ON THE SCATTERING OF A PLANE WAVE BY A PERTURBED OPEN PERIODIC WAVEGUIDE

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Abstract. We consider the scattering of a plane wave by a locally perturbed periodic (with respect to x_1) medium. If there is no perturbation it is usually assumed that the scattered wave is quasi-periodic with the same parameter as the incident plane wave. As it is well known, one can show existence under this condition but not necessarily uniqueness. Uniqueness fails for certain incident directions (if the wavenumber is kept fixed), and it is not clear which additional condition has to be assumed in this case. In this paper we will analyze three concepts. For the Limiting Absorption Principle (LAP) we replace the refractive index $n = n(x)$ by $n(x) + i\varepsilon$ in a layer of finite width and consider the limiting case $\varepsilon \to 0$. This will give an unsatisfactory condition. In a second approach we require continuity of the field with respect to the incident direction. This will give the same satisfactory condition as the third approach where we approximate the incident plane wave by an incident point source and let the location of the source tend to infinity.

1. Introduction

Let $k > 0$ be the wave number and $\hat{\theta} \in \mathbb{R}^2$ be a unit vector with $\hat{\theta}_2 < 0$ which are fixed. In polar coordinates we express $\hat{\theta}$ as $\hat{\theta} = \binom{\sin\theta}{-\cos\theta}$ for some $|\theta| < \frac{\pi}{2}$. Furthermore, let $n \in L^\infty(\mathbb{R}^2)$ be the real valued index of refraction which is assumed to be $2\pi-$periodic with respect to x_1 and equal to 1 for $|x_2| > h_0$ for some $h_0 > 0$. Let $q \in L^\infty(\mathbb{R}^2)$ have compact support in $Q := (0, 2\pi) \times (-h_0, h_0)$. It is the aim to solve

$$\Delta u + k^2 (n + q)\, u \; = \; 0 \quad \text{in } \mathbb{R}^2 \tag{1}$$

for the total field $u(x) = e^{ik\hat{\theta}\cdot x} + u^s(x)$ as the sum of the incident plane wave of direction $\hat{\theta}$ and the scattered field u^s. Furthermore, a suitable radiating condition for u^s has to be assumed. In the first part of the paper we consider the unperturbed case; that is, $q = 0$. We note that the incident field $u^i(x) = e^{ik\hat{\theta}\cdot x}$ is $\alpha-$quasi-periodic with respect to x_1 with parameter $\alpha = k\hat{\theta}_1 = k \sin\theta$. (Recall that a function $\phi = \phi(x_1)$ is $\alpha-$quasi-periodic if $\phi(x_1 + 2\pi) = e^{2\pi\alpha i}\phi(x_1)$ for all $x_1 \in \mathbb{R}$.) Therefore, it is common (see, e.g., [2, 1, 4, 6]) to assume that also the scattered field has to be quasi-periodic with the same parameter α, and then the Rayleigh expansion provides a suitable radiation condition.

As we will recall below, for fixed $k > 0$ there exist parameters α (which we will call propagative wave numbers, see Definition 2.1); that is angles θ of incident directions, for which no uniqueness holds under the Rayleigh expansion. For these particular angles it is not clear which solution is – mathematically or physically – the correct one.

There are at least three ways to derive a correct radiation condition in this case where no

Date: September 4, 2022

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

uniqueness holds. A classical way is to apply the Limiting Absorption Principle (LAP). Noting that the scattered field satisfies the inhomogeneous Helmholtz equation (for $q=0$)

$$\Delta u^s + k^2 n\, u^s \;=\; -k^2(n-1)u^i \quad \text{in } \mathbb{R}^2 \tag{2}$$

with incident plane wave $u^i(x)=e^{ik\hat{\theta}\cdot x}$ we observe that the application of the LAP to the wave number k; that is, replacing k by $k+i\varepsilon$, does not work because in that case the right hand side $f=k^2(n-1)u^i$ vanishes for $|x_2|>h_0$ but is not even bounded in the layer $W:=\mathbb{R}\times(-h_0,h_0)$. An alternative is to apply the LAP to the refractive index n; that is replace $n(x)$ by $n(x)+i\varepsilon$ inside the waveguide. Since also in this case we do not expect a $H^1(\mathbb{R}^2)-$solution for u^s we have to add a radiation condition. The "upwards propagation radiation condition" gives uniqueness in $H^1_{loc}(\mathbb{R}^2)$ even in the case of general q; that is, with refractive index $n(x)+q(x)+i\varepsilon$. In the unperturbed case $q=0$ this condition is equivalent to the Rayleigh expansion. In Section 3 we will study the question of convergence when ε tends to zero. It will turn out that this principle gives an unexpected and unsatisfactory answer in the case where no uniqueness holds.

The second approach demands continuity of the solution with respect to the angle of incidence. As we will see in Section 6 this will pick one particular solution and gives an additional condition on the field.

In Section 5 we will follow a third approach and consider first the scattering of an incident point source at $z\in\mathbb{R}^2$ with $z_2>h_0$ and later let z tend to infinity. Therefore, the incident field is given by $u^i_z(x)=\Phi(x,z)$, $x\in\mathbb{R}^2\setminus\{z\}$, where $\Phi(x,z)=\frac{i}{4}H_0^{(1)}(k|x-z|)$ denotes the fundamental solution. We recall the asymptotic behavior

$$\Phi(x,z) \;=\; \gamma\frac{e^{ik|z|}}{\sqrt{|z|}}e^{-ikx\cdot z/|z|} \;+\; \mathcal{O}(|z|^{-3/2})\,, \quad |z|\to\infty\,,$$

uniformly with respect to directions $z/|z|$ and x from bounded sets. Here, $\gamma=\frac{e^{i\pi/4}}{\sqrt{8\pi k}}$. Therefore, if $\hat{\theta}=\binom{\sin\theta}{-\cos\theta}$ with $|\theta|<\frac{\pi}{2}$ is the direction of the incident plane wave we define the source to be $z=-t\hat{\theta}$ and note that

$$\frac{1}{\gamma}\lim_{t\to\infty}\bigl[\sqrt{t}\,e^{-ikt}\,\Phi(x,-t\hat{\theta})\bigr] \;=\; e^{ik\,\hat{\theta}\cdot x} \tag{3}$$

uniformly for x from bounded sets. Therefore we expect that the solution of the scattering problem of a point source at $z=-t\hat{\theta}$ (multiplied with the factor $\frac{1}{\gamma}\sqrt{t}\,e^{-ikt}$) converges to a solution of of the scattering problem for the plane incident field of direction $\hat{\theta}$. We will prove this convergence result for the unperturbed case; that is, for $q=0$, in Section 5.

The last Section 6 is devoted to the case where q is general; that is, where the refractive index is given by $n+q$.

All three approaches use the theory of quasi-periodic scattering problems (either because the problems themselves are quasi-periodic or via the Floquet-Bloch transform) which we repeat in Section 2. Also the problems are singular in the sense that they involve invertible operators L_ε for $\varepsilon\neq 0$ which tend to an operator L_0 as $\varepsilon\to 0$ which is singular. For treating the convergence of the corresponding solutions of $L_\varepsilon u_\varepsilon=r_\varepsilon$ we apply an abstract singular perturbation result which we learned from [3], Section 1.4. We recall and extend it in Theorem 2.7 of Section 2.

Let us summarize some notations on sets and spaces. Let again $W := \mathbb{R} \times (-h_0, h_0)$ and $Q := (0, 2\pi) \times (-h_0, h_0)$ and, furthermore, $Q^\infty := (0, 2\pi) \times \mathbb{R}$ and $Q_+^{h_0} := (0, 2\pi) \times (h_0, \infty)$ and $Q_-^{h_0} := (0, 2\pi) \times (-\infty, -h_0)$ and $\Gamma_\pm := (0, 2\pi) \times \{\pm h_0\}$; that is, $Q^\infty = Q_-^{h_0} \cup \Gamma_- \cup Q \cup \Gamma_+ \cup Q_+^{h_0}$. We set $\Gamma := \Gamma_+ \cup \Gamma_-$.

Furthermore, let $H^1_{\alpha,loc}(\mathbb{R}^2) := \big\{u \in H^1_{loc}(\mathbb{R}^2) : u(\cdot, x_2) \text{ is } \alpha\text{−quasi-periodic}\big\}$ where a function u is α−quasi-periodic if $u(x_1 + 2\pi, x_2) = e^{i\alpha 2\pi} u(x_1, x_2)$ for all $x = (x_1, x_2)$. We identify $H^1_{\alpha,loc}(\mathbb{R}^2)$ sometimes with $H^1_{\alpha,loc}(Q^\infty)$; that is, identify quasi-periodic functions on $(0, 2\pi)$ with those on $\mathbb{R}$ – as we do also by identifying the space $\{f \in L^2(\mathbb{R}^2) : f \text{ vanishes outside of } Q^\infty\}$ with $L^2(Q^\infty)$. In the same way $H^1_\alpha(Q)$ is defined. The space $H^1_{per}(Q)$ denotes the subspace of $H^1(Q)$ of 2π−periodic functions with respect to x_1. Finally, the space $H^{1/2}_\alpha(\Gamma)$ is the trace space of $H^1_{\alpha,loc}(Q^\infty)$ on Γ and $H^{-1/2}_\alpha(\Gamma)$ the dual of $H^{1/2}_{-\alpha}(\Gamma)$.

2. Quasi-Periodic Problems And a Singular Perturbation Result

We first recall some notations.

Definition 2.1. *(a) $\alpha \in \mathbb{R}$ is called a cut-off value if there exists $\ell \in \mathbb{Z}$ with $|\ell + \alpha| = k$.*
(b) $\alpha \in \mathbb{R}$ is called a propagative wave number (or quasi-momentum or Floquet spectral value) if there exists a non-trivial $\hat\phi \in H^1_{\alpha,loc}(\mathbb{R}^2)$ such that

$$\Delta\hat\phi + k^2 n\,\hat\phi \;=\; 0 \quad in\ \mathbb{R}^2\,, \tag{4}$$

and $\hat\phi$ satisfies the Rayleigh expansion

$$\hat\phi(x) \;=\; \sum_{\ell\in\mathbb{Z}} \hat\phi_\ell^\pm\, e^{i(\ell+\alpha)x_1 + i\sqrt{k^2-(\ell+\alpha)^2}|x_2|} \quad for\ \pm x_2 > h_0 \tag{5}$$

for some $\hat\phi_\ell^\pm \in \mathbb{C}$ where the convergence is uniform for $|x_2| \geq h_0 + \delta$ for all $\delta > 0$. The functions $\hat\phi$ are called propagating (or guided) modes.

If we decompose k into $k = \hat\ell + \kappa$ with $\hat\ell \in \mathbb{N} \cup \{0\}$ and $\kappa \in (-1/2, 1/2]$ we observe that the cut-off values are given by $\pm\kappa + \ell$ for any $\ell \in \mathbb{Z}$.

Since with α also $\alpha + \ell$ for every $\ell \in \mathbb{Z}$ is a propagative wave number we can restrict ourselves to propagative wave numbers in $(-1/2, 1/2]$.

Under the following assumption it can easily be seen that every propagating mode $\hat\phi$ corresponding to some propagative wave number α is evanescent; that is, $\hat\phi_\ell^\pm = 0$ for all $|\ell + \alpha| \leq k$; that is, there exist $c, \delta > 0$ with $|\hat\phi(x)| \leq c\, e^{-\delta|x_2|}$ for all $|x_2| > h_0$.

Assumption 2.2. *Let $|\ell + \alpha| \neq k$ for all propagative wave numbers α and all $\ell \in \mathbb{Z}$; that is, the cut-off values are no propagative wave numbers.*

Under Assumption 2.2 it can also be shown (see, e.g. [9]) that at most a finite number of propagative wave numbers exist in $[-1/2, 1/2]$. Furthermore, if α is a propagative wave number with mode $\hat\phi$ then $-\alpha$ is a propagative wave number with mode $\overline{\hat\phi}$. Therefore, we can numerate the propagative wave numbers in $[-1/2, 1/2]$ such they are given by

$\{\hat{\alpha}_j : j \in J\}$ where $J \subset \mathbb{Z}$ is symmetric with respect to 0 and $\hat{\alpha}_{-j} = -\hat{\alpha}_j$ for $j \in J$. Furthermore, it is known that every eigenspace

$$\hat{X}_j \;:=\; \left\{\hat{\phi} \in H^1_{\hat{\alpha}_j,loc}(\mathbb{R}^2) : \hat{\phi} \text{ satisfies (4) and (5)}\right\} \tag{6}$$

is finite dimensional with some dimension $m_j > 0$. We note that the elements of $\hat{X}_j$ are in $H^2(Q^\infty)$ and even analytic for $|x_2| > h_0$. We construct an orthonormal basis in $\hat{X}_j$ as follows. Let $j \in J$ be fixed. First we choose an arbitrary inner product $(\cdot,\cdot)_{\hat{X}_j}$ in $\hat{X}_j$. Then we consider the following finite dimensional eigenvalue problem in $\hat{X}_j$.

Determine $\lambda_{\ell,j} \in \mathbb{R}$, $\ell = 1,\ldots,m_j$, and non-trivial $\hat{\phi}_{\ell,j} \in \hat{X}_j$ for $\ell = 1,\ldots,m_j$ such that

$$-2i \int\limits_{Q^\infty} \frac{\partial \hat{\phi}_{\ell,j}}{\partial x_1}\,\overline{\psi}\,dx \;=\; \lambda_{\ell,j}\left(\hat{\phi}_{\ell,j}\,,\,\psi\right)_{\hat{X}_j} \quad \text{for all } \psi \in \hat{X}_j\,. \tag{7}$$

This eigenvalue problem is self-adjoint because the left hand side defines a hermitean sesqui-linear form on the finite dimensional space $\hat{X}_j$. Let the eigenfunctions be normalized such that $\left(\hat{\phi}_{\ell,j}\,,\,\hat{\phi}_{\ell',j}\right)_{\hat{X}_j} = \delta_{\ell,\ell'}$ for $\ell,\ell' = 1,\ldots,m_j$.

