Terminal value problem for a generalized fractional ordinary differential equation

Can Li[a], Min-min Li[a], Han Zhou[b] *

[a]Department of Applied Mathematics, Xi'an University of Technology,Xian, Shaanxi 710054, P.R. China

[b]Department of Applied Mathematics, Hebei University of Technology, Tianjin, Tianjin 300401, P.R. China

Abstract

The present work is concerned with the well-posedness and efficient numerical algorithm for a terminal value problem with a generalized Caputo fractional derivative. We investigate the existence and uniqueness of the solution of the terminal value problem, and consider the continuous dependence of the solutions on the given data. To illustrate our theoretical results, we present a one step algorithm for solving the considered problems. Some numerical examples are shown to illustrate the theoretical results and the efficiency of the numerical method.

Keywords: Generalized Caputo fractional derivatives, Terminal value problem , one step method

AMS subject classifications: 26A33, 35R11, 65M06, 65M12.

1 Introduction

The fractional calculus has been employed to model many non-classic physical phenomenons due to the nonlocal nature of this kind of operators. To match the applications of mathematics, physics and engineering, various fractional derivatives were introduced [8, 15, 25, 18, 26, 27, 24]. Differential equations with fractional derivatives have been investigated extensively in the last decades. In recent years, terminal value problems for fractional ordinary differential equations have also attracted several scholars' attention [8, 9, 10, 12, 13, 14, 22]. In this work, we study the well-posedness and efficient numerical approximations for the following terminal value problem involving a generalized Caputo fractional derivative

$$
\begin{aligned}
&{}^{\rho}_{c}D^{\alpha}_{0+}\left(y(t)\right) = f(t, y(t)), \quad t \in [0, T], && (1)\\
&y(T) = y_T, && (2)
\end{aligned}
$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous, and ${}^{\rho}_{c}D^{\alpha}_{0+}\left(y(t)\right), (\alpha \in (0, 1), \rho \in \mathbb{R})$ denotes the generalized Caputo fractional derivative of order α [3, 18, 16, 21]

$$
{}^{\rho}_{c}D^{\alpha}_{0+}y(t) = {}^{\rho}\mathcal{D}^{\alpha}_{0+}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!}(t)^k\right], \tag{3}
$$

with the generalized Riemann-Liouville derivative of order $\alpha \in (n-1, n)$

$$
{}^{\rho}\mathcal{D}^{\alpha}_{0+}y(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^n {}^{\rho}\mathcal{I}^{n-\alpha}_{0+}y(t), \tag{4}
$$

where ${}^{\rho}I^{\sigma}_{0+}$ denotes the generalized fractional integral of order σ

$$
{}^{\rho}\mathcal{I}^{\sigma}_{0+}y(t) = \frac{\rho^{1-\sigma}}{\Gamma(\sigma)} \int_0^t \frac{\tau^{\rho-1}y(\tau)}{(t^{\rho} - \tau^{\rho})^{1-\sigma}}\, d\tau,\ \sigma > 0,\ \rho \in \mathbb{R}. \tag{5}
$$

*Corresponding Author. E-mail: zhouhan@hebut.edu.cn

Obviously, if $\rho = 1$, then the generalized Caputo fractional derivative reduces to the classic Caputo fractional derivative. The generalized fractional derivatives of Caputo-type (3) and Riemann-Liouville-type (4) were introduced by Katugampola [16] in order to generalize the Riemann-Liouville and Hadamard fractional derivatives. There are many different ways to define the generalized Caputo derivative (3). For example, the one is discussed by Oliveira [23] with properly choosing parameters in Erdélyi-Kober-type derivative [24, 26, 15, 18, 19, 21]. For more properties, physical applications of the generalized Caputo derivative, we may refer to the recent review papers [23, 26, 2, 18, 20].

So far, much efforts have been made by many researchers to develop the theory and numerical algorithm of differential equations with generalized Caputo derivative (3). Katugampola [17] studied the existence and uniqueness of a fractional differential equation governed by the generalized fractional derivative. Almeida et al. [3] provided a decomposition formula to solve a Cauchy problem of Eq. (1). Recently, by using nonuniform grid, Zeng et al. present a numerical method for an initial Value problem of a generalized fractional differential equation with fractional derivative (3). Furthermore, many significant contributions were made to the numerical methods for Caputo fractional initial and terminal value problems, such as [6, 7, 8, 10, 13, 11, 14, 22].

To the best of our knowledge, both the theoretical and numerical investigations on the terminal value problems (1)-(2) are rather rare. In this work, we will first discuss the well-posedness of problems (1)-(2), based on which we further develop a one step algorithm for the initial and terminal value problems of the generalized fractional differential equation (1). In addition, detailed convergence analysis of the proposed numerical algorithm is rigorously established.

The rest of this paper is organized as follows. In Section 2, we discuss the existence and uniqueness of the solution of problem (1)-(2). Then we investigate the continuous dependence of the solution on the given data in Section 3. Section 4 is devoted to the derivation of numerical approach and several numerical examples are presented to verify the efficiency of our numerical algorithm and theoretical results.

