SPATIAL SEGREGATION LIMIT OF COMPETITION SYSTEMS AND FREE BOUNDARY PROBLEMS[§]

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Abstract. We consider a PDE/ODE system for two pairs of competing species and study the spatial segregation limit as the interspecific competition rate tends to infinity. We show that the limiting problem is a one-phase Stefan problem for nonlinear diffusion equations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. We study the following competition system:

$$
\begin{cases}
u_{1t} = \nabla \cdot A_1(u_1) + f_1(x,t,u_1,u_2) - kp_1u_1v_1, & x \in \Omega,\ t>0,\\
v_{1t} = -kq_1u_1v_1, & x \in \Omega,\ t>0,\\
u_{2t} = \nabla \cdot A_2(u_2) + f_2(x,t,u_1,u_2) - kp_2u_2v_2, & x \in \Omega,\ t>0,\\
v_{2t} = -kq_2u_2v_2, & x \in \Omega,\ t>0,\\
A_i(u_i)\cdot \nu = 0, & x \in \partial\Omega,\ t>0,\ i=1,2,\\
u_i(x,0) = u_{i0}(x),\ v_i(x,0) = v_{i0}(x), & x \in \Omega,\ i=1,2,
\end{cases} \tag{1.1}
$$

where, for $i=1,2$, $A_i(u) := a_i\nabla u + u\nabla b_i$ for some functions a_i and b_i, ν denotes the outward unit normal vector on the boundary $\partial\Omega$, f_i is the growth term of u_i, p_i and q_i are positive constants, and the parameter $k>0$ denotes the interspecific competition rate.

This system involves four species (with densities u_1, v_1, u_2 and v_2). For $i=1,2$, u_i and v_i are formed to be a competition pair (the competition becomes stronger with the increase of k). u_i satisfies a PDE while v_i satisfies an ODE without diffusion. Moreover, u_1 and u_2 are coupled through the reaction terms f_1 and f_2. We will show that as k goes to infinity, the habitats of u_i and v_i segregate each other, and the limiting problem is a system of u_1 and u_2 with Stefan free boundary condition, which, as an interesting problem itself, has been extensively studied in recent years (see details in section 4). Further background and related problems of the system (1.1) can be found in [4, 12, 17, 19] etc..

Our basic assumptions are as follows:

(A1) (Coefficients). For $i=1,2$, a_i, b_i are smooth functions defined in $\overline{\Omega}\times[0,\infty)$ with $\lambda \le a_i \le \Lambda$ for some constants $\Lambda > \lambda > 0$.

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(A2) (Source terms). For $i=1,2$, $f_i \in C^1\big(\overline{\Omega}\times[0,\infty)^3\big)$, $f_1(x,t,0,\xi)=f_2(x,t,\xi,0)\equiv 0$ and $f_i(x,t,1,\xi)+\Delta b_i \le 0$ for $(x,t)\in\overline{\Omega}\times[0,\infty)$ and $\xi\in[0,1]$.

(A3) (Initial conditions). For $i=1,2$, $(u_{i0},v_{i0})\in C(\overline{\Omega})\times L^\infty(\Omega)$. Both of them take values in $[0,1]$ and $u_{i0}v_{i0}=0$ in Ω.

In section 2 we first present the existence and uniqueness for the solution $U:=(u_1,v_1,u_2,v_2)$ to (1.1) (we also write the solution as $U^{(k)}:=(u_1^{(k)},v_1^{(k)},u_2^{(k)},v_2^{(k)})$ in some places to emphasize its dependence on the parameter k), and then give some a priori bounds for the solution. These bounds are used in section 3 to show that the solution sequence $U^{(k)}$ converges, as $k\to\infty$, to $(u_1^*,v_1^*,u_2^*,v_2^*)$. Then we prove that this tetrad is uniquely determined by the unique weak solution of a Stefan problem. Finally, we present an explicit form for the Stefan problem, which is a one-phase free boundary problem for nonlinear diffusion equations. Summarizing the conclusions in sections 2 and 3 we have the following theorem.

Theorem 1.1. *Let T be any positive number. Suppose that $U^{(k)}:=(u_1^{(k)},v_1^{(k)},u_2^{(k)},v_2^{(k)})$ is the unique solution of (1.1) in $Q_T:=\Omega\times(0,T]$. Then there exists $U^*:=(u_1^*,v_1^*,u_2^*,v_2^*)$ with $(u_i^*,v_i^*)\in L^2(0,T;H^1(\Omega))\times L^\infty(Q_T)$ such that, for $i=1,2$, as $k\to\infty$,*

$$u_i^{(k)}\to u_i^* \text{ strongly in } L^2(Q_T),\ \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and a.e. in } Q_T,$$

$$v_i^{(k)}\to v_i^* \text{ weakly in } L^2(Q_T),$$

and $\Omega_{u_i^*}(t)\cap\Omega_{v_i^*}(t)=\emptyset$, where

$$\Omega_{u_i^*}(t):=\{(x,t)\in Q_T \mid u_i^*(x,t)>0\},\quad \Omega_{v_i^*}(t):=\{(x,t)\in Q_T \mid v_i^*(x,t)>0\}.$$

Moreover, for $i=1,2$, if $\Gamma_i:=\bigcup_{0\le t\le T}\Gamma_i(t)$ (with $\Gamma_i(t):=\partial\Omega_{u_i^*}(t)$) is a smooth hypersurface satisfying $\Gamma_i(t)\cap\partial\Omega=\emptyset$ for $0\le t\le T$, u_i^* is smooth in $\overline{\Omega_{u_i^*}(t)}\times[0,T]$ and $v_{i0}(x)\in(0,1]$ for $x\in\overline{\Omega_{v_i}(0)}$, then $(u_1^*(x,t),u_2^*(x,t),\Gamma_1(t),\Gamma_2(t))$ solves the following free boundary problem

$$\text{(1.2)}\qquad \begin{cases} u_{1t}=\nabla\cdot A_1(u_1)+f_1(x,t,u_1,u_2), & (x,t)\in\Omega_{u_1}(t)\times(0,T],\\ u_{2t}=\nabla\cdot A_2(u_2)+f_2(x,t,u_1,u_2), & (x,t)\in\Omega_{u_2}(t)\times(0,T],\\ u_i(x,t)=0, & (x,t)\in\Gamma_i(t)\times(0,T],\ i=1,2,\\ V_{n_i}=-\dfrac{q_ia_i(x,t)}{p_iv_{i0}(x)}\dfrac{\partial u_i}{\partial n_i}, & (x,t)\in\Gamma_i(t)\times(0,T],\ i=1,2,\\ A_i(u_i)\cdot\nu=0, & (x,t)\in\partial\Omega\times(0,T],\ i=1,2,\\ \Gamma_i(0)=\partial\Omega_{u_i}(0),\quad u_i(x,0)=u_{i0}(x), & x\in\Omega,\ i=1,2, \end{cases}$$

where n_i is the unit normal vector on $\Gamma_i(t)$ oriented from $\Omega_{u_i}(t)$ to $\Omega\backslash\Omega_{u_i}(t)$.