Remark 2.3. *In [8] it is shown (for the case of the source problem $\Delta u + k^2 n u = -f$ in the half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and additional Neumann boundary conditions for $x_2 = 0$) that the Limiting Absorption Principle (LAP) with respect to k leads to the eigenvalue problem with inner product $(\phi,\psi)_{\hat{X}_j} = 2k\int_{Q^\infty} n\,\phi\,\overline{\psi}\,dx$ while the LAP with respect to n in the layer W leads to the eigenvalue problem with inner product $(\phi,\psi)_{\hat{X}_j} = k^2\int_Q \phi\,\overline{\psi}\,dx$.*

We make a further assumption which is equivalent to the fact that the group velocities do not vanish (see [7]).

Assumption 2.4. *Let $\lambda_{\ell,j} \neq 0$ for all $\ell = 1,\ldots,m_j$ and $j \in J$; that is, there is no non-trivial $\phi \in \hat{X}_j$ with $\int_{Q^\infty} \frac{\partial\phi}{\partial x_1}\,\overline{\psi}\,dx = 0$ for all $\psi \in \hat{X}_j$.*

In all of the paper we make Assumptions 2.2, and 2.4 without mentioning this anymore.

After these preparations we will now consider quasi-periodic source problems with source functions $f \in L^2(Q^\infty)$ which are not compactly supported.

Let $f \in L^2(Q^\infty)$ such there exist $c,\delta > 0$ with $|f(x)| \le ce^{-\delta|x_2|}$ for all $|x_2| > h_0$. For any $\alpha \in \mathbb{R}$ consider the problem to determine $u \in H^1_{\alpha,loc}(Q^\infty)$ such that

$$\Delta u + k^2 n u \;=\; -f \quad \text{in } Q^\infty\,, \tag{8a}$$

and u satisfies the generalized Rayleigh condition

$$\sum_{\ell\in\mathbb{Z}} \left|(\operatorname{sign} x_2)\,\frac{du_\ell(x_2)}{dx_2} - i\sqrt{k^2-(\ell+\alpha)^2}\,u_\ell(x_2)\right|^2 \;\longrightarrow 0\,, \quad |x_2| \to \infty\,. \tag{8b}$$

Here, $u_\ell(x_2) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} u(x_1,x_2)e^{-i(\ell+\alpha)x_1}dx_1$ are the Fourier coefficients of $u(\cdot,x_2)$. The corresponding $\alpha-$quasi-periodic Dirichlet-to-Neumann operator $\Lambda_\alpha : H^{1/2}_\alpha(\Gamma) \to H^{-1/2}_\alpha(\Gamma)$ is given by

$$(\Lambda_\alpha\phi)(x_1,\pm h_0) \;:=\; \frac{i}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \sqrt{k^2-(\ell+\alpha)^2}\,\phi_\ell(\pm h_0)\,e^{i(\ell+\alpha)x_1}\,, \quad x_1 \in (0,2\pi)\,, \tag{9}$$

for $\phi \in H^{1/2}_\alpha(\Gamma)$.
The following theorem collects properties of the problem (8a), (8b). For a proof we refer to [7], Theorem 4.1–4.3 and Remark 4.4.

Theorem 2.5. *Let Assumptions 2.2 and 2.4 hold.*

(a) *For every $\alpha \in \mathbb{R}$ the problem (8a), (8b) is equivalent to the variational equation*

$$\int_Q [\nabla u \cdot \nabla \overline{\psi} - k^2 n\, u\, \overline{\psi}]\, dx - \int_\Gamma (\Lambda_\alpha u)\, \overline{\psi}\, ds = \int_Q f\, \overline{\psi}\, dx + \int_\Gamma \frac{\partial w}{\partial \nu}\, \overline{\psi}\, ds \tag{10}$$

for all $\psi \in H^1_\alpha(Q)$ where $\partial w/\partial\nu := \pm \partial w^\pm/\partial x_2$ on $\Gamma_\pm$. Here, $w^\pm \in H^1_{\alpha,loc}(Q^{h_0}_\pm)$ are the (uniquely determined) solutions of $\Delta w^\pm + k^2 w^\pm = -f$ in $Q^{h_0}_\pm$, $w^\pm = 0$ on $\Gamma_\pm$, which satisfy the generalized Rayleigh condition (8b); that is,

$$\sum_{\ell\in\mathbb{Z}} \left| (\operatorname{sign} x_2)\, \frac{dw^\pm_\ell(x_2)}{dx_2} - i\sqrt{k^2 - (\ell+\alpha)^2}\, w^\pm_\ell(x_2) \right|^2 \longrightarrow 0\,, \quad x_2 \to \pm\infty\,,$$

where $w^\pm_\ell(x_2)$ are the Fourier coefficients of $w^\pm(\cdot, x_2)$.

(b) *For every $\alpha \in \mathbb{R}$ the variational equation (10) can be written as*

$$L_\alpha u \;=\; r_\alpha \quad \text{in } H^1_\alpha(Q)$$

where $r_\alpha \in H^1_\alpha(Q)$ and $L_\alpha : H^1_\alpha(Q) \to H^1_\alpha(Q)$ are given by

$$\begin{aligned} (L_\alpha u, \psi)_{H^1(Q)} &= \int_Q [\nabla u \cdot \nabla \overline{\psi} - k^2 n\, u\, \overline{\psi}]\, dx - \int_\Gamma (\Lambda_\alpha u)\, \overline{\psi}\, ds\,, \\ (r_\alpha, \psi)_{H^1(Q)} &= \int_Q f\, \overline{\psi}\, dx + \int_\Gamma \frac{\partial w}{\partial \nu}\, \overline{\psi}\, ds \end{aligned}$$

for $u, \psi \in H^1_\alpha(Q)$. The operator L_α is a Fredholm operator with index zero and Riesz number one (that is, the null spaces of L_α and L^2_α coincide). The operator L_α is invertible if, and only if, α is not a propagative wave number. If $\alpha = \hat\alpha_j + \ell$ (for some $\ell \in \mathbb{Z}$) is a propagative wave number then the nullspaces of L_α and its adjoint L^*_α coincide and are given by the restrictions to Q of the corresponding modes in $\hat X_j$.

(c) *If $\alpha = \hat\alpha_j + \ell$ is a propagative wave number for some $\ell \in \mathbb{Z}$ and $j \in J$ then the problem (8a), (8b) is solvable if, and only, if $\int_{Q^\infty} f\, \overline{\hat\phi}\, dx = 0$ for all $\hat\phi \in \hat X_j$.*

(d) *Define $J_\alpha : H^1_{per}(Q) \to H^1_\alpha(Q)$ by $(J_\alpha \phi)(x) := e^{i\alpha x_1}\phi(x)$ and $\tilde r_\alpha \in H^1_{per}(Q)$ and the operator $\tilde L_\alpha$ from $H^1_{per}(Q)$ into itself by $\tilde r_\alpha := J_\alpha^{-1} r_\alpha$ and $\tilde L_\alpha := J_\alpha^{-1} L_\alpha J_\alpha$, respectively. If $\hat\alpha \in \mathbb{R}$ is not a cut-off value then there exists a neighborhood $U \subset \mathbb{C}$ of $\hat\alpha$ such that $\alpha \mapsto \tilde r_\alpha$ and $\alpha \mapsto \tilde L_\alpha$ are analytic as mappings from U into $H^1_{per}(Q)$ and $\mathcal{L}\big(H^1_{per}(Q)\big)$, respectively.*

We note that in the case where $f \in L^2(Q^\infty)$ has compact support in Q the generalized Rayleigh condition (8b) can be replaced by the Rayleigh expansion (5) and the function w appearing in (10) vanishes. Application of this theorem yields existence of the following quasi-periodic scattering problem.

Theorem 2.6. *For given wave number $k > 0$ and unit vector $\hat{\theta} = \binom{\sin\theta}{-\cos\theta}$ with $|\theta| < \frac{\pi}{2}$; that is, $\hat{\theta}_2 = -\cos\theta < 0$, set $\alpha := k\hat{\theta}_1 = k\sin\theta$. Then there exists $u \in H^1_{\alpha,loc}(\mathbb{R}^2)$ such that $\Delta u + k^2 n u = 0$ in $\mathbb{R}^2$, and $u^s(x) := u(x) - e^{ik\hat{\theta}\cdot x}$ satisfies the Rayleigh expansion (5).*

Proof: The scattered field u^s satisfies (8a) with $f = k^2(n-1)u^i$ where $u^i(x) = e^{ik\hat{\theta}\cdot x}$ denotes the incident field. If α is not a propagative wave number then there exists a unique solution u^s by parts (a) and (b) of Theorem 2.5. If $\alpha = \hat{\alpha}_j + \ell$ is a propagative wave number for some $\ell \in \mathbb{Z}$ and $j \in J$ then we have to show that $\int_{Q^\infty}(n-1)\, u^i \overline{\hat{\phi}}\, dx = 0$ for all $\hat{\phi} \in \hat{X}_j$. From the differential equation for $\hat{\phi}$ we obtain

$$k^2 \int\limits_{Q^\infty} (n-1)\, u^i\, \overline{\hat{\phi}}\, dx \;=\; \int_{Q^\infty} u^i\, [\Delta\overline{\hat{\phi}} + k^2 \overline{\hat{\phi}}]\, dx \;=\; \int_{Q^\infty} [\Delta u^i + k^2 u^i]\, \overline{\hat{\phi}}\, dx \;=\; 0$$

by Green's second theorem. We used that the product $u^i\, \overline{\hat{\phi}}$ is $2\pi-$periodic with respect to x_1, that u^i is bounded, and that $\hat{\phi}$ decays exponentially for $|x_2| \to \infty$. $\square$

The following theorem is a special case of a singular perturbation result in [3], Section 1.4. We add the characterization of the limiting solution and give a more direct proof for the convenience of the reader.

Theorem 2.7. *Let $\hat{\alpha} \in I$ for some open interval $I \subset \mathbb{R}$, K_α compact operators from some Hilbert space X into itself and $r_\alpha \in \mathcal{R}(L_\alpha)$ for all $\alpha \in I$ where $L_\alpha := I - K_\alpha$, and $\mathcal{R}(L_\alpha)$ denotes the range of L_α. Furthermore, let L_α be one-to-one (thus invertible) for all $\alpha \neq \hat{\alpha}$ and let $L_{\hat{\alpha}} = I - K_{\hat{\alpha}}$ have Riesz number one. Let $P : X \to \mathcal{N} := \mathcal{N}(L_{\hat{\alpha}})$ be the projection onto the nullspace of $L_{\hat{\alpha}}$ along the direct decomposition $X = \mathcal{N} \oplus \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(L_{\hat{\alpha}})$. Finally, let $\alpha \mapsto r_\alpha$ and $\alpha \mapsto K_\alpha$ be analytic in a neighborhood $U \subset \mathbb{C}$ of $\hat{\alpha}$ and let $PL'_{\hat{\alpha}}|_{\mathcal{N}}$ be an isomorphism from $\mathcal{N}$ onto itself where $L'_{\hat{\alpha}}$ denotes the derivative of L_α with respect to α at $\alpha = \hat{\alpha}$.*
Then the mapping $\alpha \mapsto u_\alpha := L_\alpha^{-1} r_\alpha$ has an extension to an analytic mapping from U into X. The limit $u_{\hat{\alpha}} = \lim_{\alpha\to\hat{\alpha}} u_\alpha$ is the unique solution of the system $L_{\hat{\alpha}} u_{\hat{\alpha}} = r_{\hat{\alpha}}$ and $PL'_{\hat{\alpha}} u_{\hat{\alpha}} = Pr'_{\hat{\alpha}}$ where $r'_{\hat{\alpha}}$ denotes the derivative of r_α at $\alpha = \hat{\alpha}$. Furthermore, there exists a closed interval $I_0 \subset I$ containing $\hat{\alpha}$ in its interior and $c > 0$ with

$$\|u_\alpha\|_X \;\leq\; c\,\big[\sup_{\beta\in I_0} \|r_\beta\|_X + \sup_{\beta\in I_0} \|\partial r_\beta/\partial\beta\|_X\big] \quad \text{for all } \alpha \in I_0\,. \tag{11}$$

Proof: Without loss of generality we assume that $\hat{\alpha} = 0$. First we show uniqueness of the system $L_0 u_0 = r_0$ and $PL'_0 u_0 = Pr'_0$. Let $u_0^{(j)}$ for $j = 1, 2$ denote two solutions. Then $u_0 = u_0^{(1)} - u_0^{(2)}$ satisfies $L_0 u_0 = 0$ and $PL'_0 u_0 = 0$; that is, $u_0 \in \mathcal{N}$ and thus $u_0 = 0$ because PL'_0 is one-to-one on $\mathcal{N}$.

For $\alpha \neq 0$ we decompose u_α into $u_\alpha = u_\alpha^N + u_\alpha^R$ with $u_\alpha^N \in \mathcal{N}$ and $u_\alpha^R \in \mathcal{R}$ and project the equation $L_\alpha u_\alpha = r_\alpha$ onto $\mathcal{N}$ and $\mathcal{R}$; that is, $PL_\alpha(u_\alpha^N + u_\alpha^R) = Pr_\alpha$ and $QL_\alpha(u_\alpha^N + u_\alpha^R) = Qr_\alpha$ where $Q = I - P$ is the projection onto $\mathcal{R}$.