2 Existence and uniqueness of the solution

Using the composite properties of the generalized fractional calculus given in [17, 23], we have the following results.

Lemma 2.1 ([17]). *For the continuous function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, the following initial value problem*

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0+}\,(y(t)) = f(t, y(t)), \quad t \in (0, T], \\ y^{(k)}(t)|_{t=0} = c_k,\ k = 0, 1, \ldots, n-1, \end{cases} \tag{6}$$

is equivalent to the Volterra integral equation of the second kind

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} c_k + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau, \tag{7}$$

where $\alpha \in (n-1, n), n \in \mathbb{N}^+$. Particularly, for $\alpha \in (0, 1)$, we have

$$y(t) = c_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau. \tag{8}$$

Lemma 2.2 ([4, 5]). *Let $u(t)$ and $v(t)$ be nonnegative and integrable functions, $g(t)$ be nonnegative and nondecreasing in $[a, b]$. If*

$$u(t) \le v(t) + g(t)\rho^{1-\alpha} \int_a^t \frac{\tau^{\rho-1} u(\tau)}{(t^\rho - \tau^\rho)^{\alpha-1}} d\tau,\ t \in [a, b], \tag{9}$$

then

$$u(t) \le\ v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{\rho^{1-k\alpha}\,(g(t)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \frac{\tau^{\rho-1} v(\tau)}{(t^\rho - \tau^\rho)^{1-k\alpha}} d\tau,\ \ t \in [a, b]. \tag{10}$$

Furthermore, if $v(t)$ is nondecreasing, then

$$u(t) \le\ v(t) E_\alpha \left(g(t)\Gamma(\alpha) \frac{(t^\rho - a^\rho)^\alpha}{\rho^\alpha} \right),\ \ t \in [a, b], \tag{11}$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function defined by [25]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},\ 0<\alpha<1. \tag{12}$$

Furthermore, with the similar argument given in [17, 23], we can prove the following theorem.

Theorem 2.1. *For $\alpha \in (n-1, n)$, $\rho \in \mathbb{R}$, the fractional differential equation (6) has a unique solution $y(t) \in C^n([0,T])$.*

Using Lemma 2.1, we can derive the relationship between the terminal boundary problem (1)-(2) and a nonlinear Volterra integral equation. The conclusion is stated in the following lemma.

Lemma 2.3. *If the function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, then the terminal value problem (1)-(2) is equivalent to the nonlinear Volterra integral equation*

$$\begin{aligned} y(t) \quad &= \quad y_T + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau. \end{aligned} \tag{13}$$

Proof. By Lemma 2.1, we conclude that the solution y satisfies the following integral equation

$$y(t) = y(0) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau. \tag{14}$$

On the other hand, the solution of the problem (1)-(2) satisfies

$$y(T) = y(0) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau,$$

which yields

$$y(0) = y(T) - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau. \tag{15}$$

Finally, the result (13) is obtained by inserting (15) into (14). □

In what follows, we discuss the existence and uniqueness of solutions of the terminal value problem (1)-(2) employing the Banach fixed point theorem. Instead of investigating the problem (1)-(2), we consider its equivalent problem (13).

Denote $\Omega_\delta = \{y \in C([0,T]) : \|y - y_T\|_{[0,T]} \le \delta\}$ equipped with the norm $\|y(t)\|_{[0,T]} = \max\limits_{t\in[0,T]} |y(t)|$, and

$$\delta = \frac{2T^{\rho\alpha}\|f\|_{[0,T]}}{\Gamma(1+\alpha)}. \tag{16}$$

The set Ω_δ is a closed subset of the Banach space consisting of all continuous functions on $[0,T]$, equipped with the norm $\|\cdot\|_{[0,T]}$, and it is nonempty because the function $y(t) = y_T$ belongs to the set Ω_δ. In Ω_δ, the integral equation (13) can be rewritten in the form of

$$y(t) = (\mathcal{P}y)(t),$$

where the integral operator $\mathcal{P}$ gives

$$\begin{aligned} (\mathcal{P}y)(t) \quad &= \quad y_T + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau. \end{aligned} \tag{17}$$

In order to prove that the problem (1)-(2) has a unique continuous solution, we only need to prove that the operator $\mathcal{P}$ has a unique fixed point in Ω_δ. With the help of the Banach fixed point theorem, we obtain the following theorem.

Theorem 2.2. *Let $D = [0,T] \times [y_T - \delta, y_T + \delta]$ with δ given by (16). Assume that the function $f : D \to \mathbb{R}$ is continuous for all $t \in [0,T]$, and satisfies the Lipschitz type condition with respect to the second variable, i.e.,*

$$|f(t,y) - f(t,z)| \le L_{Lip}|y - z|, \ \forall\ y,\, z \in \Omega_\delta. \tag{18}$$

If the Lipschitz constant satisfies $L_{Lip} < \frac{\rho^\alpha \Gamma(\alpha+1)}{2T^{\rho\alpha}}$, then $\mathcal{P}$ maps Ω_δ into itself and it is a contraction, i.e.,

$$\|\mathcal{P}y - \mathcal{P}z\|_{[0,T]} \le \|y - z\|_{[0,T]}, \quad \forall \quad y, z \in \Omega_\delta. \tag{19}$$

Therefore, the operator $\mathcal{P}$ has a unique solution $y^* \in \Omega_\delta$ which implies that the terminal value problem (1)-(2) has a unique solution $y^* \in \Omega_\delta$.