Note that the strong competition terms $-kp_iu_iv_i$ in (1.1) leave effect to this limiting problem in a way that the normal velocity V_{n_i} of the free boundary depends on v_{i0}.

Finally, in section 4, we give some remarks. First we present a simple version of the problem (1.2) without u_2 and Γ_2 (cf. Theorem 4.1 below). Then we give a brief review on recent studies for the problems (1.2) and (4.2). In some sense, this paper can also be regarded as a derivation for the widely studied problems (1.2) and (4.2).

Throughout this paper, when we write a formula for u_i or v_i, we mean that it is true for both $i=1,2$.

2. Well-posedness of (1.1) and a priori Bounds

2.1. **Existence and uniqueness of the solution.** The existence of solutions to the PDE/ODE system (1.1) does not follow directly from the standard theory of parabolic equations because of the lack of diffusions for v_1 and v_2.

Lemma 2.1. *Assume (A1)-(A3) hold and* $T > 0$*. Then the problem* (1.1) *has a unique weak solution* (u_1, v_1, u_2, v_2) *with*

$$u_i \in W_2^{2,1}(Q_T), \quad v_i \in C^{0,1}\big([0,T]; L^\infty(\Omega)\big),$$

and $0 \le u_i \le 1$, $0 \le v_i \le 1$.

Proof. The existence of the solution can be shown by the Schauder fixed point theorem as in [12]. We give a sketch here. Set

$$\mathscr{X} := L^2(Q_T) \times L^2(Q_T) \quad \text{and} \quad X := \{(w_1, w_2) \in \mathscr{X} \mid 0 \le w_1, w_2 \le 1\}.$$

1. We define an operator $\mathcal{C}_1 : X \to \mathscr{X}$ as follows. Given $(u_1, u_2) \in X$. Consider the initial value problem

$$\begin{cases} \hat{v}_{it} = -kq_i u_i \hat{v}_i, & \text{in } Q_T, \\ \hat{v}_i(x,0) = v_{i0}(x), & \text{in } \Omega. \end{cases}$$

A direct calculation yields

$$\hat{v}_i(x,t) = v_{i0}(x) e^{-kq_i \int_0^t u_i(x,s)ds} \in C^{0,1}([0,T]; L^\infty(\Omega)) \text{ and } 0 \le \hat{v}_i \le 1. \tag{2.1}$$

Hence $(\hat{v}_1, \hat{v}_2) = \mathcal{C}_1(u_1, u_2) \in X$. Moreover, one can show by (2.1) that $\mathcal{C}_1 : X \to \mathscr{X}$ is a continuous operator.

2. Next we define an operator $\mathcal{C}_2 : X \to \mathscr{X}$ in the following way. Given $(v_1, v_2) \in X$, let $(\hat{u}_1, \hat{u}_2) := \mathcal{C}_2(v_1, v_2)$ be the unique solution of the problem

$$\begin{cases} \hat{u}_{1t} = \nabla \cdot A_1(\hat{u}_1) + f_1(x,t,\hat{u}_1,\hat{u}_2) - kp_1 \hat{u}_1 v_1, & x \in \Omega, \ t > 0, \\ \hat{u}_{2t} = \nabla \cdot A_2(\hat{u}_2) + f_2(x,t,\hat{u}_1,\hat{u}_2) - kp_2 \hat{u}_2 v_2, & x \in \Omega, \ t > 0, \\ A_i(\hat{u}_i) \cdot \nu = 0, & x \in \partial\Omega, \ t > 0, \\ \hat{u}_i(x,0) = u_{i0}(x), & x \in \Omega, \quad i = 1,2. \end{cases} \tag{2.2}$$

Then $\hat{u}_i \in W_2^{2,1}(Q_T)$ and $\|\hat{u}_i\|_{W_2^{2,1}(Q_T)}$ has an upper bound, independent of (v_1, v_2). Moreover, $0 \le \hat{u}_i \le 1$ since $[0,1]^2$ is an invariant domain by the assumption (A2).

To show the continuity of $\mathcal{C}_2$, we assume $(v_{1m}, v_{2m}) \in X$ such that $v_{im} \to \tilde{v}_i$ as $m \to \infty$ in $L^2(Q_T)$. Then $(\hat{u}_{1m}, \hat{u}_{2m}) = \mathcal{C}_2(v_{1m}, v_{2m})$ is bounded in $W_2^{2,1}(Q_T) \times W_2^{2,1}(Q_T)$, so there is a subsequence of $\{m\}$ (still denotes it by $\{m\}$) and functions $\tilde{u}_1, \tilde{u}_2 \in L^2(Q_T)$ such that $(\hat{u}_{1m}, \hat{u}_{2m}) \to (\tilde{u}_1, \tilde{u}_2)$ as $m \to \infty$ in $L^2(Q_T) \times L^2(Q_T)$. For any test function $\eta \in C^\infty(\overline{Q_T})$ with $\frac{\partial \eta}{\partial \nu} = 0$, multiplying the equation of $\hat{u}_{im}$ by η and integrating it over Q_T by parts we obtain

$$\begin{aligned} \int_\Omega (\hat{u}_{im}\eta)\Big|_{t=0}^{t=T} dx &= \iint_{Q_T} \Big[\hat{u}_{im}\eta_t + f_i(x,t,\hat{u}_{1m},\hat{u}_{2m})\eta - kp_i \hat{u}_{im} v_{im} \eta\Big] dxdt \\ &\quad + \iint_{Q_T} \Big[\hat{u}_{im} \nabla a_i \cdot \nabla \eta + a_i \hat{u}_{im} \Delta \eta - \hat{u}_{im} \nabla b_i \cdot \nabla \eta\Big] dxdt. \end{aligned}$$