The operator $QL_0|_{\mathcal{R}}$ is an isomorphism from $\mathcal{R}$ onto itself as easily seen. Therefore, by a perturbation argument there exist $A_\alpha := \big[QL_\alpha|_{\mathcal{R}}\big]^{-1}$ from $\mathcal{R}$ onto itself for all α in a neighborhood $V \subset U$ of 0 and they depend analytically on $\alpha \in V$. Therefore, substituting

$u_\alpha^R = A_\alpha(Qr_\alpha - QL_\alpha u_\alpha^N)$ into the first equation yields

$$PL_\alpha(I - A_\alpha Q L_\alpha)u_\alpha^N \;=\; Pr_\alpha - PL_\alpha A_\alpha Q r_\alpha \quad \text{in } \mathcal{N}\,,$$

which we write briefly as $C_\alpha u_\alpha^N = s_\alpha$. From $PL_0 = 0$ and $Pr_0 = 0$ we conclude that $C_0 = 0$ and $s_0 = 0$. Therefore, $C_\alpha u_\alpha^N = s_\alpha$ is equivalent to $\frac{1}{\alpha}(C_\alpha - C_0)u_\alpha^N = \frac{1}{\alpha}(s_\alpha - s_0)$. The operators $\frac{1}{\alpha}(C_\alpha - C_0)$ and the elements $\frac{1}{\alpha}(r_\alpha - r_0)$ depend analytically on α in the neighborhood V of $\alpha = 0$ with $\lim_{\alpha\to 0}\frac{1}{\alpha}(C_\alpha - C_0) = C_0'$ and $\lim_{\alpha\to 0}\frac{1}{\alpha}(s_\alpha - s_0) = s_0'$. By the chain rule we compute $C_0' = PL_0'|_{\mathcal{N}}$ and $s_0' = Pr_0' - PL_0'A_0 r_0$. Since $C_0' = PL_0'|_{\mathcal{N}}$ is invertible by assumption also $\frac{1}{\alpha}(C_\alpha - C_0)$ is invertible for α in some interval I_0 and its inverses are uniformly bounded with respect to $\alpha \in I_0$, thus $\|u_\alpha^N\|_X \le c\|(s_\alpha - s_0)/\alpha\|_X \le c' \sup_\beta \|s_\beta'\|_X$. Also, it is easily seen that u_α^N converges to the unique solution $u_0^N \in \mathcal{N}$ of $C_0' u_0^N = s_0'$; that is, of $PL_0' u_0^N = Pr_0' - PL_0' A_0 r_0$.

Finally, we observe from above that u_α^R converges to $u_0^R = A_0(Qr_0 - QL_0 u_0^N) = A_0 r_0$. Therefore, u_0^N satisfies $PL_0' u_0^N = Pr_0' - PL_0' u_0^R$; that is, $PL_0' u_0 = Pr_0'$ which ends the proof. □

Remark 2.8. *From the proof of this theorem we observe that we can modify the assumptions on the mappings $\alpha \mapsto r_\alpha$ and $\alpha \mapsto K_\alpha$. If these mappings are only continuously differentiable in an open interval $J \subset I$ (as a subset of $\mathbb{R}$) which contains $\hat\alpha$ then the solution maps $\alpha \mapsto u_\alpha$ is continuous from J into X, and the estimate (11) holds. Also, if the assumption on the injectivity of L_α holds only for $\alpha \in J$ with $\alpha > \hat\alpha$ then the one-sided limit $u_{\hat\alpha} = \lim_{\alpha\to\hat\alpha, \alpha>\hat\alpha} u_\alpha$ exists and solves the system $L_{\hat\alpha} u_{\hat\alpha} = r_{\hat\alpha}$ and $PL_{\hat\alpha}' u_{\hat\alpha} = Pr_{\hat\alpha}'$.*

3. The Limiting Absorption Principle

In this section we consider the unperturbed case; that is, $q = 0$, and prove the limiting absorption principle (LAP) with respect to the refractive index; that is, we replace $n(x)$ in $W := \mathbb{R} \times (-h_0, h_0)$ by $n(x) + i\varepsilon q(x)$ for $\varepsilon > 0$ and let ε tend to zero. Here $q \in L^\infty(W)$ is any fixed non-negative function which is $2\pi-$periodic with respect to x_1 and satisfies $q(x) \ge q_0$ on some open set $\Omega \subset Q$ for some $q_0 > 0$. As an example we can take the constant function $q = 1$. Therefore, let

$$n_\varepsilon(x) \;:=\; \begin{cases} n(x) + i\varepsilon q(x) & \text{for } x \in W\,, \\ 1 & \text{for } x \in \mathbb{R}^2 \setminus W\,. \end{cases}$$

The incident plane wave is given by $u^i(x) = e^{ik\hat\theta\cdot x}$ where $\hat\theta = \binom{\sin\theta}{-\cos\theta}$ for some fixed $|\theta| < \frac{\pi}{2}$. Then u^i is $\alpha-$quasi-periodic with parameter $\alpha := k\hat\theta_1 = k\sin\theta$. Therefore, for $\varepsilon > 0$ we wish to determine $u_\varepsilon \in H^1_{\alpha,loc}(Q^\infty)$ such that

$$\Delta u_\varepsilon + k^2 n_\varepsilon\, u_\varepsilon \;=\; 0 \quad \text{in } \mathbb{R}^2 \tag{12}$$

and the scattered field $u_\varepsilon^s := u_\varepsilon - u^i$ satisfies the Rayleigh expansion (5). The scattered field satisfies $\Delta u_\varepsilon^s + k^2 n_\varepsilon u_\varepsilon^s = -k^2(n_\varepsilon - 1)u^i$, and by (10) its the variational form is given by

$$\int_Q \left[\nabla u_\varepsilon^s \cdot \nabla\overline{\psi} - k^2 n_\varepsilon\, u_\varepsilon^s\, \overline{\psi}\right] dx - \int_\Gamma (\Lambda_\alpha u_\varepsilon^s)\, \overline{\psi}\, ds \;=\; k^2 \int_Q (n_\varepsilon - 1)\, u^i\, \overline{\psi}\, dx$$

for all $\psi \in H^1_\alpha(Q)$. Green's theorem applied to u^i and ψ in Q yields

$$\int\limits_Q [\nabla u^i \cdot \nabla\overline{\psi} - k^2 n_\varepsilon\, u^i\, \overline{\psi}]\, dx - \int\limits_\Gamma \frac{\partial u^i}{\partial \nu}\, \overline{\psi}\, ds \;=\; -k^2 \int\limits_Q (n_\varepsilon - 1)\, u^i\, \overline{\psi}\, dx$$

and thus by adding both equations

$$\begin{aligned}
(13)\qquad & \int\limits_Q [\nabla u_\varepsilon \cdot \nabla\overline{\psi} - k^2 n_\varepsilon\, u_\varepsilon\, \overline{\psi}]\, dx - \int\limits_\Gamma (\Lambda_\alpha u_\varepsilon)\, \overline{\psi}\, ds \\
=\;& \int\limits_\Gamma \left[\frac{\partial u^i}{\partial \nu} - \Lambda_\alpha u^i\right] \overline{\psi}\, ds \;=\; \int\limits_{\Gamma_+} \left[\frac{\partial u^i}{\partial x_2} - \Lambda_\alpha u^i\right] \overline{\psi}\, ds \\
=\;& 2ik\, \hat{\theta}_2\, e^{ik\hat{\theta}_2 h_0} \int\limits_0^{2\pi} e^{i\alpha x_1}\, \overline{\psi(x_1,h_0)}\, dx_1 \;=\; -2ik\, \cos\theta\, e^{-ikh_0\cos\theta} \int\limits_0^{2\pi} e^{i\alpha x_1}\, \overline{\psi(x_1,h_0)}\, dx_1
\end{aligned}$$

for all $\psi \in H^1_\alpha(Q)$. Here we used that for $x_2 < -h_0$ the incident field satisfies the Rayleigh condition, thus $\frac{\partial u^i}{\partial \nu} = \Lambda_\alpha u^i$ on Γ_-. Furthermore, for $x_2 > h_0$ the $\alpha-$quasi-periodic solution of the Dirichlet problem with boundary data u^i on Γ_+ is given by $e^{i\alpha x_1 + |\hat{\theta}_2|(x_2 - 2h_0)}$, thus $\Lambda_\alpha u^i = ik\,|\hat{\theta}_2| e^{i\alpha x_1 + ik\hat{\theta}_2 h_0}$ on Γ_+.

Lemma 3.1. *For all $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in H^1_{\alpha,loc}(Q^\infty)$ of (12), (5) or, equivalently, (13).*

Proof: Since by Theorem 2.5 this equation can be written as $L_\varepsilon u_\varepsilon = r$ in $H^1_\alpha(Q)$ where L_ε is a Fredholm operator of index zero it suffices to prove uniqueness. For $u^i = 0$ we substitute $\psi = u_\varepsilon$ into the variational equation and obtain

$$\begin{aligned}
0 \;&=\; \int\limits_Q \left[|\nabla u_\varepsilon|^2 - k^2 n_\varepsilon |u_\varepsilon|^2\right] dx \;-\; \int\limits_\Gamma (\Lambda_\alpha u_\varepsilon)\, \overline{u_\varepsilon}\, ds \\
&=\; \int\limits_Q \left[|\nabla u_\varepsilon|^2 - k^2 n_\varepsilon |u_\varepsilon|^2\right] dx \;-\; i \sum_{\sigma\in\{+,-\}} \sum_{\ell\in\mathbb{Z}} \sqrt{k^2 - (\ell+\alpha)^2}\, |u_{\varepsilon,\ell}(\sigma h_0)|^2\,.
\end{aligned}$$

Taking the imaginary part

$$0 \;=\; -\varepsilon k^2 \int\limits_Q q\, |u_\varepsilon|^2 dx \;-\; \sum_{\sigma\in\{+,-\}} \sum_{|\ell+\alpha|<k} \sqrt{k^2 - (\ell+\alpha)^2}\, |u_{\varepsilon,\ell}(\sigma h_0)|^2$$

yields $u_\varepsilon = 0$ in Ω. Unique continuation implies that u_ε vanishes in all of Q. □

Theorem 3.2. *Let Assumptions 2.2 and 2.4 hold, and let $u_\varepsilon \in H^1_{\alpha,loc}(Q^\infty)$ be the unique solution of the quasi-periodic scattering problem (12), (5) for the plane incident wave of direction $\hat{\theta} = \binom{\sin\theta}{-\cos\theta}$ for some fixed $|\theta| < \frac{\pi}{2}$. Here, $\alpha := k\hat{\theta}_1 = k\sin\theta$. Then u_ε converges to some u_0 in $H^1(Q)$ which is a solution of (12), (5) for $\varepsilon = 0$, and u_0 is the only solution which satisfies in addition $\int_Q q\, u_0\, \overline{\hat{\phi}}\, dx = 0$ for all modes $\hat{\phi} \in \hat{X}_j$ in the case that $\alpha = \hat{\alpha}_j + \ell$ (for some $\ell \in \mathbb{Z}$ and $j \in J$) is a propagative wave number.*

Proof: We note that now α is fixed and ε takes the role of the parameter which tends to zero. We write (13) again in the form $L_\varepsilon u_\varepsilon = r$ where $L_\varepsilon : H^1_\alpha(Q) \to H^1_\alpha(Q)$ and $r \in H^1_\alpha(Q)$ are given by (compare with part (b) of Theorem 2.5)

$$\begin{aligned}
(L_\varepsilon u, \psi)_{H^1_\alpha(Q)} &:= \int\limits_Q [\nabla u \cdot \nabla \overline{\psi} - k^2 n_\varepsilon\, u\, \overline{\psi}]\, dx - \int\limits_\Gamma (\Lambda_\alpha u)\, \overline{\psi}\, ds\,, \\
(r, \psi)_{H^1_\alpha(Q)} &:= -2ik\, \cos\theta\, e^{-ikh_0\cos\theta} \int\limits_0^{2\pi} e^{i\alpha x_1}\, \overline{\psi(x_1, h_0)}\, dx_1
\end{aligned}$$

for $u, \psi \in H^1_\alpha(Q)$.

If α is no propagative wave number then L_0 is invertible and one has convergence of u_ε to the unique solution u_0 of $L_0 u_0 = r$ in $H^1(Q)$ as ε tends to zero.

Let now α be a propagative wave number. It is the aim to apply Theorem 2.7 in the modification of Remark 2.8 with $X = H^1_\alpha(Q)$. Then we know from Theorem 2.5 that the Riesz number of L_0 is one and the nullspaces $\mathcal{N}$ of L_0 and its adjoint L_0^* coincide and are given by the restrictions to Q of the space of corresponding propagating modes. Furthermore, L_ε depends obviously analytically on ε. It remains to show that r is in the range of L_0 and that $PL_0'|_{\mathcal{N}}$ is an isomorphism from $\mathcal{N}$ onto itself (where L_0' denotes the derivative with respect to ε at $\varepsilon = 0$). Since the nullspaces of L_0 and its adjoint L_0^* coincide we have to show that $(r, \hat{\phi})_{H^1_\alpha(Q)} = 0$ for all propagating modes $\hat{\phi}$ corresponding to the propagative wave number α. We have

$$(r, \hat{\phi})_{H^1_\alpha(Q)} = -2ik\, \cos\theta\, e^{-ikh_0\cos\theta} \int\limits_0^{2\pi} e^{i\alpha x_1}\, \overline{\hat{\phi}(x_1, h_0)}\, dx_1 = 0$$

because the Fourier coefficients of the propagating modes $\hat{\phi}$ vanish for all $|\ell + \alpha| < k$, in particular for $\ell = 0$ because $|\alpha| = k|\sin\theta| < k$. Furthermore,

$$\big(L_0' v, \psi\big)_{H^1(Q)} = -ik^2 \int\limits_Q q\, v\, \overline{\psi}\, dx\,, \quad v, \psi \in H^1_\alpha(Q)\,,$$

which shows that $PL_0'|_{\mathcal{N}}$ is an isomorphism from $\mathcal{N}$ onto itself. Application of Theorem 2.7 yields convergence of u_ε to u_0 as ε tends to zero, and u_0 solves the $k\sin\theta-$quasi-periodic scattering problem and, in addition, $\int_Q q\, u_0\, \overline{\hat{\phi}}\, dx = 0$ for all modes $\hat{\phi}$. $\square$

This result is quite unsatisfactory because the orthogonality condition $\int_Q q\, u_0\, \overline{\hat{\phi}}\, dx = 0$ depends on q. The scattering problem for the limiting case $\varepsilon = 0$, however, is independent of q. Therefore, also the extra condition in the case of a propagative wave number should be independent of q.