Proof. Firstly, we show that $\mathcal{P}y \in \Omega_\delta$ if $y \in \Omega_\delta$. Applying the definition of $\mathcal{P}$, we arrive at

$$\begin{aligned}
&|\mathcal{P}y - y_T| \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau - \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau \right| \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \left| (t^\rho - \tau^\rho)^{\alpha-1} - (T^\rho - \tau^\rho)^{\alpha-1} \right| \tau^{\rho-1} f(\tau, y(\tau))\, d\tau \\
&+ \int_t^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau))\, d\tau.
\end{aligned}$$

For $0 < \alpha < 1$, using the fact $(a^\alpha - b^\alpha) \le (a-b)^\alpha$ when $a \ge b \ge 0$, we can estimate that

$$\begin{aligned}
&\int_0^t \left| (t^\rho - \tau^\rho)^{\alpha-1} - (T^\rho - \tau^\rho)^{\alpha-1} \right| \tau^{\rho-1} d\tau \\
&= \frac{1}{\rho} \int_0^t \left| (t^\rho - \tau^\rho)^{\alpha-1} - (T^\rho - \tau^\rho)^{\alpha-1} \right| d(\tau^\rho) \\
&= \frac{1}{\rho\alpha} \left[(T^\rho - t^\rho)^\alpha + (T^{\rho\alpha} - \tau^{\rho\alpha}) \right] \\
&\le \frac{2}{\rho\alpha} (T^\rho - t^\rho)^\alpha.
\end{aligned}$$

Then it follows that

$$\|\mathcal{P}y - y_T\|_{[0,T]} \le \frac{4\|f\|_{[0,T]} T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} := \delta,$$

which implies $(\mathcal{P}y) \in \Omega_\delta$.

Secondly, we show that $\mathcal{P}$ is a contraction on Ω_δ, with δ defined by (16). For any $y, z \in \Omega_\delta$, $t \in [0,T]$, we can check that

$$\begin{aligned}
|(\mathcal{P}y)(t) - (\mathcal{P}z)(t)| \ &\le\ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \Bigg(\int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |f(\tau, y(\tau)) - f(\tau, z(\tau))| d\tau \\
&\qquad + \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |f(\tau, y(\tau)) - f(\tau, z(\tau))| d\tau \Bigg) \\
&\le\ \frac{2L_{Lip} T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y - z\|_{[0,T]} < \|y - z\|_{[0,T]},
\end{aligned}$$

which means the operator $\mathcal{P}$ is a contraction on Ω_δ. Finally, the conclusion follows from the Banach fixed point theorem. □

With the similar argument, we can also get the existence and uniqueness of solutions of the following two-point boundary value problem

$$\begin{cases} {}^\rho_c D^\alpha_{0+} (y(t)) = f(t, y(t)), & t \in (0,T),\ 0 < \alpha < 1, \\ y(0) = 0, \quad y(T) = 0. \end{cases} \tag{20}$$

Analogous to Lemma 2.3, we also have the following result for the problem (20).

Lemma 2.4. *If $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is continuous, then two point boundary problem (20) is equivalent to the nonlinear Fredholm integral equation*

$$\begin{aligned} y(t) = {} & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau \\ & - \frac{t\rho^{1-\alpha}}{T\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau. \end{aligned} \tag{21}$$

Theorem 2.3. *Assume that the function $f : D \to \mathbb{R}$ is continuous for all $t \in [0,T]$, and f also fulfills a Lipschitz condition with respect to the second variable. If the Lipschitz constant L_{Lip} satisfies $L_{Lip} < \frac{\rho^\alpha \Gamma(\alpha+1)}{2T^{\rho\alpha}}$, then (20) has a unique solution $y \in C^1[0,T]$.*

Proof. Define the operator

$$\begin{aligned} (\mathbf{P}y)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau \\ & \quad - \frac{t\rho^{1-\alpha}}{T\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, f(\tau, y(\tau))\, d\tau. \end{aligned} \tag{22}$$

We can check that the operator defined above is a contraction, i.e.,

$$\begin{aligned} |(\mathbf{P}y)(t) - (\mathbf{P}z)(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \Bigg(\int_0^t (t^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1} |f(\tau, y(\tau)) - f(\tau, z(\tau))| d\tau \\ & + \frac{t}{T} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1}\, \tau^{\rho-1}\, |f(\tau, y(\tau)) - f(\tau, z(\tau))| d\tau \Bigg) \\ &\leq \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y - z\|_{[0,T]} < \|y - z\|_{[0,T]}. \end{aligned}$$

Then, by using the Banach fixed point theorem, we obtained that $\mathbf{P}$ has a unique fixed point. □