Taking limit as $m \to \infty$ we have

$$\begin{aligned} \int_\Omega (\tilde{u}_i\eta)\Big|_{t=0}^{t=T} dx &= \iint_{Q_T} \Big[\tilde{u}_i\eta_t + f_i(x,t,\tilde{u}_1,\tilde{u}_2)\eta - kp_i \tilde{u}_i \tilde{v}_i \eta\Big] dxdt \\ &\quad + \iint_{Q_T} \Big[\tilde{u}_i \nabla a_i \cdot \nabla \eta + a_i \tilde{u}_i \Delta \eta - \tilde{u}_i \nabla b_i \cdot \nabla \eta\Big] dxdt. \end{aligned}$$

Therefore $(\tilde{u}_1, \tilde{u}_2)$ is a weak solution of (2.2) with $v_i = \tilde{v}_i$, so we have $(\tilde{u}_1, \tilde{u}_2) = \mathcal{C}_2(\tilde{v}_1, \tilde{v}_2)$. This proves the continuity of $\mathcal{C}_2$.

3. From above we see that $\mathcal{C} := \mathcal{C}_2 \circ \mathcal{C}_1$ maps X into X continuously. Moreover, $\mathcal{C}$ maps bounded sets of X into compact sets of X as $\mathcal{C}_2$ does. So, the Schauder fixed point theorem can be applied to give the fixed point of $\mathcal{C}$, which is the solution of (1.1).

4. The uniqueness of the weak solution is a consequence of the following lemma. □

Lemma 2.2. *Let (u_1, v_1, u_2, v_2) and $(\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2)$ be the solution of (1.1), as in the previous lemma, with initial data $(u_{10}, v_{10}, u_{20}, v_{20})$ and $(\tilde{u}_{10}, \tilde{v}_{10}, \tilde{u}_{20}, \tilde{v}_{20})$, respectively. Then for any $t \geq 0$ we have*

$$\int_\Omega \Theta(x,t)dx \leq e^{Mt} \int_\Omega \Theta(x,0)dx,$$

where $M > 0$ is a constant depending only on f_1, f_2, q_1, q_2 with $\varphi_i := u_i - \tilde{u}_i$, $\psi_i := v_i - \tilde{v}_i$,

$$\Theta(x,t) := q_1|\varphi_1(x,t)| + p_1|\psi_1(x,t)| + q_2|\varphi_2(x,t)| + p_2|\psi_2(x,t)|.$$

Proof. For any given $t > 0$, by our assumption (A2), there exist $M_i > 0$ such that

$$|f_i(x,t,s_1,s_2) - f_i(x,t,\tilde{s}_1,\tilde{s}_2)| \leq M_i(|s_1 - \tilde{s}_1| + |s_2 - \tilde{s}_2|) \text{ for } (x,t) \in Q_t := \Omega \times (0,t],\ s_i, \tilde{s}_i \in [0,1].$$

Subtracting the equation for u_i and $\tilde{u}_i$, multiplying the result by $\operatorname{sign}(\varphi_i)$ and then integrating it over Q_t we deduce

$$\int_\Omega |\varphi_i(x,t)|dx \leq \int_\Omega |\varphi_i(x,0)|dx - kp_i \iint_{Q_t} |\varphi_i| v_i + kp_i \iint_{Q_t} |\psi_i|\tilde{u}_i + M_i \iint_{Q_t} \big(|\varphi_1| + |\varphi_2|\big). \tag{2.3}$$

Here we used the fact

$$\iint_{Q_t} \varphi_{it} \cdot \operatorname{sign}(\varphi_i) = \int_\Omega |\varphi_i(x,t)|dx - \int_\Omega |\varphi_i(x,0)|dx \quad \text{and} \quad \iint_{Q_t} \nabla \cdot A_i(\varphi_i) \cdot \operatorname{sign}(\varphi_i) = 0$$

as in the proof of [4, Theorem 2.2]. Subtracting the equation for v_i and $\tilde{v}_i$, multiplying the result by $\operatorname{sign}(\psi_i)$ and then integrating it over Q_t we obtain

$$\int_\Omega |\psi_i(x,t)|dx \leq \int_\Omega |\psi_i(x,0)|dx + kq_i \iint_{Q_t} |\varphi_i| v_i - kq_i \iint_{Q_t} |\psi_i| \tilde{u}_i. \tag{2.4}$$

Multiplying (2.3) and (2.4) by q_i and p_i, then taking the sum of them we have

$$\begin{aligned} \int_\Omega q_i|\varphi_i(x,t)|dx &\leq \int_\Omega \big[q_i|\varphi_i(x,t)| + p_i|\psi_i(x,t)|\big]dx \\ &\leq \int_\Omega \big[q_i|\varphi_i(x,0)| + p_i|\psi_i(x,0)|\big]dx + q_i M_i \iint_{Q_t} \big(|\varphi_1| + |\varphi_2|\big). \end{aligned}$$

Set $E(t) := \iint_{Q_t} \big(q_1|\varphi_1| + q_2|\varphi_2|\big)$, then the above inequality implies that

$$E'(t) \leq \int_\Omega \Theta(x,0)dx + (q_1 M_1 + q_2 M_2) \iint_{Q_t} \big(|\varphi_1| + |\varphi_2|\big) \leq \int_\Omega \Theta(x,0)dx + ME(t),$$

where $M := (q_1 M_1 + q_2 M_2)/\min\{q_1, q_2\}$. The conclusion then follows from the Gronwall's inequality. □

2.2. **A priori bounds.** Now we present some a priori bounds for the solution of (1.1). These bounds will play important roles later.

Lemma 2.3. *Let (u_1, v_1, u_2, v_2) be the solution of (1.1), then there exists $C > 0$, independent of k such that*

$$k \iint_{Q_T} u_i v_i dxdt \le C, \quad \iint_{Q_T} |\nabla u_i|^2 dxdt \le C. \tag{2.5}$$

Proof. To prove the first bound, one only needs to integrate the equation for u_i over Q_T.