4. Continuity With Respect to the Direction of Incidence

We continue with the unperturbed case; that is, $q = 0$, and the scattering of a plane wave $u^i_\varphi(x) = e^{ik\hat{\varphi}\cdot x}$ for some $\hat{\varphi} = \binom{\sin\varphi}{-\cos\varphi}$ with $|\varphi| < \frac{\pi}{2}$ such that $\alpha := k\hat{\varphi}_1 = k\sin\varphi$ is not a propagative wave number in the sense of Definition 2.1. Then Theorem 2.5 yields uniqueness and existence of a $\alpha-$quasi-periodic solution u_φ of $\Delta u_\varphi + k^2 n u_\varphi = 0$ such that

$u_\varphi - u^i_\varphi$ satisfies the Rayleigh expansion (5). Let now $\hat\theta = \binom{\sin\theta}{-\cos\theta}$ with $|\theta| < \frac{\pi}{2}$ such that $\hat\alpha := k\hat\theta_1 = k\sin\theta$ is a propagative wave number and consider φ in a neighborhood of θ. It is the aim to prove that the unique solution u_φ converges to a solution u_θ of the problem for $\hat\theta$ and give a characterization.

We recall from (13) that the scattering problem for the incident direction $\hat\varphi$ is equivalent to the variational equation

$$\begin{aligned}(14)\qquad & \int\limits_Q [\nabla u_\varphi \cdot \nabla\overline{\psi} - k^2 n\, u_\varphi\, \overline{\psi}]\, dx - \int\limits_\Gamma (\Lambda_\alpha u_\varphi)\, \overline{\psi}\, ds \\ &= -2ik\, \cos\varphi\, e^{-ikh_0\cos\varphi} \int\limits_0^{2\pi} e^{i\alpha x_1}\, \overline{\psi(x_1,h_0)}\, dx_1 \quad \text{for all } \psi \in H^1_\alpha(Q)\end{aligned}$$

where $\alpha = k\sin\varphi$. With this variational formulation of the scattering problem we are able to prove the following convergence result.

Theorem 4.1. *Let Assumptions 2.2 and 2.4 hold and let $\hat\alpha := k\sin\theta$ for some $|\theta| < \frac{\pi}{2}$ be a propagative wave number; that is, $\hat\alpha = k\sin\theta = \hat\ell + \hat\alpha_j$ for some $\hat\ell \in \mathbb{Z}$ and $j \in J$. Furthermore, let u_φ be the unique solution of the $k\sin\varphi-$quasi-periodic scattering problem of the plane wave incidence of direction $\hat\varphi = \binom{\sin\varphi}{-\cos\varphi}$ for φ in a neighborhood of θ. Then u_φ converges in $H^1(Q)$ to some u_θ as φ tends to θ, and u_θ is a $\hat\alpha-$quasi-periodic solution of the scattering problem corresponding to the incident field of direction $\hat\theta$ and the only solution which also satisfies $\int_{Q^\infty} \frac{\partial u_\theta}{\partial x_1}\, \overline{\hat\phi}\, dx = 0$ for all propagating modes $\hat\phi \in \hat X_j$.*

Proof: We transform (14) into the $2\pi-$periodic form by setting $\tilde u_\varphi(x) = e^{-ik\sin\varphi\, x_1} u_\varphi(x)$ and substitute the form of the Dirichlet-Neumann map. This yields

$$\begin{aligned}(15)\qquad & \int\limits_Q \Big[\nabla\tilde u_\varphi \cdot \nabla\overline{\psi} - 2ik\sin\varphi\, \frac{\partial\tilde u_\varphi}{\partial x_1}\, \overline{\psi} - k^2(n - \sin^2\varphi)\, \tilde u_\varphi\, \overline{\psi}\Big]\, dx \\ & -i \sum_{\sigma\in\{+,-\}} \sum_{\ell\in\mathbb{Z}} \sqrt{k^2 - (\ell + k\sin\varphi)^2}\, \tilde u_{\varphi,\ell}(\sigma h_0)\, \overline{\psi_\ell(\sigma h_0)} \\ &= -2ik\cos\varphi\, e^{-ikh_0\cos\varphi} \int\limits_0^{2\pi} \overline{\psi(x_1,h_0)}\, dx_1 \quad \text{for all } \psi \in H^1_{per}(Q)\,.\end{aligned}$$

Here, $\tilde u_{\varphi,\ell}(\pm h_0)$ are the Fourier coefficients of $\tilde u_\varphi(\cdot,\pm h_0)$. We write this as $\tilde L_\varphi \tilde u_\varphi = \tilde r_\varphi$ in $H^1_{per}(Q)$ where $\tilde L_\varphi := J_\alpha^{-1} L_\alpha J_\alpha$ as in Theorem 2.5. Since $\hat\alpha = k\sin\theta$ is a propagative wave number it is not a cut-off value by Assumption 2.2. Therefore, by Theorem 2.5 the operator $\tilde L_\varphi$ satisfies the smoothness assumptions of Theorem 2.7 in a neighborhood of θ, and also the right hand side $\tilde r_\varphi$ depends obviously analytically on φ. Furthermore, $\tilde L_\theta$ has Riesz number one and the nullspaces $\mathcal{N}$ of L_θ and its adjoint L_θ^* coincide and are given by the restrictions to Q of the space of corresponding propagating modes (transformed to the periodic case). The derivatives with respect to φ are given by

$$
\begin{aligned}
(\tilde L'_\varphi \tilde v,\tilde\psi)_{H^1(Q)} &= -2ik\cos\varphi\int\limits_Q \big[\frac{\partial\tilde v}{\partial x_1}+ik\sin\varphi\,\tilde v\big]\,\overline{\tilde\psi}\,dx \\
&\qquad + ik\cos\varphi\sum_{\sigma\in\{+,-\}}\sum_{\ell\in\mathbb{Z}}\frac{\ell+k\sin\varphi}{\sqrt{k^2-(\ell+k\sin\varphi)^2}}\,\tilde v_\ell(\sigma h_0)\,\overline{\tilde\psi_\ell(\sigma h_0)}\,, \\
(\tilde r'_\varphi,\tilde\psi)_{H^1(Q)} &= 2ik\sin\varphi\, e^{-ikh_0\cos\varphi}\big[1-ikh_0\cos\varphi\big]\int\limits_0^{2\pi}\overline{\tilde\psi(x_1,h_0)}\,dx_1
\end{aligned}
$$

for $\tilde v,\tilde\psi\in H^1_{per}(Q)$. To show that $P\tilde L'_\theta$ is one-to-one on $\mathcal{N}$ we compute $(\tilde L'_\varphi\tilde v,\tilde\psi)_{H^1(Q)}$ for $\tilde v,\tilde\psi\in\mathcal{N}$. As mentioned above, $\tilde v(x)=e^{-ik\sin\theta\,x_1}v(x)$ and $\tilde\psi(x)=e^{-ik\sin\theta\,x_1}\psi(x)$ in Q with propagating modes $v,\psi\in\hat X_j$ which have expansions outside of Q in the forms

$$
\begin{aligned}
v(x) &= \frac{1}{\sqrt{2\pi}}\sum_{|\ell+k\sin\theta|>k} v_\ell(\pm h_0)\,e^{i(\ell+k\sin\theta)x_1-\sqrt{(\ell+k\sin\theta)^2-k^2}(|x_2|-h_0)}\,,\quad \pm x_2>h_0\,, \\
\psi(x) &= \frac{1}{\sqrt{2\pi}}\sum_{|\ell+k\sin\theta|>k} \psi_\ell(\pm h_0)\,e^{i(\ell+k\sin\theta)x_1-\sqrt{(\ell+k\sin\theta)^2-k^2}(|x_2|-h_0)}\,,\quad \pm x_2>h_0\,,
\end{aligned}
$$

respectively. A direct computation yields that

$$
\int\limits_Q\frac{\partial v}{\partial x_1}\,\overline{\psi}\,dx = \int\limits_Q\big[\frac{\partial\tilde v}{\partial x_1}+ik\sin\theta\,\tilde v\big]\,\overline{\tilde\psi}\,dx \quad\text{and} \tag{16a}
$$

$$
\begin{aligned}
\int\limits_{h_0}^{\infty}\int\limits_0^{2\pi}\frac{\partial v}{\partial x_1}\,\overline{\psi}\,dx_1\,dx_2 &= \frac{i}{2}\sum_{|\ell+k\sin\theta|>k}\frac{\ell+k\sin\theta}{\sqrt{(\ell+k\sin\theta)^2-k^2}}\,v_\ell(h_0)\,\overline{\psi_\ell(h_0)} \\
&= -\frac{1}{2}\sum_{|\ell+k\sin\theta|>k}\frac{\ell+k\sin\theta}{\sqrt{k^2-(\ell+k\sin\theta)^2}}\,v_\ell(h_0)\,\overline{\psi_\ell(h_0)}
\end{aligned} \tag{16b}
$$

and analogously for the integral over $(0,2\pi)\times(-\infty,-h_0)$. Therefore,

$$
(\tilde L'_\theta\tilde v,\tilde\psi)_{H^1(Q)} = -2ik\cos\theta\int\limits_{Q^\infty}\frac{\partial v}{\partial x_1}\,\overline{\psi}\,dx
$$

for $\tilde v,\tilde\psi\in\mathcal{N}$. Therefore, $P\tilde L'_\theta\tilde v=0$ for some $\tilde v\in\mathcal{N}$ implies that $\int_{Q^\infty}\frac{\partial v}{\partial x_1}\,\overline{\psi}\,dx=0$ for all $\psi\in\hat X_j$ which implies that v vanishes identically by Assumption 2.4.

Application of Theorem 2.7 yields continuity of $\varphi\mapsto\tilde u_\varphi$ in $H^1(Q)$ and $P\tilde L'_\theta\tilde u_\theta=P\tilde r'_\theta$; that is, $(\tilde L'_\theta\tilde u_\theta,\tilde\psi)_{H^1(Q)}=(\tilde r'_\theta,\tilde\psi)_{H^1(Q)}$ for all $\tilde\psi\in\mathcal{N}$. As above we go back to the quasi-periodic fields u_θ and ψ. We observe that for $x_2 > h_0$ and $x_2 < -h_0$ the total field u_θ is given by

$$\begin{aligned} u_\theta(x) &= e^{ik\sin\theta\, x_1}\big[e^{-ik\cos\theta\, x_2} - e^{ik\cos\theta(x_2-2h_0)}\big] \\ &\quad + \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} u_{\theta,\ell}(h_0)\, e^{i(\ell+k\sin\theta)x_1 - \sqrt{(\ell+k\sin\theta)^2-k^2}(x_2-h_0)}\,, \quad x_2 > h_0\,, \\ u_\theta(x) &= \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} u_{\theta,\ell}(-h_0)\, e^{i(\ell+k\sin\theta)x_1 - \sqrt{(\ell+k\sin\theta)^2-k^2}(-x_2-h_0)}\,, \quad x_2 < -h_0\,, \end{aligned}$$

where $u_{\theta,\ell}(\pm h_0) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} u_\theta(x_1,\pm h_0)e^{-i(\ell+k\sin\theta)x_1}dx_1$. From this and the fact that $\int_0^{2\pi}\psi(x_1,h_0)e^{-ik\sin\theta\, x_1}dx_1$ vanishes the propagating modes we conclude as before that

$$(\tilde{L}'_\theta\tilde{u}_\theta,\psi)_{H^1(Q)} \;=\; -2ik\cos\theta\int\limits_{Q^\infty}\frac{\partial u_\theta}{\partial x_1}\,\overline{\psi}\,dx \quad\text{and}\quad (\tilde{r}'_\theta,\psi)_{H^1(Q)} \;=\; 0$$

for all propagating modes $\psi\in\hat{X}_j$ which proves $\int_{Q^\infty}\frac{\partial u_\theta}{\partial x_1}\,\overline{\psi}\,dx = 0$ for all modes. □

We note that this condition on u_θ is independent of h_0 in contrast to the condition obtained by the LAP.

5. Approximation by Point Sources

We begin with the scattering problem of a point source at $z\in\mathbb{R}^2$ with $z_2 > h_0$. The total field $u_z(x) = \Phi(x,z) + u_z^s(x)$ is required to satisfy

$$\Delta u_z + k^2(n+q)\,u_z \;=\; 0 \quad \text{in } \mathbb{R}^2\setminus\{z\}\,, \tag{17}$$

the following open waveguide radiation condition, and $u_z^s := u_z - \Phi(\cdot,z)\in H^1_{loc}(\mathbb{R}^2)$.