3 Continuous dependence of the solution

In this section, we will discuss the continuous dependence of the solution on the data for problem (1)-(2. As we have proved that the problem (1)-(2) is equivalent to the nonlinear Volterra integral equation of the second kind, the theoretical analysis is carried out on the model (13). To analyze the continuous dependence of the solution on the given data, we should consider suffer perturbations on the terminal value y_T, the parameters α, ρ and the right-hand side term f. More specifically, we will consider the following perturbed problems

$$\begin{aligned} {}^{\rho}_{c}D^{\alpha}_{0+}\,(z(t)) &= f(t, z(t)), \quad t \in [0,T], & (23) \\ z(T) &= z_T, & (24) \end{aligned}$$

$$\begin{aligned} {}^{\rho}_{c}D^{\alpha-\epsilon}_{0+}\,(z(t)) &= f(t, z(t)), \quad t \in [0,T], & (25) \\ z(T) &= y_T, & (26) \end{aligned}$$

$$\begin{aligned} {}^{\rho-\varepsilon}_{c}D^{\alpha}_{0+}\,(z(t)) &= f(t, z(t)), \quad t \in [0,T], & (27) \\ z(T) &= y_T, & (28) \end{aligned}$$

$$\begin{aligned} {}^{\rho}_{c}D^{\alpha}_{0+}\,(y(t)) &= \tilde{f}(t, z(t)), \quad t \in [0,T], & (29) \\ z(T) &= y_T, & (30) \end{aligned}$$

which are equivalent to the following integral equations

$$z(t) = z_T + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, z(\tau))\, d\tau \\ - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, z(\tau))\, d\tau, \tag{31}$$

$$z(t) = y_T + \frac{\rho^{1-\alpha+\epsilon}}{\Gamma(\alpha-\epsilon)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-\epsilon-1} \tau^{\rho-1} f(\tau, z(\tau))\, d\tau \\ - \frac{\rho^{1-\alpha}}{\Gamma(\alpha-\epsilon)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-\epsilon-1} \tau^{\rho-1} f(\tau, z(\tau))\, d\tau, \tag{32}$$

$$z(t) = y_T + \frac{(\rho-\varepsilon)^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho-\varepsilon} - \tau^{\rho-\varepsilon})^{\alpha-1} \tau^{\rho-\varepsilon-1} f(\tau, z(\tau))\, d\tau \\ - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^{\rho-\varepsilon} - \tau^{\rho-\varepsilon})^{\alpha-1} \tau^{\rho-\varepsilon-1} f(\tau, z(\tau))\, d\tau, \tag{33}$$

$$y(t) = y_T + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \hat{f}(\tau, z(\tau))\, d\tau \\ - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \hat{f}(\tau, z(\tau))\, d\tau, \tag{34}$$

respectively. Under the assumption of Theorem 2.2, we have the following estimates for above perturbed problems.

Theorem 3.1. *Let y and z be the unique solutions of problems (1)-(2) and (23)-(24), respectively. Then*

$$\|y - z\| \leq \frac{1}{C_{\alpha,\rho,L_{Lip}}} |y_T - z_T|, \tag{35}$$

where $C_{\alpha,\rho,L_{Lip}} = 1 - \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}$.

Proof. From (13) and (31), for any $t \in [0, T]$, we have

$$\begin{aligned} |y(t) - z(t)| &\leq |(y_T - z_T)| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left| \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \left[f(\tau, y(\tau)) - f(\tau, z(\tau)) \right] d\tau \right| \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |f(\tau, y(\tau)) - f(\tau, z(\tau))|\, d\tau, \\ &\leq |y_T - z_T| + \frac{L_{Lip}\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |y(\tau) - z(\tau)|\, d\tau \\ &\quad + \frac{L_{Lip}\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |y(\tau) - z(\tau)|\, d\tau. \end{aligned} \tag{36}$$

Thus

$$\begin{aligned} \|y - z\| &\leq |y_T - z_T| + \frac{L_{Lip}\rho^{1-\alpha}}{\Gamma(\alpha)} \|y - z\| \times \\ &\quad \left(\int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1}\, d\tau + \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1}\, d\tau \right) \\ &= |y_a - z_a| + \frac{L_{Lip}\rho^{1-\alpha}}{\Gamma(\alpha)} \|y - z\| \left(\frac{t^{\rho\alpha}}{\rho\alpha} + \frac{T^{\rho\alpha}}{\rho\alpha} \right) \\ &\leq |y_T - z_T| + \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y - z\|. \end{aligned}$$

In view of the Lipschitz constant L_{Lip} given in Theorem 2.2, we have

$$C_{\alpha,\rho,L_{Lip}} = 1 - \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} > 0, \tag{37}$$

which yields (35). $\square$

Remark 3.1. *Furthermore, we have from (36) that*

$$|y(t)-z(t)| \leq |y_T - z_T| + \frac{L_{Lip}\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \, |y(\tau)-z(\tau)| \, d\tau.$$

For $\rho \geq 1$, using Lemma 2.2, we can get

$$|y(t)-z(t)| \leq |y_T - z_T| \, E_\alpha \left(L_{Lip} \frac{t^{\rho\alpha}}{\rho^\alpha} \right), \; t \in [0,T], \tag{38}$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function defined by (12).