We now prove the second inequality. Multiplying the equation for u_i by u_i and integrating it over Q_T we have

$$\frac{1}{2}\iint_{Q_T} (u_i^2)_t dxdt = \iint_{Q_T} \left[\nabla \cdot (A_i(u_i)u_i) - A_i(u_i)\cdot \nabla u_i\right] + \iint_{Q_T} [f_i(x,t,u_1,u_2) - kp_i u_i v_i]u_i.$$

Using the boundary condition in (1.1) we have

$$\frac{1}{2}\int_\Omega u_i^2(x,t)\Big|_{t=0}^{t=T} dx = -\iint_{Q_T} A_i(u_i)\cdot \nabla u_i + \iint_{Q_T} [f_i(x,t,u_1,u_2) - kp_i u_i v_i]u_i.$$

By the assumption (A1) we have

$$\begin{aligned} \lambda \iint_{Q_T} |\nabla u_i|^2 &\le \iint_{Q_T} a_i |\nabla u_i|^2 + kp_i \iint_{Q_T} u_i^2 v_i \\ &= -\iint_{Q_T} u_i \nabla b_i \cdot \nabla u_i + \iint_{Q_T} f_i(x,t,u_1,u_2)u_i + \frac{1}{2}\int_\Omega u_i^2(x,t)\Big|_{t=0}^{t=T} dx \\ &\le \frac{\lambda}{2}\iint_{Q_T} |\nabla u_i|^2 + \frac{1}{2\lambda}\iint_{Q_T} u_i^2 |\nabla b_i|^2 + F_i|\Omega|T + \frac{1}{2}|\Omega|, \end{aligned}$$

where $|\Omega|$ denotes the measure of Ω and $F_i := \max\{f_i(x,t,s_1,s_2) \mid x \in \overline{\Omega}, t \in [0,T], s_1, s_2 \in [0,1]\}$. Therefore,

$$\iint_{Q_T} |\nabla u_i|^2 \le C := \frac{1}{\lambda^2}\iint_{Q_T} |\nabla b_i|^2 + \frac{2}{\lambda}F_i|\Omega|T + \frac{1}{\lambda}|\Omega|.$$

This completes the proof. □

We will use the Riesz-Fréchet-Kolmogoroff theorem to give the convergence for $U^{(k)}$ as $k \to \infty$. For this purpose we need the estimates for the difference between u_i, v_i and their shifts.

Let $\hat{r} > 0$ be a small constant. For any $r \in (0, \hat{r})$, denote $\Omega_r := \{x \in \Omega \mid B(x,r) \subset \Omega\}$, where $B(x,r)$ is an open ball with radius r centered at x.

Lemma 2.4. *There exists $C > 0$ such that, for any $\xi \in \overline{B(0,r)}$,*

$$\int_0^T \int_{\Omega_r} |u_i(x+\xi,t) - u_i(x,t)|^2 \le C|\xi|^2, \tag{2.6}$$

and for any $\tau \in (0,T)$,

$$\int_0^{T-\tau}\int_\Omega |u_i(x,t+\tau) - u_i(x,t)|^2 \le C\tau. \tag{2.7}$$

Proof. The inequality (2.6) follows from (2.5) immediately. Indeed, by (2.5) we have

$$\int_0^T \int_{\Omega_r} [u_i(x+\xi,t) - u_i(x,t)]^2 = \int_0^T \int_{\Omega_r} \left[\int_0^1 \nabla u_i(x+s\xi,t)\cdot \xi ds\right]^2 \le C|\xi|^2$$

for some constant $C > 0$ independent of ξ and r. Next we prove (2.7).

$$\begin{aligned}
&\int_0^{T-\tau}\int_\Omega [u_i(x,t+\tau)-u_i(x,t)]^2 \\
= &\int_0^{T-\tau}\int_\Omega [u_i(x,t+\tau)-u_i(x,t)]\int_0^\tau u_{it}(x,t+s)ds \\
= &\int_0^{T-\tau}\int_\Omega [u_i(x,t+\tau)-u_i(x,t)]\int_0^\tau [\nabla\cdot \widetilde{A}_i + \tilde{f}_i - kp_i u_i(x,\kappa)v_i(x,\kappa)]ds,
\end{aligned}$$

where $\kappa = t+s$, $\tilde{f}_i := f_i(x,\kappa,u_1(x,\kappa),u_2(x,\kappa))$ and $\widetilde{A}_i := a_i(x,\kappa)\nabla u_i(x,\kappa)+u_i(x,\kappa)\nabla b_i(x,\kappa)$.

For $\hat{t}=0$ or $\hat{t}=\tau$, note that

$$\begin{aligned}
&\int_0^\tau\int_\Omega\int_0^{T-\tau} a_i(x,\kappa)|\nabla u_i(x,t+s)|\cdot|\nabla u_i(x,t+\hat{t})|dtdxds \\
\le\ &\frac{\Lambda}{2}\int_0^\tau\int_\Omega\left(\int_0^{T-\tau}\big[|\nabla u_i(x,t+s)|^2+|\nabla u_i(x,t+\hat{t})|^2\big]dt\right)dxds \\
\le\ &\Lambda\int_0^\tau\int_\Omega\int_0^T|\nabla u_i(x,t)|^2dtdxds \le \Lambda\tau\iint_{Q_T}|\nabla u_i(x,t)|^2 \le C\tau,
\end{aligned}$$

and by $u_i(x,t)\in[0,1]$, we have

$$\int_0^\tau\int_\Omega\int_0^{T-\tau} u_i(x,\kappa)|\nabla b_i|\cdot|\nabla u_i(x,t+\hat{t})|dtdxds \le \tau\|\nabla b_i\|_{L^2(Q_T)}\|\nabla u_i\|_{L^2(Q_T)} \le C\tau.$$

So, for some constant C independent of τ, we firstly have

$$\begin{aligned}
&\int_0^\tau\int_\Omega\int_0^{T-\tau}[u_i(x,t+\tau)-u_i(x,t)]\nabla\cdot\widetilde{A}_i\ dtdxds \\
= &-\int_0^\tau\int_\Omega\int_0^{T-\tau}\nabla[u_i(x,t+\tau)-u_i(x,t)]\cdot\big[a_i(x,\kappa)\nabla u_i(x,\kappa)+u_i(x,\kappa)\nabla b_i(x,\kappa)\big]dtdxds \\
\le\ &C\tau.
\end{aligned}$$