Definition 5.1. *Let $\psi_+,\psi_-\in C^\infty(\mathbb{R})$ be any (fixed) functions with $\psi_\pm(x_1) = 1$ for $\pm x_1 \geq \sigma_0$ (for some $\sigma_0 > 2\pi+1$) and $\psi_\pm(x_1) = 0$ for $\pm x_1 \leq \sigma_0 - 1$. Denote by D a disc centered at the origin which contains the source z and the support of q and by $W_H := \mathbb{R}\times(-H,H)$ the layer of width $2H$ for any $H>0$.*

A solution $u\in H^1_{loc}(\mathbb{R}^2\setminus\overline{D})$ of $\Delta u + k^2nu = 0$ in $\mathbb{R}^2\setminus\overline{D}$ satisfies the open waveguide radiation condition with respect to given inner products $(\cdot,\cdot)_{\hat{X}_j}$ in $\hat{X}_j$ if

(a) *u has a decomposition in the form $u = u_{rad} + u_{prop}$ where $u_{rad}\in H^1(W_H\setminus\overline{D})$ for all $H > h_0$ and*

$$u_{prop}(x) \;=\; \sum_{j\in J}\Big[\psi_+(x_1)\sum_{\lambda_{\ell,j}>0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x) \;+\; \psi_-(x_1)\sum_{\lambda_{\ell,j}<0} a_{\ell,j}\,\hat{\phi}_{\ell,j}(x)\Big] \tag{18}$$

for $x\in\mathbb{R}^2\setminus\overline{D}$ and some $a_{\ell,j}\in\mathbb{C}$. Here, $\lambda_{\ell,j}\in\mathbb{R}$ and $\hat{\phi}_{\ell,j}\in\hat{X}_j$ for $j\in J$ are the eigenvalues and corresponding eigenfunctions, respectively, of the eigenvalue problem (7) in $\hat{X}_j$.

(b) *The radiating part u_{rad} satisfies the generalized angular spectrum radiation condition*

$$\int\limits_{-\infty}^{\infty}\left|(\operatorname{sign} x_2)\,\frac{\partial(\mathcal{F}u_{rad})(\omega,x_2)}{\partial x_2} - i\sqrt{k^2-\omega^2}\,(\mathcal{F}u_{rad})(\omega,x_2)\right|^2 d\omega \;\longrightarrow\; 0 \tag{19}$$

as $|x_2| \to \infty$ where $(\mathcal{F}u_{rad})(\cdot, x_2)$ denotes the Fourier transform of $u_{rad}(\cdot, x_2)$ with respect to x_1.
We normalize the Fourier transform as $(\mathcal{F}\phi)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s)\, e^{-ist}\, ds$ for $t \in \mathbb{R}$.

We transform this scattering problem to a problem with a compactly supported source. Indeed, for some $\varepsilon > 0$ we choose a function $\eta \in C^\infty(\mathbb{R}^2)$ with $\eta(y) = 1$ for $|y| \le \varepsilon/2$ and $\eta(y) = 0$ for $|y| \ge \varepsilon$. We decompose u_z as $u_z = \eta_z \Phi(\cdot, z) + \hat{u}_z^s$ with $\hat{u}_z^s := u_z - \eta_z \Phi(\cdot, z)$ where we have set $\eta_z(x) = \eta(x - z)$. Then $\hat{u}_z^s$ satisfies (note that $(1 - n + q)\, \eta_z$ vanishes identically if $z_2 > h_0 + \varepsilon$)

$$\Delta \hat{u}_z^s + k^2 (n + q)\, \hat{u}_z^s \;=\; -\, f_z \quad \text{in } \mathbb{R}^2\,, \tag{20}$$

where the right hand side $f_z := 2\, \nabla \eta_z \cdot \nabla_x \Phi(\cdot, z) + \Delta \eta_z\, \Phi(\cdot, z)$ is supported in the annulus $\{x \in \mathbb{R}^2 : \varepsilon/2 < |x - z| < \varepsilon\}$ which we assume to be in D.
It has been shown in [9] for the case of a half plane problem that the radiation condition of Definition 5.1 for compactly supported source functions $f \in L^2(Q)$ is a consequence of the limiting absorption principle. In [7] it is shown that the source problem (20) for any source function $f \in L^2(Q)$ has a unique solution satisfying the open waveguide radiation condition. Furthermore, we note that the solution $\hat{u}_z^s$ of (20) satisfies the radiation condition if, and only if, the solution u_z of (17) satisfies the radiation condition because $\hat{u}_z^s - u_z$ vanishes for $|x - z| > \varepsilon$.

From now on we consider again the unperturbed case $q = 0$. In this case the coefficients $a_{\ell,j} = a_{\ell,j}(z)$ are given explicitly by

$$a_{\ell,j}(z) \;:=\; \frac{2\pi i}{|\lambda_{\ell,j}|} \int\limits_{K(z,\varepsilon)} f_z(x)\, \overline{\hat{\phi}_{\ell,j}(x)}\, dx\,, \tag{21}$$

see again [7]. It is the aim to prove the following convergence result.

Theorem 5.2. *Let Assumptions 2.2 and 2.4 hold and let $\hat{\theta} = \binom{\sin\theta}{-\cos\theta} \in \mathbb{R}^2$ be a fixed unit vector with $|\theta| < \frac{\pi}{2}$; that is, $\hat{\theta}_2 < 0$. In addition, let $\hat{\alpha} := k\hat{\theta}_1 = k \sin\theta$ not be a cut-off value in the sense of Definition 2.1. Let u_t be the unique solution of the unperturbed (that is, for $q = 0$) scattering problem of the point source at $z = -t\hat{\theta}$ for $t|\hat{\theta}_2| > 2h_0$ such that $u_t^s := u_t - \Phi(\cdot, -t\hat{\theta}) \in H^1_{loc}(\mathbb{R}^2)$ and u_t satisfies the open waveguide radiation condition of Definition 5.1. Then*

$$\frac{1}{\gamma} \lim_{t \to \infty} \left[\sqrt{t}\, e^{-ikt} u_t\right] \;=\; v_\theta \quad \textit{in } H^1(Q_R) \tag{22}$$

for any $R > 0$ where $Q_R := (-R, R) \times (-h_0, h_0)$, and where $v_\theta \in H^1_{\hat{\alpha},loc}(\mathbb{R}^2)$ solves the $\hat{\alpha}-$quasi-periodic scattering problem $\Delta v_\theta + k^2 n v_\theta = 0$ in $\mathbb{R}^2$ such that the scattered field $v_\theta^s(x) := v_\theta(x) - e^{ik\hat{\theta}\cdot x}$ satisfies the Rayleigh expansion (5) for $\alpha = \hat{\alpha} = k\sin\theta$.
If $\hat{\alpha} = k \sin\theta = \ell + \hat{\alpha}_j$ is a propagative wave number (for some $\ell \in \mathbb{Z}$ and $j \in J$) with corresponding space $\hat{X}_j$ of propagating modes then the total field v_θ is the only solution which satisfies in addition

$$\int\limits_{Q^\infty} \frac{\partial v_\theta}{\partial x_1}\, \overline{\hat{\phi}}\, dx \;=\; 0 \quad \textit{for all } \hat{\phi} \in \hat{X}_j\,. \tag{23}$$

We note that the convergence of the total fields in (22) corresponds exactly to the convergence of the incident fields in (3). Therefore, this theorem justifies rigorously the assumption that one searches right away for $k\sin\theta-$quasi-periodic solutions of the scattering problem. We note however that this result holds also for the case that $k\sin\theta$ is a propagative wave number. In this case there is no uniqueness of the scattering problem by the plane wave of direction $\hat\theta$ of incidence, and Theorem 5.2 formulates the extra orthogonality condition (23) which coincides with the condition of Theorem 4.1.

We were not able to prove Theorem 5.2 in the case that $k\sin\theta-\ell$ is one of the cut-off values $\pm\kappa$ for some $\ell\in\mathbb{Z}$.

Proof of Theorem 5.2[1]: For the moment we consider any $z\in\mathbb{R}^2$ with $z_2>h_0+\varepsilon$. From (20) (for $q=0$) we note that the radiating part $u_{z,rad}$ of u_z^s solves

$$\Delta u_{z,rad}+k^2 n\, u_{z,rad} \;=\; -\; f_z \;-\; g_z \quad\text{in } \mathbb{R}^2\,, \tag{24}$$

where

$$f_z \;=\; 2\,\nabla\eta_z\cdot\nabla_x\Phi(\cdot,z)-\Delta\eta_z\,\Phi(\cdot,z) \;=\; (\Delta+k^2)\big[(\eta_z-1)\Phi(\cdot,z)\big] \quad\text{and} \tag{25a}$$

$$g_z \;=\; (\Delta+k^2 n)u_{z,prop} \;=\; \sum_{j\in J}\sum_{\ell=1}^{m_j} a_{\ell,j}(z)\,\varphi_{\ell,j} \quad\text{with} \tag{25b}$$

$$\varphi_{\ell,j}(x) \;=\; \begin{cases} 2\,\psi_+'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1}+\psi_+''(x_1)\,\hat\phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j}>0\,,\\ 2\,\psi_-'(x_1)\,\frac{\partial\hat\phi_{\ell,j}(x)}{\partial x_1}+\psi_-''(x_1)\,\hat\phi_{\ell,j}(x) & \text{if } \lambda_{\ell,j}<0\,. \end{cases}$$

Now we use the Floquet-Bloch transform F to transform (24) to a family of quasi-periodic problems. For functions $v\in C_0^\infty(\mathbb{R}^2)$ the transform is defined as

$$(Fv)(x,\alpha) \;:=\; \sum_{\ell\in\mathbb{Z}} v(x_1+2\pi\ell,x_2)\,e^{-i\alpha 2\pi\ell}\,,\qquad x\in\mathbb{R}^2\,.$$

Then it is known (see, e.g., [10]) that F has an extension to an isomorphism from $H^1(W)$ onto

$$L^2\big(I,H^1_\alpha(Q)\big) \;:=\; \left\{u\in L^2\big(Q\times I\big):\; \begin{array}{l} u(\cdot,\alpha)\in H^1_\alpha(Q) \text{ for almost all } \alpha \text{ and}\\ \alpha\mapsto\|u(\cdot,\alpha)\|_{H^1(Q)} \text{ is in } L^2(I)\end{array}\right\}$$

where $I=(-1/2,1/2)$ (or any other interval of length 1). The inverse is given by $u=\int_I (Fu)(\cdot,\alpha)d\alpha$ in W where $(Fu)(\cdot,\alpha)$ is extended $\alpha-$quasi-periodically to W.

We know from [7] that the Floquet-Bloch transformed equation

$$\Delta u_{\alpha,z}+k^2 n\, u_{\alpha,z} \;=\; -(Ff_z)(\cdot,\alpha)\;-\;(Fg_z)(\cdot,\alpha) \quad\text{in } Q^\infty \tag{26}$$

for $u_{\alpha,z}=(Fu_{z,rad})(\cdot,\alpha)$ is solvable for all $\alpha\in\mathbb{R}$ (without exception) and that $\alpha\mapsto u_{\alpha,z}$ has an extension to a mapping in $W^{1,1}\big(I,H^1(Q)\big)$ and is even analytic in neighborhoods of points $\hat\alpha$ which are no cut-off values. By part (b) of Theorem 2.5 this equation can be written as a variational equation in the form

$$\int\limits_Q[\nabla u_{\alpha,z}\cdot\nabla\overline{\psi}-k^2 n\,u_{\alpha,z}\,\overline{\psi}]\,dx-\int\limits_\Gamma(\Lambda_\alpha u_{\alpha,z})\,\overline{\psi}\,ds=\int\limits_Q(Fg_z)(\cdot,\alpha)\,\overline{\psi}\,dx+\int\limits_\Gamma\frac{\partial w_{\alpha,z}}{\partial\nu}\,\overline{\psi}\,ds \tag{27}$$

[1] We note already here that we will interrupt the proof by four Lemmas.

for all $\psi \in H^1_\alpha(Q)$ or shortly as $L_\alpha u_{\alpha,z} = r_{\alpha,z}$ in $H^1_\alpha(Q)$ where $r_{\alpha,z} \in H^1_\alpha(Q)$ denotes the Riesz representation of the right hand side. Note that $F f_z$ vanishes in Q and therefore appears only implicitly in $w^+_{\alpha,z}$. The functions $w^\pm_{\alpha,z} \in H^1_{\alpha,loc}(Q^{h_0}_\pm)$ are the $\alpha-$quasi-periodic solutions of

$$\begin{aligned}\Delta w^+_{\alpha,z} + k^2 w^+_{\alpha,z} &= -(F f_z)(\cdot,\alpha) - (F g_z)(\cdot,\alpha)\\ &= -(\Delta + k^2) F\big((\eta_z - 1)\Phi(\cdot,z)\big) - (F g_z)(\cdot,\alpha)\\ &= -(\Delta + k^2) F\big((\eta_z - 1)\Phi(\cdot,z) + \Phi(\cdot,z^*)\big) - (F g_z)(\cdot,\alpha)\end{aligned}$$

in $Q^{h_0}_+$ with $w^+_{\alpha,z} = 0$ for $x_2 = h_0$ and

$$\Delta w^-_{\alpha,z} + k^2 w^-_{\alpha,z} = -(F g_z)(\cdot,\alpha) \quad \text{in } Q^{h_0}_-$$

with $w^-_{\alpha,z} = 0$ for $x_2 = -h_0$, satisfying the generalized Rayleigh condition (8b). Here we used the definition of f_z and the fact that η_z vanishes in $Q^{h_0}_-$. The point $z^* = (z_1, 2h_0 - z_2)^\top$ is the reflection of z at the line $x_2 = h_0$.