Theorem 3.2. *Let y and z be the unique solutions of problems (1)-(2) and (25)-(26), respectively. Then, for $0 < \alpha - \epsilon < 1$, we have*

$$\|y-z\| = \frac{2C_K T \, \|f\|}{C_{\alpha,\rho,L_{Lip}}} \, \epsilon, \tag{39}$$

where $C_{\alpha,\rho,L_{Lip}}$ is given by (37).

Proof. Using (13) and (32), for any $t \in [0,T]$, we have

$$\begin{aligned} |y(t)-z(t)| \;\; &= \;\; \bigg| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau)) \, d\tau \\ & \quad - \frac{\rho^{1-\alpha+\epsilon}}{\Gamma(\alpha-\epsilon)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-\epsilon-1} \tau^{\rho-1} f(\tau, z(\tau)) \, d\tau \\ & \quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} f(\tau, y(\tau)) \, d\tau \\ & \quad + \frac{\rho^{1-\alpha+\epsilon}}{\Gamma(\alpha-\epsilon)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-\epsilon-1} \tau^{\rho-1} f(\tau, z(\tau)) \, d\tau \bigg|, \end{aligned}$$

hence,

$$\begin{aligned} & |y(t)-z(t)| \\ & \leq \int_0^t \left| \frac{1}{\Gamma(\alpha)} f(\tau, y(\tau)) - \frac{\rho^\epsilon (t^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau)) \right| \left| \rho^{1-\alpha} \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \right| \, d\tau \\ & + \int_0^T \left| \frac{1}{\Gamma(\alpha)} f(\tau, y(\tau)) - \frac{\rho^\epsilon (T^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau)) \right| \left| \rho^{1-\alpha} \tau^{\rho-1} (T^\rho - \tau^\rho)^{\alpha-1} \right| \, d\tau \\ & := I + II. \end{aligned} \tag{40}$$

For the first term on the right-hand side of (40), we have

$$\begin{aligned} I \;\; &= \;\; \int_0^t \left| \frac{1}{\Gamma(\alpha)} f(\tau, y(\tau)) - \frac{\rho^\epsilon (t^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau)) \right| \left| \rho^{1-\alpha} \tau^{\rho-1} (t^\rho - \tau^\rho)^{\alpha-1} \right| \, d\tau \\ &= \;\; \int_0^t \bigg| \frac{1}{\Gamma(\alpha)} f(\tau, y(\tau)) - \frac{1}{\Gamma(\alpha)} f(\tau, z(\tau)) + \frac{1}{\Gamma(\alpha)} f(\tau, z(\tau)) \\ & \qquad - \frac{\rho^\epsilon (t^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau)) \bigg| \left| \rho^{1-\alpha} \tau^{\rho-1} (T^\rho - \tau^\rho)^{\alpha-1} \right| \, d\tau \\ &\leq \;\; \frac{L_{Lip} T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y-z\| + \int_0^T \left| \frac{1}{\Gamma(\alpha)} f(\tau, z(\tau)) - \frac{\rho^\epsilon (t^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau)) \right| \\ & \quad \times \left| \rho^{1-\alpha} \tau^{\rho-1} (T^\rho - \tau^\rho)^{\alpha-1} \right| \, d\tau. \end{aligned}$$

Defining the function $K(x) = \dfrac{(t^\rho - \tau^\rho)^x}{\Gamma(x+1)}$, and using the mean value theorem, we obtain

$$\left| \frac{(t^\rho - \tau^\rho)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t^\rho - \tau^\rho)^{\alpha-\epsilon-1}}{\Gamma(\alpha-\epsilon)} \right| \leq C_K \epsilon,$$

where $C_K = \max_{x\in[\alpha-\epsilon-1,\alpha-\epsilon]} K'(x)$. Hence, we get the bound of the term I as follows

$$I \leq \frac{L_{Lip}T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \|y-z\| + C_K T \|f\| \epsilon.$$

For the second term on the right-hand side of (40), we have

$$\begin{aligned} II &= \int_0^T \left|\frac{1}{\Gamma(\alpha)} f(\tau, y(\tau)) - \frac{\rho^\epsilon (T^\rho - \tau^\rho)^{-\epsilon}}{\Gamma(\alpha-\epsilon)} f(\tau, z(\tau))\right| \left|\rho^{1-\alpha}\tau^{\rho-1}(T^\rho-\tau^\rho)^{\alpha-1}\right| \, d\tau \\ &\leq \frac{L_{Lip}T^{\rho\alpha}}{\Gamma(\alpha+1)} \|y-z\| + C_K T \|f\| \epsilon, \end{aligned}$$

and therefore $\|y-z\| \leq \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \|y-z\| + 2C_K T \|f\| \epsilon$, which implies (39). $\square$

Remark 3.2. *From (40), we can get*

$$|y(t)-z(t)| \leq 2C_K T \|f\| \; \epsilon + \frac{L_{Lip}}{\rho^\alpha\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \; |y(\tau)-z(\tau)| \, d\tau.$$

Using Lemma 2.2, for $\rho \geq 1$, we have

$$|y(t)-z(t)| \leq 2C_K T \|f\| E_\alpha \left(L_{Lip}\frac{t^{\rho\alpha}}{\rho^\alpha}\right) \epsilon, \;\; t\in[0,T],$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function defined by (12).