Secondly, we have

$$\left|\int_0^{T-\tau}\int_\Omega[u_i(x,t+\tau)-u_i(x,t)]\int_0^\tau \tilde{f}_i\ ds\right| \le F_i|\Omega|T\tau,$$

where F_i is the same constant as in the proof of the previous lemma. Thirdly, by $u_i, v_i\in[0,1]$ we have

$$\left|-kp_i\int_0^\tau\int_\Omega\int_0^{T-\tau}[u_i(x,t+\tau)-u_i(x,t)]u_i(x,\kappa)v_i(x,\kappa)\ dtdxds\right| \le kp_i|\Omega|T\tau.$$

Combining these inequalities together we obtain (2.7). □

Lemma 2.5. *For $i=1,2$, there exists a positive function $G_i(\xi)$ such that $G_i(\xi)\to 0$ as $\xi\to 0$, and a positive constant $C_1>0$ such that, for any $\xi\in B(0,r)$,*

$$\int_0^T\int_{\Omega_r}|v_i(x+\xi,t)-v_i(x,t)| \le G_i(\xi), \tag{2.8}$$

and for any $\tau\in(0,T)$,

$$\int_0^{T-\tau}\int_\Omega|v_i(x,t+\tau)-v_i(x,t)| \le C_1\tau. \tag{2.9}$$

Proof. From the equation of v_i we easily obtain that

$$v_i(x,t) = v_{i0}(x)e^{-kq_i\int_0^t u_i(x,s)ds}, \quad t>0.$$

The estimates follow easily from this formula (cf. [12, 20]). □

3. Spatial segregation limit and free boundary problems

In this section we fix a positive number T and consider the limit as $k\to\infty$ of the solution sequence $(u_1^{(k)}, v_1^{(k)}, u_2^{(k)}, v_2^{(k)})$ to (1.1).

3.1. **Convergence of the solution sequence.** From the previous Lemmas 2.1 and 2.3 we see that the family $\{u_i^{(k)} \mid k>0\}$ is bounded in $L^2(0,T;H^1(\Omega))$ and the family $\{v_i^{(k)} \mid k>0\}$ is bounded in $L^\infty(Q_T)$. Therefore, it follows from Lemma 2.4 and the Riesz-Fréchet-Kolmogoroff theorem ([2], Theorem IV.25 and Corollary IV.26) that $\{u_i^{(k)} \mid k>0\}$ and $\{v_i^{(k)} \mid k>0\}$ are precompact in $L^2(Q_T)$. Therefore there exist

$$u_i^* \in L^2(0,T;H^1(\Omega))\cap L^\infty(Q_T), \quad v_i^*\in L^\infty(Q_T)$$

and a subsequence $\{k_j\}$ of $\{k\}$ with $k_j\to\infty$ such that, as $j\to\infty$, we have

$$u_i^{(k_j)} \to u_i^*,\ v_i^{(k_j)} \to v_i^* \text{ strongly in } L^2(Q_T),\ \text{a.e. in } Q_T, \tag{3.1}$$

and

$$u_i^{(k_j)} \rightharpoonup u_i^* \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{3.2}$$

Lemma 3.1. *Assume u_i^* and v_i^* are the limits obtained in* (3.1). *Then*

$$u_i^*,\ v_i^* \in [0,1], \quad u_i^* v_i^* = 0 \textit{ for } (x,t)\in Q_T. \tag{3.3}$$

Proof. The conclusion $u_i^*(x,t), v_i^*(x,t)\in[0,1]$ for $(x,t)\in Q_T$ follows from the limits in (3.1) and the conclusion $u_i^{(k)}, v_i^{(k)}\in[0,1]$ in Lemma 2.1. By the first inequality of (2.5) we have

$$\iint_{Q_T} u_i^* v_i^* = \lim_{j\to\infty}\iint_{Q_T} u_i^{(k_j)} v_i^{(k_j)} \le \lim_{j\to\infty}\frac{C}{k_j} = 0.$$

This proves $u_i^* v_i^* = 0$ for $(x,t)\in Q_T$. □

In what follows we show that the limit $(u_1^*, v_1^*, u_2^*, v_2^*)$ is uniquely determined by the weak solution of the following problem

$$\begin{cases} Z_{1t} = \nabla\cdot\mathscr{D}_1(Z_1) + h_1\Big(x,t,\frac{Z_1}{q_1},\frac{Z_2}{q_2}\Big), & x\in\Omega,\ 0<t\le T,\\ Z_{2t} = \nabla\cdot\mathscr{D}_2(Z_2) + h_2\Big(x,t,\frac{Z_1}{q_1},\frac{Z_2}{q_2}\Big), & x\in\Omega,\ 0<t\le T,\\ \mathscr{D}_1(Z_1)\cdot\nu = 0,\quad \mathscr{D}_2(Z_2)\cdot\nu=0, & x\in\partial\Omega,\ 0<t\le T,\\ Z_1(x,0)=Z_{10}(x),\quad Z_2(x,0)=Z_{20}(x), & x\in\Omega, \end{cases} \tag{3.4}$$

where, for any function $\zeta,\rho\in L^2(0,T;H^1(\Omega))$,

$$\mathscr{D}_i(\zeta) := \begin{cases} A_i(\zeta), & \zeta(x,t)>0,\\ 0, & \zeta(x,t)\le 0,\end{cases} \qquad h_1(x,t,\zeta,\rho) := \begin{cases} q_1 f_1(x,t,\zeta,\rho), & \zeta(x,t)>0,\ \rho(x,t)>0,\\ q_1 f_1(x,t,\zeta,0), & \zeta(x,t)>0,\ \rho(x,t)\le 0,\\ 0, & \zeta(x,t)\le 0,\end{cases}$$

and

$$h_2(x,t,\zeta,\rho) := \begin{cases} q_2 f_2(x,t,\zeta,\rho), & \zeta(x,t)>0,\ \rho(x,t)>0,\\ q_2 f_2(x,t,0,\rho), & \zeta(x,t)\le 0,\ \rho(x,t)>0,\\ 0, & \rho(x,t)\le 0.\end{cases}$$

In addition, for any function $w(x,t)$, we denote

$$(w)_+(x,t) := \max\{w(x,t),0\},\ (w)_-(x,t) := -\min\{w(x,t),0\}.$$

Definition 3.2. *A pair (Z_1,Z_2) with $Z_i \in L^\infty(Q_T)$, $(Z_i)_+ \in L^2(0,T;H^1(\Omega))$ is called a weak solution of (3.4) if*

$$\iint_{Q_T} Z_i\eta_t + \int_\Omega Z_{i0}(x)\eta(x,0)dx = \iint_{Q_T}\Big[\mathscr{D}_i(Z_i)\cdot\nabla\eta - h_i\Big(x,t,\frac{Z_1}{q_1},\frac{Z_2}{q_2}\Big)\eta\Big] \tag{3.5}$$

for all test functions $\eta \in C^\infty(\overline{Q_T})$ with $\eta(x,T)=0$.