Lemma 5.3. *$\partial w^\pm_{\alpha,z}/\partial x_2$ are given by*

$$\begin{aligned}\frac{\partial w^+_{\alpha,z}(x_1,h_0)}{\partial x_2} &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}} e^{i\sqrt{k^2-(\ell+\alpha)^2}(z_2-h_0)}\, e^{i(\ell+\alpha)(x_1-z_1)}\\ &\quad + \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \int_{h_0}^{\infty} (F g_z)_\ell(y_2,\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\, dy_2\, e^{i(\ell+\alpha)x_1},\\ \frac{\partial w^-_{\alpha,z}(x_1,-h_0)}{\partial x_2} &= -\frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}} \int_{h_0}^{\infty} (F g_z)_\ell(-y_2,\alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\, dy_2\, e^{i(\ell+\alpha)x_1}\end{aligned}$$

for $x_1 \in (0,2\pi)$ where $(F g_z)_\ell(y_2,\alpha) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(F g_z)(y,\alpha)e^{-i(\ell+\alpha)y_1}dy_1$ are the Fourier coefficients of $(F g_z)(\cdot,y_2,\alpha)$.

Proof: We write $(\eta_z - 1)\Phi(\cdot,z) + \Phi(\cdot,z^*) = -G^+(\cdot,z) + \eta_z\Phi(\cdot,z)$ where $G^+(x,z) = \Phi(x,z) - \Phi(x,z^*)$ denotes the Green's function for the half space $\{x \in \mathbb{R}^2 : x_2 > h_0\}$. Furthermore, the Floquet-Bloch transform $(F G^+(\cdot,z))(x,\alpha)$ is just the $\alpha-$quasi-periodic Green's function in $Q^{h_0}_+$, given by

$$\begin{aligned}&\big(F G^+(\cdot,z)\big)(x,\alpha)\\ &= \frac{i}{4\pi}\sum_{\ell\in\mathbb{Z}} \frac{1}{\sqrt{k^2-(\ell+\alpha)^2}}\left[e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2-z_2|} - e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2+z_2-2h_0)}\right] e^{i(\ell+\alpha)(x_1-z_1)}.\end{aligned}$$

Indeed, this follows from the connection between the Fourier transform $\mathcal{F}$ and the Floquet-Bloch transform F

$$(\mathcal{F}\phi)(\ell+\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(s)\, e^{-is(\ell+\alpha)}\, ds = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}(F\phi)(t,\alpha)\, e^{-i(\ell+\alpha)t}\, dt$$

(just decompose the region of integration into $\bigcup_{\ell\in\mathbb{Z}}(2\pi\ell, 2\pi\ell + 2\pi)$), writing this as $(F\phi)(t,\alpha) = \frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}(\mathcal{F}\phi)(\ell+\alpha)\, e^{i(\ell+\alpha)t}$, and using formulas 3. and 4. in [5], Section 6.677.

Therefore, $F\big(G^+(\cdot,z)-\eta_z\Phi(\cdot,z)\big)$ is smooth near $x=z$ and vanishes for $x_2=h_0$ and satisfies the Rayleigh expansion (5) because η_z vanishes near $x_2=h_0$ and for $|x|>|z|+\varepsilon$. Therefore,

$$w^+_{\alpha,z} \;=\; F\big(G^+(\cdot,z)-\eta_z\Phi(\cdot,z)\big)(\cdot,\alpha) \;+\; v^+_{\alpha,z} \quad \text{in } Q^{h_0}_+$$

where $v^+_{\alpha,z}$ is the radiating solution of $\Delta v^+_{\alpha,z}+k^2v^+_{\alpha,z}=-(Fg_z)(\cdot,\alpha)$ in $Q^{h_0}_+$ with $v^+_{\alpha,z}=0$ for $x_2=h_0$. Expanding $v^+_{\alpha,z}$ into a Fourier series and solving the one dimensional boundary value problem $\frac{d}{dx_2}v^+_{\ell,\alpha,z}(x_2)+(k^2-(\ell+\alpha)^2)v^+_{\ell,\alpha,z}(x_2)=-(Fg_z)_\ell(x_2,\alpha)$ for $x_2>h_0$ and $v^+_{\ell,\alpha,z}(h_0)=0$ and the generalized Rayleigh condition (8b) for its Fourier coefficients gives

$$v^+_{\ell,\alpha,z}(x_2)=\frac{i}{2}\int\limits_{h_0}^{\infty}(Fg_z)_\ell(y_2,\alpha)\,\frac{e^{i\sqrt{k^2-(\ell+\alpha)^2}|x_2-y_2|}-e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2+y_2-2h_0)}}{\sqrt{k^2-(\ell+\alpha)^2}}\,dy_2\,.$$

This proves the form for $w^+_{\alpha,z}$. Since $w^-_{\alpha,z}$ plays the role of $v^+_{\alpha,z}$ in $Q^{h_0}_-$ the representation is shown analogously. $\square$

With this result we rewrite (27) as

$$\begin{aligned}(L_\alpha u_{\alpha,z},\psi)_{H^1(Q)} \;&=\; \int\limits_Q (Fg_z)(\cdot,\alpha)\,\overline{\psi}\,dx+\frac{1}{\sqrt{2\pi}}\sum_{\ell\in\mathbb{Z}}e^{i\sqrt{k^2-(\ell+\alpha)^2}(z_2-h_0)}\,e^{-i(\ell+\alpha)z_1}\,\overline{\psi_\ell(h_0)}\\ &\quad+\sum_{\sigma\in\{+,-\}}\sum_{\ell\in\mathbb{Z}}\overline{\psi_\ell(\sigma h_0)}\int\limits_{h_0}^{\infty}(Fg_z)_\ell(\sigma y_2,\alpha)\,e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\,dy_2\,,\end{aligned}\tag{28}$$

where the operator L_α from $H^1_\alpha(Q)$ into itself is again defined as

$$(L_\alpha v,\psi)_{H^1(Q)} \;:=\; \int\limits_Q[\nabla v\cdot\nabla\overline{\psi}-k^2n\,v\,\overline{\psi}]\,dx-\int\limits_\Gamma(\Lambda_\alpha v)\,\overline{\psi}\,ds\,,\quad v,\psi\in H^1_\alpha(Q)\,.$$

At this point we define the sources z to be $z=z(t)=-t\,\hat\theta$ for $t>0$ where $\hat\theta=\binom{\sin\theta}{-\cos\theta}$ for $|\theta|<\frac{\pi}{2}$ is the fixed direction of the incident plane wave with $\hat\theta_2=-\cos\theta<0$. We choose $t>0$ such that $z_2(t)=-t\hat\theta_2=t\cos\theta>2h_0$. Then $z_2(t)\to\infty$ as $t\to\infty$. We change the symbols slightly and write $u_{\alpha,t}$ and g_t and $a_{\ell,j}(t)$ for $u_{\alpha,z(t)}$ and $g_{z(t)}$ and $a_{\ell,j}(z(t))$, respectively.

It is now the aim to study the inverse Floquet-Bloch transform $u_t(x)=\int_{-1/2}^{1/2}u_{\alpha,t}(x)d\alpha$ when t tends to infinity. We will decompose u_t into components and split the region into parts and discuss the contributions separately.

From the definitions (25b) and (21) of g_z and $a_{\ell,j}(z)$, respectively, the exponential decay of $\hat\phi_{\ell,j}$, and the fact that the support of f_z is contained in the disc $\{x\in\mathbb{R}^2:|x-z|\le\varepsilon\}$ we first note that $|a_{\ell,j}(t)|\le c\,e^{-\delta t}$ and thus

$$\|u_{t,prop}\|_{H^1(Q_R)} \;\le\; c_R\sum_{j\in J}\sum_{\ell=1}^{m_j}|a_{\ell,j}(t)| \;\le\; c\,e^{-\delta t}\tag{29}$$

for some $c>0$ and

$$|(Fg_t)(x,\alpha)| \;+\; |\partial(Fg_t)(x,\alpha)/\alpha| \;\le c\,e^{-\delta(t+|x_2|)}\,,\quad x\in Q^\infty\setminus Q\,,\tag{30}$$

for all $t > 0$, $R > 0$, and $\alpha \in [-1/2, 1/2]$.

We split the first series on the right hand side of (28) into propagating and evanescent parts. Decompose k again into the form $k = \hat{\ell} + \kappa$ with $\hat{\ell} \in \mathbb{N}_0$ and $\kappa \in (-1/2, 1/2]$. Then $\pm\kappa$ are the cut-off values. We can always decompose $[-1/2, 1/2]$ in the form $[-1/2, 1/2] = I_1 \cup I_2 \cup I_3$ with closed intervals I_m such that their interiors are pairwise disjoint and find corresponding sets $\mathcal{L}_m \subset \{-\hat{\ell}, \ldots, \hat{\ell}\}$ such that $|\ell + \alpha| \leq k$ for all $\alpha \in I_m$ and $\ell \in \mathcal{L}_m$ and $|\ell + \alpha| \geq k$ for all $\alpha \in I_m$ and $\ell \notin \mathcal{L}_m$ for $m = 1, 2, 3$. For example, if $\kappa \geq 0$ then $I_1 = [-\kappa, \kappa]$ with $\mathcal{L}_1 = \{-\hat{\ell}, \ldots, \hat{\ell}\}$, $I_2 = [-1/2, -\kappa]$ with $\mathcal{L}_2 = \{-\hat{\ell} + 1, \ldots, \hat{\ell}\}$, and $I_3 = [\kappa, 1/2]$ with $\mathcal{L}_3 = \{-\hat{\ell}, \ldots, \hat{\ell} - 1\}$. Some of the intervals can degenerate into points (as I_3 in the preceding example if $\kappa = 1/2$ or I_1 if $\kappa = 0$) and some of the sets $\mathcal{L}_m$ can be empty (as $\mathcal{L}_2$ and $\mathcal{L}_3$ in the preceding example if $\hat{\ell} = 0$). The cut-off values are contained in the boundary points of I_m.

For $\alpha \in I_m$ (where $m \in \{1, 2, 3\}$ is kept fixed) we rewrite (28) in the form

$$
\begin{aligned}
\big(L_\alpha u_{\alpha,t}, \psi\big)_{H^1(Q)} \;&=\; \frac{1}{\sqrt{2\pi}} \sum_{\ell \in \mathcal{L}_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2-(\ell+\alpha)^2}|\hat{\theta}_2|]}\, e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0}\, \overline{\psi_\ell(h_0)} \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{\ell \notin \mathcal{L}_m} e^{t[i(\ell+\alpha)\hat{\theta}_1 - \sqrt{(\ell+\alpha)^2-k^2}|\hat{\theta}_2|]}\, e^{\sqrt{(\ell+\alpha)^2-k^2}h_0}\, \overline{\psi_\ell(h_0)} \qquad (31) \\
&\quad + \int_Q (Fg_t)(\cdot, \alpha)\, \overline{\psi}\, dx \;+ \sum_{\sigma \in \{+,-\}} \sum_{\ell \in \mathbb{Z}} \overline{\psi_\ell(\sigma h_0)} \int_{h_0}^{\infty} (Fg_t)_\ell(\sigma y_2, \alpha)\, e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\, dy_2
\end{aligned}
$$

for all $\psi \in H^1_\alpha(Q)$. We recall that if α is not a propagative wave number then this equation is uniquely solvable. If $\alpha = \hat{\alpha}_j$ is a propagative wave number in I_m then, by the choice of $a_{\ell,j}(t)$, this equation is also solvable because $r_{\hat{\alpha}_j,t}$ is orthogonal to $\hat{X}_j$; that is, the right hand side of (31) vanishes for modes $\psi = \hat{\phi}_j \in \hat{X}_j$ corresponding to $\hat{\alpha}_j$. This has been shown in [7].

The right hand side of (31) suggests to decompose $u_{\alpha,t}$ for $\alpha \in I_m$ into a sum of the form

$$
u_{\alpha,t} \;=\; \frac{i}{4\pi} \sum_{\ell \in \mathcal{L}_m} e^{it[(\ell+\alpha)\hat{\theta}_1 + \sqrt{k^2-(\ell+\alpha)^2}|\hat{\theta}_2|]}\, \frac{1}{\sqrt{k^2-(\ell+\alpha)^2}}\, v_{\ell,\alpha} \;+\; u^{(1)}_{\alpha,t} \qquad (32)
$$

with functions $v_{\ell,\alpha} \in H^1_\alpha(Q)$ for $\ell \in \mathcal{L}_m$ which are independent of t and solutions of

$$
\begin{aligned}
\big(L_\alpha v_{\ell,\alpha}, \psi\big)_{H^1(Q)} \;&=\; -2i\sqrt{2\pi}\,\sqrt{k^2-(\ell+\alpha)^2}\, e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0}\, \overline{\psi_\ell(h_0)} \\
&=\; -2i\sqrt{k^2-(\ell+\alpha)^2}\, e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0} \int_0^{2\pi} \overline{\psi(x_1, h_0)}\, e^{i(\ell+\alpha)x_1} dx_1 \qquad (33)
\end{aligned}
$$

for all $\psi \in H^1_\alpha(Q)$. The solutions exist for all $\alpha \in I_m$ because for every propagative wave number $\alpha = \hat{\alpha}_j \in I_m$ the right hand side of (33) vanishes for every $\psi = \hat{\phi} \in \hat{X}_j$. Indeed, in this case $\hat{\phi}$ is evanescent; that is, the Fourier coefficients $\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \hat{\phi}(x_1, h_0, \hat{\alpha})\, e^{-i(\ell+\hat{\alpha})x_1} dx_1$ vanish for $|\ell + \hat{\alpha}| < k$; that is, for all $\ell \in \mathcal{L}_m$. This proves existence of a solution for all $\alpha \in I_m$. The functions $v_{\ell,\alpha}$ are solutions of $\alpha-$quasi-periodic scattering problems for plane wave incidence as the next lemma shows.