Theorem 3.3. *Let y and z be the unique solutions of problems (1)-(2) and (27)-(28), respectively. Then*

$$\|y-z\| = \frac{2\tilde{C}_K \|f\|}{C_{\alpha,\rho,L_{Lip}}} \; \varepsilon, \tag{41}$$

where $C_{\alpha,\rho,L_{Lip}}$ is given by (37).

Proof. Combining (13) and (33), for any $t\in[0,T]$, yields

$$\begin{aligned} &\Gamma(\alpha)\,|y(t)-z(t)| \\ &\leq \int_0^t \left|\rho^{1-\alpha}(t^\rho-\tau^\rho)^{\alpha-1}\,\tau^{\rho-1} f(\tau,y(\tau)) - (\rho-\varepsilon)^{1-\alpha}(t^{\rho-\varepsilon}-\tau^{\rho-\varepsilon})^{\alpha-1}\,\tau^{\rho-\varepsilon-1} f(\tau,z(\tau))\right| d\tau \\ &+ \int_0^T \left|\rho^{1-\alpha}(T^\rho-\tau^\rho)^{\alpha-1}\,\tau^{\rho-1} f(\tau,y(\tau)) - (\rho-\varepsilon)^{1-\alpha}(T^{\rho-\varepsilon}-\tau^{\rho-\varepsilon})^{\alpha-1}\,\tau^{\rho-\varepsilon-1} f(\tau,z(\tau))\right| \, d\tau \\ &:= \tilde{I} + \tilde{II}. \end{aligned} \tag{42}$$

For the term $\tilde{I}$, applying the mean value theorem, we have

$$\begin{aligned} \tilde{I} &\leq \int_0^t \left|\rho^{1-\alpha}(t^\rho-\tau^\rho)^{\alpha-1}\,\tau^{\rho-1} f(\tau,y(\tau)) - \rho^{1-\alpha}(t^\rho-\tau^\rho)^{\alpha-1}\,\tau^{\rho-1}\, f(\tau,z(\tau))\right| d\tau \\ &+ \int_0^t \left|\rho^{1-\alpha}(t^\rho-\tau^\rho)^{\alpha-1}\,\tau^{\rho-1}\, f(\tau,z(\tau)) - (\rho-\varepsilon)^{1-\alpha}(t^{\rho-\varepsilon}-\tau^{\rho-\varepsilon})^{\alpha-1}\,\tau^{\rho-\varepsilon-1} f(\tau,z(\tau))\right| d\tau \\ &\leq \frac{L_{Lip}T^{\rho\alpha}}{\rho^\alpha \alpha} \|y-z\| + \tilde{C}_K \|f\| \varepsilon, \end{aligned}$$

where $\tilde{C}_K = \max_{x\in[\rho-\varepsilon,\rho]} \left|\tilde{K}'(x)\right|$ and

$$\tilde{K}(x) = x^{1-\alpha}(t^x - \tau^x)^{\alpha-1}\,\tau^{x-1}.$$

By similar arguments, we get

$$\tilde{I} \leq \frac{L_{Lip}T^{\rho\alpha}}{\rho^\alpha\alpha} \|y-z\| + \tilde{C}_K \|f\| \varepsilon.$$

Hence $\|y-z\| \leq \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \|y-z\| + 2\tilde{C}_K \|f\| \varepsilon$. Finally, we obtain (41). $\square$

Remark 3.3. *From (42), we have*

$$|y(t) - z(t)| \leq 2\hat{C}_K T \|f\| \ \epsilon + \frac{L_{Lip}}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \, |y(\tau) - z(\tau)| \, d\tau.$$

Using Lemma 2.2, for $\rho \geq 1$, we have

$$|y(t) - z(t)| \leq 2\hat{C}_K T \|f\| \, E_\alpha \left(L_{Lip} \frac{t^{\rho\alpha}}{\rho^\alpha} \right) \ \epsilon, \ \ t \in [0, T],$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function defined by (12).

Theorem 3.4. *Let y and z be the unique solutions of problems (1)-(2) and (29)-(30), respectively. Then*

$$\|y - z\| = \frac{1}{C_{\alpha,\rho,L_{Lip}}} \frac{2T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \left\| f - \tilde{f} \right\|,$$

where $C_{\alpha,\rho,L_{Lip}}$ is given by (37).