Our main result in this subsection is the following lemma.

Lemma 3.3. *Let $Z_i^* := q_iu_i^* - p_iv_i^*$ with u_i^* and v_i^* being obtained in (3.1). Then the pair (Z_1^*,Z_2^*) is the unique weak solution of the problem (3.4) with initial data $Z_{i0} = q_iu_{i0} - p_iv_{i0}$.*

Proof. Clearly, $Z_i^* \in L^\infty(Q_T)$ and $(Z_i)_+ = q_iu_i^* \in L^2(0,T;H^1(\Omega))$. Using the equations in (1.1) we have

$$(q_iu_i^{(k_j)} - p_iv_i^{(k_j)})_t = q_i\nabla\cdot A_i(u_i^{(k_j)}) + q_if_i(x,t,u_1^{(k_j)},u_2^{(k_j)}),\quad x\in\Omega, t\in[0,T].$$

Multiplying these equations by the test function η, integrating by parts and using the limits in (3.1) and (3.2) we conclude that

$$\begin{aligned}\iint_{Q_T} Z_i^*\eta_t + \int_\Omega Z_{i0}(x)\eta(x,0)dx &= \iint_{Q_T}\Big[A_i(q_iu_i^*)\cdot\nabla\eta - q_if_i(x,t,u_1^*,u_2^*)\eta\Big]\\ &= \iint_{Q_T}\Big[\mathscr{D}_i(Z_i^*)\cdot\nabla\eta - h_i\Big(x,t,\frac{Z_1^*}{q_1},\frac{Z_2^*}{q_2}\Big)\eta\Big].\end{aligned}$$

Hence (Z_1^*,Z_2^*) is a weak solution to (3.4). The uniqueness of this solution can be proved in a similar way as in [3, 12, 18, 20] etc.. We omit the details here. □

We remark that $(u_1^*,v_1^*,u_2^*,v_2^*)$ is uniquely determined by the unique weak solution of the problem (3.4) with $Z_{i0} = q_iu_{i0} - p_iv_{i0}$ $(i=1,2)$. If fact, once we obtain a solution (Z_1,Z_2) to (3.4), by Lemma 3.3 we have $Z_i \equiv Z_i^*$, and so $u_i^* = (Z_i)_+/q_i$ and $v_i^* = (Z_i)_-/p_i$ are determined by (Z_1,Z_2). The same reason also leads to the following supplementary result.

Corollary 3.4. *The limits in (3.1) and (3.2) hold for all subsequences, namely, as $k\to\infty$,*

$$\begin{aligned}&u_i^{(k)} \to u_i^*,\ v_i^{(k)} \to v_i^* \text{ strongly in } L^2(Q_T),\ \text{a.e. in } Q_T,\\ &u_i^{(k)} \to u_i^* \text{ weakly in } L^2(0,T;H^1(\Omega)).\end{aligned}$$

3.2. **Explicit Stefan free boundary problem.** The problem (3.4) is actually a Stefan type, but it is not given in an explicit form. In this subsection we show that under suitable regularity assumptions the system (3.4) can be explicitly written as a free boundary problem.

Assume (Z_1,Z_2) is the unique solution of (3.4) with some initial data (Z_{10},Z_{20}). Set

$$u_i := (Z_i)_+/q_i,\quad v_i := (Z_i)_-/p_i \text{ in } Q_T,\ i=1,2.$$

For each $t\in[0,T]$, denote

$$\begin{cases}\Omega_{u_i}(t) := \{x\in\Omega | Z_i(x,t) > 0\},\\ \Omega_{v_i}(t) := \{x\in\Omega | Z_i(x,t) < 0\},\\ \Gamma_i(t) := \Omega\setminus(\Omega_{u_i}(t)\cup\Omega_{v_i}(t)).\end{cases} \tag{3.6}$$

Theorem 3.5. *Assume* (Z_1, Z_2) *is the unique weak solution of* (3.4) *with initial data* (Z_{10}, Z_{20}). *Let* $\Omega_{u_i}(t)$, $\Omega_{v_i}(t)$ *and* $\Gamma_i(t)$ *be defined as above. Suppose that, for each* $i = 1, 2$, $\Gamma_i(t)$ *satisfies* $\Gamma_i(t) \cap \partial\Omega = \emptyset$ *for any* $t \in [0, T]$ *and it is a smooth, closed, orientable hypersurface. Let* n_i *be the unit normal vector on* $\Gamma_i(t)$ *from* $\Omega_{u_i}(t)$ *to* $\Omega_{v_i}(t)$ *and assume* $\Gamma_i(t)$ *moves smoothly with speed* V_{n_i}. *Suppose further that* u_i *is smooth in* $\overline{\Omega_{u_i}(t) \times (0, T]}$, *and* $v_{i0}(x) \in (0, 1]$ *for* $x \in \overline{\Omega_{v_i}(0)}$. *Then* $(u_1, u_2, \Gamma_1, \Gamma_2)$ *satisfies* (1.2) *and* $Z_i(x, 0) = Z_{i0}(x)$ *a.e. in* Ω.

Proof. By the above definitions we have

$$u_i = 0 \text{ on } \Gamma_i(t) \times [0, T], \quad Z_i = q_i u_i \text{ in } \Omega_{u_i}(t) \times [0, T], \quad Z_i = -p_i v_i \text{ in } \Omega_{v_i}(t) \times [0, T].$$

Note that v_i does not necessarily to be zero on $\Gamma_i(t) \times [0, T]$ since it is not assumed that v_i is continuous in $\overline{\Omega_{v_i}(t) \times (0, T]}$.