Lemma 5.4. *$v_{\ell,\alpha}$ is the restriction to Q of a solution of the $\alpha-$quasi-periodic scattering problem of the incident plane wave of direction $\hat{\theta}_\ell = \frac{1}{k}\big(\ell+\alpha, -\sqrt{k^2-(\ell+\alpha)^2}\big)^\top$ to determine the total field $v_{\ell,\alpha}$ as the sum $v_{\ell,\alpha}(x) = e^{i(\ell+\alpha)x_1 - i\sqrt{k^2-(\ell+\alpha)^2}x_2} + v^s_{\ell,\alpha}(x)$ such that*

$$\Delta v_{\ell,\alpha} \; + \; k^2 n\, v_{\ell,\alpha} \; = \; 0 \quad \text{in } \mathbb{R}^2\,, \tag{34}$$

and the scattered field $v^s_{\ell,\alpha}$ satisfies the Rayleigh expansion (5) outside of Q.

Proof: We consider the scattering problem and make an ansatz for the solution in the form

$$v_{\ell,\alpha}(x) \;=\; \tilde{v}^s_{\ell,\alpha}(x) \;+ \begin{cases} e^{i(\ell+\alpha)x_1}\big[e^{-i\sqrt{k^2-(\ell+\alpha)^2}x_2} - e^{i\sqrt{k^2-(\ell+\alpha)^2}(x_2-2h_0)}\big], & x_2 > h_0, \\ 0\,, & x_2 < -h_0\,, \end{cases}$$

where

$$\tilde{v}^s_{\ell,\alpha}(x) \;=\; \frac{1}{\sqrt{2\pi}} \sum_{\ell' \in \mathbb{Z}} a_{\ell'}(\pm h_0)\, e^{i(\ell'+\alpha)x_1 + i\sqrt{k^2-(\ell'+\alpha)^2}(|x_2|-h_0)} \quad \text{for } \pm x_2 > h_0\,. \tag{35}$$

Then $v_{\ell,\alpha} = \tilde{v}^s_{\ell,\alpha}$ on $\Gamma = \Gamma_+ \cup \Gamma_-$ and thus

$$\begin{aligned} \frac{\partial v_{\ell,\alpha}(x_1,h_0)}{\partial x_2} &= (\Lambda_\alpha v_{\ell,\alpha})(x_1,h_0) \;-\; 2i\sqrt{k^2-(\ell+\alpha)^2}\, e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0}\, e^{i(\ell+\alpha)x_1}\,, \\ \frac{\partial v_{\ell,\alpha}(x_1,-h_0)}{\partial x_2} &= (\Lambda_\alpha v_{\ell,\alpha})(x_1,-h_0)\,. \end{aligned}$$

Therefore, the variational form of (34) is

$$\begin{aligned} &\int_Q \big[\nabla v_{\ell,\alpha}\cdot\nabla\overline{\psi} - k^2 n v_{\ell,\alpha}\,\overline{\psi}\big]\,dx - \int_\Gamma (\Lambda v_{\ell,\alpha})\,\overline{\psi}\,ds \\ = \;\; & -2i\sqrt{k^2-(\ell+\alpha)^2}\, e^{-i\sqrt{k^2-(\ell+\alpha)^2}h_0} \int_0^{2\pi} e^{i(\ell+\alpha)x_1}\overline{\psi(x_1,h_0)}\,dx_1 \quad \text{for all } \psi \in H^1_{per}(Q) \end{aligned}$$

which coincides with (33). $\square$

Lemma 5.5. *Let $\hat{\alpha}$ be a fixed value in the interior of I_m and $v_{\ell,\alpha}$ as in the previous lemma for $\ell \in \mathcal{L}_m$ and $\alpha \in I_m$. Then the solution map $\alpha \mapsto v_{\ell,\alpha}$ can be extended to an analytic map from an open neighborhood $U \subset \mathbb{C}$ of $\hat{\alpha}$ into $H^1(Q)$. Furthermore, if $\hat{\alpha} = \hat{\alpha}_j$ is a propagative wave number then this extension into $\hat{\alpha}_j$ satisfies*

$$\int_{Q^\infty} \frac{\partial v_{\ell,\hat{\alpha}_j}}{\partial x_1}\,\overline{\hat{\phi}}\,dx \;=\; 0 \tag{36}$$

for all corresponding modes $\hat{\phi} \in \hat{X}_j$.

We omit the proof because it follows from Theorem 4.1 if one writes $(\ell+\alpha)/k$ as $(\ell+\alpha)/k = k\sin\varphi$ in the incident plane wave of direction $\hat{\theta}_\ell$.

Next we consider the remaining term

$$u^{(1)}_{\alpha,t} \;:=\; u_{\alpha,t} \;-\; \frac{i}{4\pi}\sum_{\ell\in\mathcal{L}_m} e^{it[(\ell+\alpha)\hat\theta_1+\sqrt{k^2-(\ell+\alpha)^2}|\hat\theta_2|]}\,\frac{1}{\sqrt{k^2-(\ell+\alpha)^2}}\,v_{\ell,\alpha}$$

of (32) which satisfies

$$\big(L_\alpha u^{(1)}_{\alpha,t},\psi\big)_{H^1_\alpha(Q)} \;=\; \frac{1}{\sqrt{2\pi}}\sum_{\ell\notin\mathcal{L}_m} e^{t[i(\ell+\alpha)\hat\theta_1-\sqrt{(\ell+\alpha)^2-k^2}|\hat\theta_2|]}\,e^{\sqrt{(\ell+\alpha)^2-k^2}h_0}\,\overline{\psi_\ell(h_0)}$$

$$+\int_Q (Fg_t)(x,\alpha)\,\overline{\psi(x)}\,dx+\sum_{\sigma\in\{+,-\}}\sum_{\ell\in\mathbb{Z}}\overline{\psi_\ell(\sigma h_0)}\int_{h_0}^{\infty}(Fg_t)_\ell(\sigma y_2,\alpha)\,e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\,dy_2$$

for all $\psi\in H^1_{per}(Q)$ which we write briefly as $L_\alpha u^{(1)}_{\alpha,t}=r^{(1)}_{\alpha,t}+r^{(2)}_{\alpha,t}+r^{(3)}_{\alpha,t}$.

Lemma 5.6. *There exists $c>0$ such that*

$$\int_{I_m}\|u^{(1)}_{\alpha,t}\|_{H^1(Q)}\,d\alpha \;\leq\; \frac{c}{t}\quad \textit{for all}\quad t|\hat\theta_2|\geq 2h_0\,.$$

Proof: We decompose I_m into a finite union of closed intervals $I\subset I_m$ with non-intersecting interiors where I is one of the following two types.
First case: Let $I\subset I_m$ does not contain any of the propagative wave numbers $\hat\alpha_j$. Then $\tilde L_\alpha^{-1}$ is uniformly bounded with respect to $\alpha\in I$. We estimate the three terms $r^{(j)}_{\alpha,t}$ on the right hand side. The inequality of Cauchy-Schwarz and the trace theorem yields for every $\alpha\in I_m$

$$\|r^{(1)}_{\alpha,t}\|_{H^1(Q)}\;\leq\; c\bigg(\sum_{\ell\notin\mathcal{L}_m} e^{-2\sqrt{(\ell+\alpha)^2-k^2}\,(t|\hat\theta_2|-h_0)}\bigg)^{1/2}$$

where $c>0$ is independent of α and t. Furthermore, (30) implies

$$\|r^{(2)}_{\alpha,t}\|_{H^1(Q)}\;\leq\;\|(Fg_t)(\cdot,\alpha)\|_{L^2(Q)}\;\leq\; c\,e^{-\delta t}\quad\text{for all } \alpha\in I_m\,.$$

For $r^{(3)}_{\alpha,t}$ we consider first $|\ell|\geq k+1$. Then $|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}|=e^{-\sqrt{(\ell+\alpha)^2-k^2}(y_2-h_0)}$ and thus

$$\Bigg[\sum_{|\ell|\geq k+1}|\psi_\ell(h_0)|\int_{h_0}^{\infty}\big|(Fg_t)_\ell(y_2,\alpha)\big|\,\big|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\big|\,dy_2\Bigg]^2$$

$$\leq\;\sum_{|\ell|\geq k+1}|\psi_\ell(h_0)|^2\sum_{|\ell|\geq k+1}\int_{h_0}^{\infty}\big|(Fg_t)_\ell(y_2,\alpha)\big|^2dy_2\int_{h_0}^{\infty}e^{-2\sqrt{(\ell+\alpha)^2-k^2}(y_2-h_0)}dy_2$$

$$\leq\; c_1\|\psi\|^2_{H^1(Q)}\|(Fg_t)(\cdot,\alpha)\|^2_{L^2(Q^{h_0}_+)}\;\leq\; c_2\,e^{-2\delta t}\|\psi\|^2_{H^1_{per}(Q)}$$

for all $\alpha \in [-1/2, 1/2]$ by (30). The remaining finite sum is estimated as

$$
\begin{aligned}
& \sum_{|\ell| \le k+1} |\psi_\ell(h_0)| \int\limits_{h_0}^{\infty} \big|(Fg_t)_\ell(y_2,\alpha)\big| \, \big|e^{i\sqrt{k^2-(\ell+\alpha)^2}(y_2-h_0)}\big| \, dy_2 \\
\le \;& \sum_{|\ell| \le k+1} |\psi_\ell(h_0)| \int\limits_{h_0}^{\infty} \big|(Fg_t)_\ell(y_2,\alpha)\big| \, dy_2 \;\le\; \frac{\sqrt{2k+3}}{\sqrt{2\pi}} \, \|\psi\|_{H^1(Q)} \int\limits_0^{2\pi} \int\limits_{h_0}^{\infty} \big|(Fg_t)(y,\alpha)\big| \, dy_2 \, dy_1 \\
\le \;& c \, \|\psi\|_{H^1(Q)} \, e^{-\delta t} \quad \text{for all } \alpha \in [-1/2, 1/2]
\end{aligned}
$$

where we used (30) again. The restrictions of these estimates to $\alpha \in I$ and the uniform boundedness of $\tilde{L}_\alpha^{-1}$ yields the existence of $c > 0$ with

$$
\|u^{(1)}_{\alpha,t}\|_{H^1(Q)} \;\le\; c \Big(\sum_{\ell \notin \mathcal{L}_m} e^{-t\sqrt{(\ell+\alpha)^2-k^2}\,|\hat{\theta}_2|} \Big)^{1/2} \;+\; c\, e^{-\delta t} \tag{37}
$$

for all $\alpha \in I$ and $t|\hat{\theta}_2| \ge 2h_0$

Second case: Let $I \subset I_m$ contain no cut-off value (that is, $I \subset \operatorname{int} I_m$; that is, $|\ell + \alpha| \neq k$ for all $\ell \in \mathbb{Z}$ and $\alpha \in I$). In this case we wish to apply Theorem 2.7 (in the modification of Remark 2.8) to the equation $\tilde{L}_\alpha \tilde{u}^{(1)}_{\alpha,t} = \tilde{r}^{(1)}_{\alpha,t} + \tilde{r}^{(2)}_{\alpha,t} + \tilde{r}^{(3)}_{\alpha,t}$ in the space $H^1_{per}(Q)$ of periodic functions. We have to show that $\tilde{r}^{(j)}_{\alpha,t}$ is differentiable with respect to α and have to bound the derivative. First we note that in this case of I there exists $c_0 > 0$ with $\sqrt{(\ell+\alpha)^2 - k^2} \ge c_0(|\ell|+1)$ for all $\alpha \in I$ and $\ell \notin \mathcal{L}_m$. We begin with $\tilde{r}^{(1)}_{\alpha,t}$. For $\alpha \in I$ and $\ell \notin \mathcal{L}_m$ we have

$$
\begin{aligned}
& \left| \frac{\partial}{\partial \alpha} e^{it(\ell+\alpha)\hat{\theta}_1 - \sqrt{(\ell+\alpha)^2-k^2}(t|\hat{\theta}_2|-h_0)} \right| \\
= \;& \left| it\hat{\theta}_1 - \frac{\ell+\alpha}{\sqrt{(\ell+\alpha)^2-k^2}} \, (t|\hat{\theta}_2| - h_0) \right| e^{-\sqrt{(\ell+\alpha)^2-k^2}(t|\hat{\theta}_2|-h_0)} \\
\le \;& c\, t\, e^{-c_0(|\ell|+1)(t|\hat{\theta}_2|-h_0)} \,.
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \sum_{\ell \notin \mathcal{L}_m} \left| \frac{\partial}{\partial \alpha} e^{it(\ell+\alpha)\hat{\theta}_1 - \sqrt{(\ell+\alpha)^2-k^2}(t|\hat{\theta}_2|-h_0)} \right| |\psi_\ell(h_0)| \\
\le \;& c\, t\, \|\psi\|_{H^1_{per}(Q)} \Big(\sum_{\ell \notin \mathcal{L}_m} e^{-2c_0(|\ell|+1)(t|\hat{\theta}_2|-h_0)} \Big)^{1/2} \;\le\; c\, t\, e^{-c_0(t|\hat{\theta}_2|-h_0)} \, \|\psi\|_{H^1_{per}(Q)} \,;
\end{aligned}
$$

that is, $\|\partial \tilde{r}^{(1)}_{\alpha,t}/\partial\alpha\|_{H^1_{per}(Q)} \le c\,t\,e^{-c_0(t|\hat{\theta}_2|-h_0)} \le c\,t\,e^{-c_0 t|\hat{\theta}_2|/2}$ for all $\alpha \in I$ and $t|\hat{\theta}_2| \ge 2h_0$. The estimates of $\|\partial \tilde{r}^{(j)}_{\alpha,t}/\partial\alpha\|_{H^1_{per}(Q)}$ for $j = 2,3$ follow the same arguments as for $\|r^{(j)}_{\alpha,t}\|_{H^1(Q)}$ using in addition that $|\ell+\alpha|/|\sqrt{k^2-(\ell+\alpha)^2}|$ is uniformly bounded with respect to $\ell \in \mathbb{Z}$ and $\alpha \in I$.

