda + 2^\lambda\Big)(n^2 - n) + \frac{1}{2}(n^2 + n).$$

(2) If $\lambda < 0$, then

$$\Big(\frac{1}{2}\, 3^\lambda + 2^\lambda\Big)(n^2 - n) + \frac{1}{2}(n^2 + n) \le W^\lambda(G\overline{G}) \le \frac{3}{2}\, 2^\lambda (n^2 - n) + \frac{1}{2}(n^2 + n).$$

Proof. Assume that G has m edges.

Let us denote by u, v the vertices in $V(G)$ and by u', v' their corresponding vertices in $V(\overline{G})$.

There are m couples of vertices $\{u, v\}$ at distance 1 in G, and $\binom{n}{2} - m$ pairs of vertices at distance greater than 1 in G (and so, at distance 2 or 3). Also, there are $\binom{n}{2} - m$ pairs of vertices $\{u', v'\}$ at distance 1 in $\overline{G}$, and m pairs of vertices at distance greater than 1 in $\overline{G}$ (and so, at distance 2 or 3). Furthermore, there are n pairs of vertices $\{u, u'\}$ at distance 1 in $G\overline{G}$, and $n^2 - n$ pairs of vertices $\{u, v'\}$ at distance 2.

If $\lambda > 0$, then

$$W^\lambda(G\overline{G}) \le \binom{n}{2} + n + 3^\lambda \binom{n}{2} + 2^\lambda (n^2 - n) = \Big(\frac{1}{2}\, 3^\lambda + 2^\lambda\Big)(n^2 - n) + \frac{1}{2}(n^2 + n),$$

$$W^\lambda(G\overline{G}) \ge \binom{n}{2} + n + 2^\lambda \binom{n}{2} + 2^\lambda (n^2 - n) = \frac{3}{2}\, 2^\lambda (n^2 - n) + \frac{1}{2}(n^2 + n).$$

We obtain the converse inequalities if $\lambda < 0$. □

If we take $\lambda = 1$ in Theorem 22, we obtain the following consequence for the classical Wiener index.

Corollary 6. If G is a graph with n vertices, then

$$\frac{7}{2}n^2 - \frac{5}{2}n \le W(G\overline{G}) \le 4n^2 - 3n.$$

4 | CONCLUSIONS

In this research we obtain new results on the hyperbolicity constant of the complementary prism networks. The main results in the paper are the following:

Let G be a connected graph.

- If $\operatorname{diam} V(G) = 1$, then $\delta(G) = \delta(G\overline{G}) \le 1$.
- If $\operatorname{diam} V(G) = 2$, then $5/4 \le \delta(G\overline{G}) \le 2$.
- If $\operatorname{diam} V(G) = 3$, then $3/2 \le \delta(G\overline{G}) \le 2$.
- If $\operatorname{diam} V(G) = 4$, then $7/4 \le \delta(G\overline{G}) \le 2$.
- If $\operatorname{diam} V(G) \ge 5$, then $\delta(G\overline{G}) = 2$.

Some topological indices have been successfully applied in several branches of science such as Chemistry, Biology, Computer Science, among others. We focus on studying two general classes of topological indices on the complementary prism networks: $A(G) = \sum_{uv\in E(G)} a(d_u, d_v)$ and $B(G) = \sum_{u\in V(G)} b(d_u)$. It should be noted that the use of our general approach allows us to find new properties of the most important topological indices:

- if $a(x, y) = xy$, then A is the second Zagreb index M_2;
- if $a(x, y) = 2/(x + y)$, then A is the harmonic index H;
- if $a(x, y) = 2\sqrt{xy}/(x + y)$, then A is the geometric-arithmetic index GA;
- if $a(x, y) = xy/(x + y)$, then A is the inverse sum indeg index ISI;
- if $b(t) = t^2$, then B is the first Zagreb index M_1;
- if $b(t) = 1/t$, then B is the inverse index ID.

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Conflicts of Interest

The authors declare no conflict of interest. The founding sponsors had no role in the design of the study; in the collection, analysis, or interpretation of data; in the writing of the manuscript, and in the decision to publish the results.

Author contributions

All authors contributed equally in the creation and writing of this paper.

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