Proof. Using (13) and (34), for any $t \in [0, T]$, we have

$$\begin{aligned} |y(t) - z(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \left| f(\tau, y(\tau)) - \hat{f}(\tau, z(\tau)) \right| d\tau \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^T (T^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \left| f(\tau, y(\tau)) - \hat{f}(\tau, z(\tau)) \right| d\tau \\ &:= (I) + (II). \end{aligned}$$

After simple arguments, we have

$$\begin{aligned} (I) &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \, |f(\tau, y(\tau)) - f(\tau, z(\tau))| \, d\tau \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \left| f(\tau, z(\tau)) - \hat{f}(\tau, z(\tau)) \right| d\tau \\ &\leq \frac{L_{Lip} t^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y - z\| + \frac{t^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \left\| f - \tilde{f} \right\|, \end{aligned}$$

and

$$(II) \leq \frac{L_{Lip} T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \|y - z\| + \frac{T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \left\| f - \tilde{f} \right\|.$$

Hence $\|y - z\| \leq \frac{2L_{Lip}T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \|y - z\| + \frac{2T^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \left\| f - \tilde{f} \right\|$, which follows the result of the theorem. □

Remark 3.4. *From (43), we have*

$$|y(t) - z(t)| \leq + \frac{2T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \left\| f - \tilde{f} \right\| + \frac{L_{Lip}}{\rho^\alpha \Gamma(\alpha)} \int_0^t (t^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \, |y(\tau) - z(\tau)| \, d\tau.$$

Using Lemma 2.2, for $\rho \geq 1$, we can get

$$|y(t) - z(t)| \leq 2\hat{C}_K T \|f\| \, E_\alpha \left(L_{Lip} \frac{t^{\rho\alpha}}{\rho^\alpha} \right) \ \epsilon, \ \ t \in [0, T],$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function defined by (12).

4 Numerical results

4.1 Numerical method for initial value problem

We first describe a simple numerical algorithm for fractional initial value problem

$$ {}^\rho_c D^\alpha_{0+} \left(y(t) \right) = f(t, y(t)), \quad t \in (0, T], \tag{43}$$
$$ y(0) = y_0. \tag{44}$$

As stated in Lemma 2.1, the problem (43)-(44) is equivalent to (8). Then, instead of solving problem (43)-(44) directly, we design an algorithm for solving (8). The interval $[0, T]$ is divided into a uniform mesh $\mathcal{T} = \{t_k = kh, k = 0, 1, \ldots, K\}$, $K \in \mathbb{N}^+$, with a mesh size $h = \frac{T}{K}$. If applying the left rectangle rule (with the kernel $(t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1}$ being the weight) on the integral in (7), i.e.,

$$\int_0^{t_k} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} f(\tau, y(\tau))d\tau \approx \sum_{j=0}^{k-1} f(t_j, y(t_j)) \int_{t_j}^{t_{j+1}} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} d\tau,$$

then we get

$$\int_0^{t_k} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} f(\tau, y(\tau))d\tau = \frac{h^{\rho\alpha}}{\rho\alpha} \sum_{j=0}^{k-1} b_{j,k} f(t_j, y(t_j)), \tag{45}$$

where f is continuous and the weights $b_{j,k+1}$ are given by

$$b_{j,k} = [k^\rho - j^\rho]^\alpha - [k^\rho - (j+1)^\rho]^\alpha, \quad j = 0, 1, \ldots, k-1.$$

Then the one step method for solving (7) is given by

$$y_h(t_k) = \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{t_k^j}{j!} y_0^{(k)} + \frac{h^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \sum_{j=0}^{k-1} b_{j,k} f(t_j, y_h(t_j)), \tag{46}$$

where $\lceil\cdot\rceil$ denotes the integer-valued function. Next, we estimate the local truncation error and convergence order for the scheme (46). First, we present several lemmas which will be used later.

Lemma 4.1 ([5]). *Let $\{x_k\}_{k=0}^K$ be a sequence of non-negative real numbers. If*

$$x_k \le \psi_k + M h^{\beta(1-\lambda)} \sum_{j=1}^{k-1} \frac{j^{\beta-1} x_k}{(k^\beta - j^\beta)^\lambda}, \quad t \in [0, T], \tag{47}$$

where $0 < \lambda < 1$, $\beta \ge 1$, M is a positive constant, then

$$x_k \le \psi_k E_{1-\lambda}\left(\frac{M\Gamma(1-\lambda)}{\beta}(kh)^{\beta(1-\lambda)}\right), \ 0 \le k \le K, \tag{48}$$

where $E_{1-\alpha}(z)$ denotes the Mittag-Leffler function defined by (12).

Lemma 4.2. *Suppose $f(t, y(t)) \in C[0, T]$, then we have*

$$\left|\int_0^{t_k} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} f(\tau, y(\tau))d\tau - \frac{h^{\rho\alpha}}{\rho\alpha} \sum_{j=0}^{k-1} b_{j,k} f(t_j, y(t_j))\right| \le C_\alpha h, \tag{49}$$

where $C_\alpha = \frac{2T^{\rho\alpha}\|f'\|_\infty}{\rho\alpha}$.