For any test function $\eta \in C^{2,1}(\overline{Q_T})$ with $\eta(\cdot, T) = 0$ in Ω we have

$$\frac{d}{dt} \int_{\Omega_{u_i}(t)} u_i \eta dx = \int_{\Omega_{u_i}(t)} (u_i \eta)_t dx + \int_{\Gamma_i(t)} V_{n_i} u_i \eta d\sigma.$$

Hence

$$\int_0^T \int_{\Omega_{u_i}(t)} (u_i \eta)_t dx dt + \int_0^T \int_{\Gamma_i(t)} V_{n_i} \eta u_i d\sigma dt = -\int_{\Omega_{u_i}(0)} u_i(x, 0) \eta(x, 0) dx.$$

Using the assumption $u_i|_{\Gamma_i(t)} = 0$ we have

$$\int_0^T \int_{\Omega_{u_i}(t)} (u_i \eta)_t dx dt = -\int_{\Omega_{u_i}(0)} u_i(x, 0) \eta(x, 0) dx.$$

In a similar way we have

$$\int_0^T \int_{\Omega_{v_i}(t)} (v_i \eta)_t dx dt = \int_0^T \int_{\Gamma_i(t)} v_i V_{n_i} \eta d\sigma dt - \int_{\Omega_{v_i}(0)} v_i(x, 0) \eta(x, 0) dx.$$

Then the first term of (3.5) on the left side can be rewritten as

$$\begin{aligned} \iint_{Q_T} Z_i \eta_t dx dt &= \int_0^T \int_{\Omega_{u_i}(t)} q_i u_i \eta_t dx dt - \int_0^T \int_{\Omega_{v_i}(t)} p_i v_i \eta_t dx dt \\ &= -\int_0^T \int_{\Omega_{u_i}(t)} q_i u_{it} \eta dx dt + \int_0^T \int_{\Omega_{v_i}(t)} p_i v_{it} \eta dx dt \\ &\quad - \int_0^T \int_{\Gamma_i(t)} p_i v_i V_{n_i} \eta d\sigma dt - \int_\Omega Z_i(x, 0) \eta(x, 0) dx. \end{aligned} \tag{3.7}$$

Now we consider the right side of (3.5). For any test function η as above, since

$$\int_{\Omega_{u_i}(t)} \nabla \eta \cdot A_i(u_i) dx = \int_{\Gamma_i(t)} \eta A_i(u_i) \cdot n_i d\sigma + \int_{\partial\Omega} \eta A_i(u_i) \cdot \nu d\sigma - \int_{\Omega_{u_i}(t)} \eta \nabla \cdot A_i(u_i) dx,$$

we obtain

$$\begin{aligned} &\iint_{Q_T} \Big[\mathscr{D}_i(Z_i) \cdot \nabla \eta - h_i\Big(x, t, \frac{Z_1}{q_1}, \frac{Z_2}{q_2}\Big) \eta\Big] \\ = \;& q_i \int_0^T \int_{\Gamma_i(t)} \eta A_i(u_i) \cdot n_i d\sigma dt + q_i \int_0^T \int_{\partial\Omega} \eta A_i(u_i) \cdot \nu d\sigma dt \\ & - q_i \int_0^T \int_{\Omega_{u_i}(t)} \eta \big[\nabla \cdot A_i(u_i) + f_i(x, t, u_1, u_2)\big]. \end{aligned} \tag{3.8}$$

Substituting the above equalities into (3.5) we have

$$\begin{aligned}&\int_0^T\int_{\Omega_{u_i}(t)}[u_{it}-\nabla\cdot A_i(u_i)-f_i]\eta-\int_0^T\int_{\Omega_{v_i}(t)}\frac{p_i}{q_i}v_{it}\eta\\&+\int_0^T\int_{\Gamma_i(t)}\Big(a_i\frac{\partial u_i}{\partial n_i}+\frac{p_i}{q_i}v_iV_{n_i}\Big)\eta d\sigma dt+\int_0^T\int_{\partial\Omega}\eta A_i(u_i)\cdot\nu d\sigma dt\\&=\frac{1}{q_i}\int_\Omega\big[Z_{i0}(x)-Z_i(x,0)\big]\eta(x,0)dx.\end{aligned}$$

By choosing test function η with compact support in $\Omega_{v_i}(t)\times(0,T]$ we derive $v_{it}=0$ and so $v_i(x,t)\equiv v_{i0}(x)$ for $(x,t)\in\Omega_{v_i}(t)\times[0,T]$. Then we take test function η with compact support in $\Omega_{u_i}(t)\times(0,T]$ to deduce

$$u_{it}=\nabla\cdot A_i(u_i)+f_i(x,t,u_1,u_2),\quad (x,t)\in\Omega_{u_i}(t)\times(0,T].$$

Therefore,

$$\begin{aligned}&\int_\Omega\frac{1}{q_i}\big[Z_{i0}(x)-Z_i(x,0)\big]\eta(x,0)dx\\=&\int_0^T\int_{\Gamma_i(t)}\Big(a_i\frac{\partial u_i}{\partial n_i}+\frac{p_i}{q_i}v_{i0}V_{n_i}\Big)\eta d\sigma dt+\int_0^T\int_{\partial\Omega}\eta A_i(u_i)\cdot\nu d\sigma dt.\end{aligned}\tag{3.9}$$

By our assumption, $\Gamma_i(t)=\partial\Omega_{u_i}(t)$ for any $t\in[0,T]$. We take test functions which vanish on $\big(\partial\Omega\times[0,T]\big)\cup\big(\Omega\times\{0\}\big)$ but do not vanish on $\Gamma_i(t)$ to deduce

$$V_{n_i}=-\frac{q_ia_i(x,t)}{p_iv_{i0}(x)}\frac{\partial u_i}{\partial n_i},\quad x\in\Gamma_i(t),\ t\in(0,T].$$

Then if we take the test functions vanishing on $\Omega\times\{0\}$ we see that

$$A_i(u_i)\cdot\nu=0,\quad (x,t)\in\partial\Omega\times(0,T].$$

Finally we consider test functions with $\eta(\cdot,0)\not\equiv 0$, then we obtain $Z_i(x,0)=Z_{i0}(x)$ *a.e.* in Ω. □