Therefore, application of Remark 2.8 yields an estimate of the form (37) where the second term is replaced by $c_2\, t\, e^{-c_3 t}$ for some $c_2, c_3 > 0$. Since we can decompose I_m as a finite union of closed intervals I of the first or second type with non-intersecting interiors[2] we have an estimate of the form

$$\|u_{\alpha,t}^{(1)}\|_{H^1(Q)} \;\le\; c_1\biggl(\sum_{\ell\notin\mathcal{L}_m} e^{-t\sqrt{(\ell+\alpha)^2-k^2}\,|\hat\theta_2|}\biggr)^{1/2} \;+\; c_2\,t\,e^{-c_3 t}$$

for all $\alpha\in I_m$ and $t|\hat\theta_2|\ge 2h_0$. Therefore, by the inequality of Cauchy-Schwarz,

$$\int\limits_{I_m}\|u_{\alpha,t}^{(1)}\|_{H^1(Q)}\,d\alpha \;\le\; c\biggl(\sum_{\ell\notin\mathcal{L}_m}\int\limits_{I_m} e^{-t\sqrt{(\ell+\alpha)^2-k^2}\,|\hat\theta_2|}d\alpha\biggr)^{1/2} \;+\; c_2\,t\,e^{-c_3 t}\,.$$

For large values of $|\ell|$, say $|\ell|\ge k+1$, we use the estimate $\sqrt{(\ell+\alpha)^2-k^2}\ge c_0|\ell|$ which yields that the series over $|\ell|\ge k+1$ decays exponentially to zero as t tends to infinity. For fixed $\ell\notin\mathcal{L}_m$ with $|\ell|\le k+1$ we make the substitution $\beta=\psi(\alpha)=\sqrt{(\ell+\alpha)^2-k^2}$. Then

$$\int\limits_{I_m} e^{-t\sqrt{(\ell+\alpha)^2-k^2}\,|\hat\theta_2|}d\alpha \;=\; \int\limits_{\psi(I_m)} e^{-t\beta|\hat\theta_2|}\,\frac{\beta}{\sqrt{\beta^2+k^2}}d\beta \;\le\; \frac{1}{k}\int\limits_{\psi(I_m)}\beta\,e^{-t\beta|\hat\theta_2|}\,d\beta$$

which tends to zero as $1/t^2$. Indeed, if $\psi(I_m)=[a,b]$ with $b>a\ge 0$ then this follows from

$$\int\limits_a^b \beta\,e^{-s\beta}\,d\beta \;=\; \frac{1}{s}\bigl(a\,e^{-sa}-b\,e^{-sb}\bigr) \;-\; \frac{1}{s^2}\bigl(e^{-sb}-e^{-sa}\bigr)\,.$$

This ends the proof. □

We go back to the decomposition (32) of $u_{\alpha,t}$ for $\alpha\in I_m$ and consider the integrals (for $\ell\in\mathcal{L}_m$)

$$\frac{i}{4\pi}\int\limits_{I_m} e^{it[(\ell+\alpha)\,\hat\theta_1+\sqrt{k^2-(\ell+\alpha)^2}|\hat\theta_2|]}\,\frac{1}{\sqrt{k^2-(\ell+\alpha)^2}}\,v_{\ell,\alpha}\,d\alpha$$

in $H^1(Q_R)$ (for some fixed $R>0$) with the method of stationary phase. We recall from Lemmas 5.4 and 5.5 that $v_{\ell,\alpha}$ is the total $\alpha-$quasi-periodic field corresponding to the incident plane wave $e^{i(\ell+\alpha)x_1-i\sqrt{k^2-(\ell+\alpha)^2}x_2}$ which is analytic with respect to α in the interior of I_m and is also in $W^{1,1}\bigl(I_m,H^1(Q_R)\bigr)$.

We define $\psi(s)=s\,\hat\theta_1+\sqrt{k^2-s^2}\,|\hat\theta_2|$ for $|s|\le k$. Then it easily seen that $\tilde s=k\hat\theta_1$ is the only critical point (that is, $\psi'(\tilde s)=0$) and $\psi(\tilde s)=k$ and $\psi''(\tilde s)=-\frac{1}{k\hat\theta_2^2}<0$. There is exactly one $\tilde\ell\in\mathbb{Z}$ and $\tilde\alpha\in(-1/2,1/2]$ with $\tilde s=k\hat\theta_1=\tilde\ell+\tilde\alpha$. We note that $\tilde\alpha\neq\pm\kappa$; that is, $\tilde\alpha$ is not a cut-off value by assumption on $k\hat\theta_1$.

Then there exists exactly one interval $I_{\tilde m}$ such that $\tilde\alpha$ is in the interior of $I_{\tilde m}$ and $\tilde\ell\in\mathcal{L}_{\tilde m}$.

[2] Note that the cut-off values are no propagative wave numbers by assumption.

Since $v_{\tilde{\ell},\alpha}$ is smooth in $I_{\tilde{m}}$ the method of stationary phase is applicable to the integral over $I_{\tilde{m}}$ which gives

$$\begin{aligned}
&\frac{i}{4\pi}\int\limits_{I_{\tilde{m}}} e^{it[(\tilde{\ell}+\alpha)\,\hat{\theta}_1+\sqrt{k^2-(\tilde{\ell}+\alpha)^2}|\hat{\theta}_2|]}\,\frac{1}{\sqrt{k^2-(\tilde{\ell}+\alpha)^2}}\,v_{\tilde{\ell},\alpha}\,d\alpha \\
&= \frac{i}{4\pi\sqrt{k^2-(\tilde{\ell}+\tilde{\alpha})^2}}\,e^{itk-i\pi/4}\sqrt{\frac{2\pi k\,\hat{\theta}_2^2}{t}}\,v_{\tilde{\ell},\tilde{\alpha}} \;+\; o(1/\sqrt{t}) \\
&= \gamma\,\frac{e^{ikt}}{\sqrt{t}}\,v_{\tilde{\ell},\tilde{\alpha}} \;+\; o(1/\sqrt{t})
\end{aligned}$$

as $t\to\infty$. For $\ell\in\mathcal{L}_{\tilde{m}}\setminus\{\tilde{\ell}\}$ the function $\alpha\mapsto(\ell+\alpha)\,\hat{\theta}_1+\sqrt{k^2-(\ell+\alpha)^2}\,|\hat{\theta}_2|$ is monotonous. Substituting $\beta=(\ell+\alpha)\,\hat{\theta}_1+\sqrt{k^2-(\ell+\alpha)^2}\,|\hat{\theta}_2|$ and using partial integration yields that these integrals decay as $\mathcal{O}(1/t)$. Therefore, by (32) and Lemma 5.6,

$$\int\limits_{I_{\tilde{m}}} u_{\alpha,t}\,d\alpha \;=\; \gamma\,\frac{e^{itk}}{\sqrt{t}}\,v_{\tilde{\ell},\tilde{\alpha}} \;+\; o(1/\sqrt{t})$$

as $t\to\infty$ in $H^1(Q_R)$. For the intervals I_m with $m\neq\tilde{m}$ and $\ell\in\mathcal{L}_m$ partial integration yields again that these integrals decay as $\mathcal{O}(1/t)$. Therefore, the integration can be done over all of $[-1/2,1/2]$, and the inverse Floquet-Bloch transform gives

$$u_t \;=\; \int\limits_{-1/2}^{1/2} u_{\alpha,t}\,d\alpha \;=\; \gamma\,\frac{e^{itk}}{\sqrt{t}}\,v_{\tilde{\ell},\tilde{\alpha}} \;+\; o(1/\sqrt{t})$$

in $H^1(Q_R)$. From Lemma 5.4 we observe that $v_{\tilde{\ell},\tilde{\alpha}}$ is the solution of the $\tilde{\alpha}-$quasi-periodic scattering problem for the incident plane wave $u^{inc}(x)=e^{i(\tilde{\ell}+\tilde{\alpha})x_1-i\sqrt{k^2-(\tilde{\ell}+\tilde{\alpha})^2}x_2}=e^{ik\hat{\theta}\cdot x}$; that is, $v_{\ell,\tilde{\alpha}}=v_\theta$ with the field v_θ from Theorem 5.2. If $k\hat{\theta}_1$ is a propagative wave number $\hat{\alpha}_j+\ell$ for some $\ell\in\mathbb{Z}$ then $\int_{Q^\infty}\frac{\partial v_\theta}{\partial x_1}\,\overline{\hat{\phi}}\,dx=0$ for all corresponding modes $\hat{\phi}\in\hat{X}_j$ by Lemma 5.5. Finally we note that the propagating part $u_{t,prop}$ tends to zero exponentially by (29) and $u_t=\tilde{u}_t^s$ on Q. This ends the proof of Theorem 5.2. $\square$

6. The Case of a Locally Perturbed Periodic Index

Now we consider the more general problem that the periodic refractive index n is perturbed by some function $q\in L^\infty(\mathbb{R}^2)$ with support in Q. The following result on uniqueness and existence has been shown in [7].

Theorem 6.1. *Let Assumptions 2.2 and 2.4 hold and, in the case $q\neq 0$, the additional assumption that no bound states exist; that is, no non-trivial $w\in H^1(\mathbb{R}^2)$ with $\Delta w+k^2(n+q)w=0$ in $\mathbb{R}^2$ exist; that is, k^2 is not in the point spectrum of $-\frac{1}{n+q}\Delta$. Then for all $f\in L^2(Q)$ there exists a unique solution $u\in H^1_{loc}(\mathbb{R}^2)$ of $\Delta u+k^2(n+q)u=-f$ in $\mathbb{R}^2$ which satisfies the open waveguide radiation condition of Definition 5.1. Furthermore, the solution operator $f\mapsto u|_Q$ is bounded from $L^2(Q)$ into $H^1(Q)$.*

It is the aim to prove the following extension of Theorem 5.2.

Theorem 6.2. *Let Assumptions 2.2 and 2.4 hold and let $\hat\theta = \binom{\sin\theta}{-\cos\theta} \in \mathbb{R}^2$ be a fixed unit vector with $|\theta| < \frac{\pi}{2}$; that is, $\hat\theta_2 < 0$. In addition, let $\hat\alpha := k\hat\theta_1 = k\sin\theta$ not be a cut-off value in the sense of Definition 2.1 and assume that there exist no bound states. Let $w_t = \Phi(\cdot, -t\hat\theta) + w_t^s$ be the unique solution of the scattering problem*

$$\Delta w_t + k^2(n+q)\,w_t \;=\; 0 \quad in\ \mathbb{R}^2 \setminus \{-t\hat\theta\}\,, \tag{38}$$

of the point source at $z = -t\hat\theta$ for $t|\hat\theta_2| > 2h_0$ such that $w_t^s \in H^2_{loc}(\mathbb{R}^2)$ and w_t satisfies the open waveguide radiation condition of Definition 5.1. Then

$$\frac{1}{\gamma}\lim_{t\to\infty}\bigl[\sqrt{t}\,e^{-ikt}w_t\bigr] \;=\; v_\theta + w \quad in\ H^1(Q_R) \tag{39}$$

for any $R > 0$ where again $Q_R := (-R,R)\times(-h_0,h_0)$. Here, $v_\theta \in H^1_{\hat\alpha,loc}(\mathbb{R}^2)$ is exactly the limit function as in Theorem 5.2; that is, v_θ solves the $\hat\alpha-$quasi-periodic scattering problem $\Delta v_\theta + k^2 n v_\theta = 0$ in $\mathbb{R}^2$ for such that the scattered field $v_\theta^s(x) := v_\theta(x) - e^{ik\hat\theta\cdot x}$ satisfies the Rayleigh expansion (5) for $|x_2| > h_0$. If $k\hat\theta_1 = k\sin\theta$ is a propagative wave number then v_θ satisfies in addition the orthogonality condition (36).
The function $w \in H^1_{loc}(\mathbb{R}^2)$ solves the source problem $\Delta w + k^2(n+q)w = -k^2 q\,v_\theta$ in $\mathbb{R}^2$, and satisfies the open waveguide radiation condition of Definition 5.1.

Proof: We define $u_t = u_t^s + \Phi(\cdot, -t\hat\theta)$ as in Theorem 5.2 to be the unique solution of the unperturbed scattering problem $\Delta u_t + k^2 n u_t = 0$ in $\mathbb{R}^2 \setminus \{-t\hat\theta\}$ for the point source incidence at $z = -t\hat\theta$ such that $u_t^s \in H^1_{loc}(\mathbb{R}^2)$ and u_t satisfies the open waveguide radiation condition of Definition 5.1. Then Theorem 5.2 implies that $\frac{1}{\gamma}e^{-ikt}\sqrt{t}u_t$ converges in $H^1(Q)$ to the solution v_θ of the $\hat\alpha-$quasi-periodic scattering problem for the plane wave of incidence $\hat\theta$. In the case that $k\hat\theta_1$ is a propagative wave number v_θ satisfies in addition the orthogonality condition (36). Then $\tilde w_t = w_t - u_t$ satisfies $\Delta\tilde w_t + k^2(n+q)\tilde w_t = -k^2 q u_t$ in $\mathbb{R}^2$ and the open waveguide radiation condition of Definition 5.1. The convergence of $\frac{1}{\gamma}e^{-ikt}\sqrt{t}u_t$ to v_θ in $H^1(Q)$ yields convergence of $\frac{1}{\gamma}e^{-ikt}\sqrt{t}\tilde w_t$ to w in $H^1(Q)$ because of the continuous dependence of the solution on the right hand side. This ends the proof. $\square$

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