Proof of Theorem 1.1. The conclusions follow from Lemmas 3.1, 3.3, Corollary 3.4 and Theorem 3.5. □

4. Some remarks

4.1. **The case with one pair of competitors.** If there is only one pair of competitors involved in (1.1), then the problem reduces to

$$\begin{cases}u_t=\nabla\cdot A(u)+f(x,t,u)-kpuv, & x\in\Omega,\ t>0,\\ v_t=-kquv, & x\in\Omega,\ t>0,\\ A(u)\cdot\nu=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\Omega,\end{cases}\tag{4.1}$$

where $A(u):=a\nabla u+u\nabla b$, $a,b,$ f and $(u_0,v_0)\in C(\overline{\Omega})\times L^\infty(\Omega)$ satisfy the analogue of the assumptions $(A1)-(A3)$, respectively. A similar approach as above shows the following theorem.

Theorem 4.1. *Let T be any positive number. Assume $(u^{(k)},v^{(k)})$ is the unique solution of* (4.1) *in Q_T. Then there exists $(u^*,v^*)\in L^2(0,T;H^1(\Omega))\times L^\infty(Q_T)$ such that, as $k\to\infty$,*

$$u^{(k)}\to u^*\ \textit{strongly in}\ L^2(Q_T),\ \textit{weakly in}\ L^2(0,T;H^1(\Omega))\ \textit{and a.e. in}\ Q_T,$$

$$v^{(k)}\to v^*\ \textit{weakly in}\ L^2(Q_T),$$

and $\Omega_{u^*}(t) \cap \Omega_{v^*}(t) = \emptyset$, *where*

$$\Omega_{u^*}(t) := \{(x,t) \in Q_T \mid u^*(x,t) > 0\}, \quad \Omega_{v^*}(t) := \{(x,t) \in Q_T \mid v^*(x,t) > 0\}.$$

Moreover, if $\Gamma := \bigcup_{0\le t\le T} \Gamma(t)$ *(with* $\Gamma(t) := \partial\Omega_{u^*}(t)$*) is a smooth hypersurface satisfying* $\Gamma(t) \cap \partial\Omega = \emptyset$ *for* $0 \le t \le T$, u^* *is smooth in* $\overline{\Omega_{u^*}(t)} \times [0,T]$ *and* $v_0(x) \in (0,1]$ *for* $x \in \overline{\Omega_{v^*}(0)}$, *then* $(u^*(x,t), \Gamma(t))$ *solves the following free boundary problem*

$$\text{(4.2)} \qquad \begin{cases} u_t = \nabla \cdot A(u) + f(x,t,u), & (x,t) \in \Omega_u(t) \times (0,T], \\ u(x,t) = 0, & (x,t) \in \Gamma(t) \times (0,T], \\ V_n = -\dfrac{qa(x,t)}{pv_0(x)} \dfrac{\partial u}{\partial n}, & (x,t) \in \Gamma(t) \times (0,T], \\ A(u) \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T], \\ \Gamma(0) = \partial\Omega_u(0), \quad u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where n *is the unit normal vector on* $\Gamma(t)$ *oriented from* $\Omega_u(t)$ *to* $\Omega \backslash \Omega_u(t)$.

4.2. **Review on recent studies for** (4.2) **and** (1.2)**.** From our theorems we see that the free boundary problems (4.2) and (1.2) can be regarded as the approximation (when the competition rate is very large) of the systems (4.1) and (1.1), respectively. The free boundary problems, on the other hand, have attracted wide attention in the last few years.

In 2010, Du and Lin [7] studied a special case of the problem (4.2) in one dimension:

$$\text{(4.3)} \qquad \begin{cases} u_t = u_{xx} + f(u), & x \in (0,h(t)),\ t > 0, \\ u(x,t) = 0, & x = h(t),\ t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x = 0, t > 0, \\ u(x,0) = u_0(x), & x \in [0,h(0)], \end{cases}$$

with logistic nonlinearity: $f(u) = u(1-u)$. Among others, they presented a spreading-vanishing dichotomy result for the asymptotic behavior of the solutions. Later, many authors considered various extended versions of this problem. For example, [9] studied this problem for general nonlinearity, including monostable, bistable and combustiuon types of nonlinearities; [21, 22] studied this problem with Dirichlet and Robin boundary conditions at $x = 0$; [5, 23, 24] studied this problem with temporal or spatial nonlinearities; [13, 14] studied the equation with advection term: $u_t = u_{xx} - \beta u_x + f(u)$. In addition, [6, 10] studied the high dimension version of (4.3), that is, the problem (4.2) with $a = const.$, $b = 0$, $v_0 = const.$ and $\Omega = \mathbb{R}^N$. As far as we know, many authors are still working on the problem (4.2) now.

In [15, 16], Guo and Wu studied a special case of the problem (1.2) in one dimension:

$$\text{(4.4)} \qquad \begin{cases} u_{1t} = d_1 u_{1xx} + r_1 u_1(1 - u_1 - pu_2), \quad x \in (0, s_1(t)),\ t > 0, \\ u_{2t} = d_2 u_{2xx} + r_2 u_2(1 - u_2 - qu_1), \quad x \in (0, s_2(t)),\ t > 0, \\ u_1(x,t) \equiv 0 \text{ for } x \ge s_1(t),\ t > 0, \quad u_2(x,t) \equiv 0 \text{ for } x \ge s_2(t),\ t > 0, \\ s_1'(t) = -\mu_1 u_{1x}(s_1(t),t), \quad s_2'(t) = -\mu_2 u_{2x}(s_2(t),t), \quad t > 0. \end{cases}$$

With suitable initial data they also studied the asymptotic behavior for the solutions and obtained spreading-vanishing dichotomy result. In addition, some other authors also studied the special version of (1.2) in one dimension. For example, [11, 8] studied the case where $\Omega_{u_2}(t) = \mathbb{R}^1$, and [1] studied the case where $\Omega_{u_1}(t) = \Omega_{u_2}(t) = (g(t), h(t))$. In some sense, this paper can also be regarded as a derivation for the widely studied problems (1.2) and (4.2).

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