Proof. By some calculations, we have

$$\begin{aligned}
&\left|\int_0^{t_k} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} f(\tau, y(\tau))d\tau - \frac{h^{\rho\alpha}}{\rho\alpha} \sum_{j=0}^{k-1} b_{j,k} y(t_j, y(t_j)),\right| \\
&= \left|\int_0^{t_k} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} \big(f(\tau, y(\tau)) - f(t_j, y(t_j))\big)d\tau\right| \\
&\le \left|\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} \big|(\tau - t_j) f'(\theta_j, y(\theta_j))\big| d\tau\right|, \ \theta_j \in (t_j, t_{j+1}) \\
&\le \|f'\|_\infty \sum_{j=0}^{k-1} h\left(\int_{t_j}^{t_{j+1}} (t_k^\rho - \tau^\rho)^{\alpha-1}\tau^{\rho-1} d\tau\right), \\
&= \frac{\|f'\|_\infty h}{\rho\alpha} \sum_{j=0}^{k-1} \left[(t_k^\rho - t_j^\rho)^\alpha - (t_k^\rho - t_{j+1}^\rho)^\alpha\right] \\
&\le C_\alpha h.
\end{aligned} \tag{50}$$

□

Denotes the maximum of the errors as $\|e_h\|_\infty = \max_{1\le k\le K}|y(x_k)-y_h(x_k)|$. Using Lemma 4.2, we obtain the error estimate of the scheme (46) in the following theorem.

Theorem 4.1. *If $f(t,y(t)) \in C[0,T]$, then the convergence order of numerical scheme (46) is one, i.e., $\|e_h\|_\infty \le Ch$.*

Remark 4.1. *Recall the function $f(t,y(t))$ satisfies a Lipschitz condition with respect to the second variable with the Lipschitz constant L_{Lip}, subtracting (8) from (46), we obtain*

$$\begin{aligned}\Big|y(x_k)-y_h(t_k)\Big| &\le L_{Lip}\Bigg[\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_k^\rho-\tau^\rho)^{\alpha-1}\tau^{\rho-1}\big|y(x_k)-y_h(t_k)\big|d\tau\Bigg]+O(h),\\ &\le L_{Lip}h^{\rho\alpha}\sum_{j=0}^{k-1}\frac{j^{\rho-1}}{(k^\rho-j^\rho)^{1-\alpha}}\big|y(x_k)-y_h(t_k)\big|+O(h).\end{aligned}$$

Using the discrete weakly singular Gronwall's inequality presented in Lemma 4.1, we can also get $\|e_h\|_\infty \le Ch$.

The numerical experiment consists of two parts. In the first part, we test the efficiency and the accuracy of the numerical algorithm (46) for the fractional initial value problem (43)-(44). The numerical errors are measured by the maximum norm $\|e_h\|_\infty$ in the following first two examples. Then, we present some numerical examples to check our theoretical findings discussed in Section 3.

Example 4.1. *Consider the following linear fractional differential equation*

$$ {}^\rho_c D^\alpha_{0+}\left(y(t)\right) = q(t)y(t)+g(t), \quad 0<t\le 1, \tag{51}$$

with the initial condition $y(0)=0$, where $q(t)=e^{-t}$, $g(t)=\frac{\Gamma\left(1+\frac{2+\rho\alpha}{\rho}\right)\rho^\alpha}{\Gamma\left(1+\frac{2+\rho\alpha}{\rho}-\alpha\right)}t^2-e^{-t}y(t)$. Employing the relation

$$ {}^\rho_c D^\alpha_{0+}y(t) = {}^\rho\mathcal{D}^\alpha_{0+}\left[y(t)-y(0)\right], \tag{52}$$

and the following formula [16, 23]

$$ {}^\rho D^\alpha_{0+}\left(t^\beta\right) = \begin{cases}\frac{\Gamma\left(1+\frac{\beta}{\rho}\right)\rho^\alpha}{\Gamma\left(1+\frac{\beta}{\rho}-\alpha\right)}t^{\beta-\alpha\rho}, \rho>0,\ \alpha-\frac{\beta}{\rho}\notin\mathbb{N}^+,\\ 0, \qquad\qquad \rho>0,\ \alpha-\dfrac{\beta}{\rho}\in\mathbb{N}^+,\end{cases} \tag{53}$$

we obtain that the solution of problem (51) is $y(t)=t^{2+\rho\alpha}$.

Table 1: Maximum errors and convergence orders of Example 4.1 solved by the scheme (46) with $T=1$, $\rho=0.4$.

h	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.9$	
	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order
$1/10$	2.7932e-001		2.6453e-001		2.7799e-001	
$1/20$	1.4484e-001	0.9474	1.4273e-001	0.8900	1.5653e-001	0.8286
$1/40$	7.3042e-002	0.9877	7.4455e-002	0.9389	8.4314e-002	0.8926
$1/80$	3.6329e-002	1.0076	3.8053e-002	0.9684	4.4096e-002	0.9351
$1/160$	1.7934e-002	1.0184	1.9215e-002	0.9858	2.2633e-002	0.9622

Table 2: Maximum errors and convergence orders of Example 4.1 solved by the scheme (46) with $T=1$, $\alpha=0.5$.

h	$\rho=0.5$		$\rho=1$		$\rho=2$	
	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order
$1/10$	2.5107e-001		1.9443e-001		1.0989e-001	
$1/20$	1.3339e-001	0.9124	9.7160e-002	1.0008	4.8702e-002	1.1739
$1/40$	6.8712e-002	0.9571	4.7924e-002	1.0196	2.2114e-002	1.1390
$1/80$	3.4787e-002	0.9820	2.3579e-002	1.0232	1.0286e-002	1.1042
$1/160$	1.7451e-002	0.9952	1.1619e-002	1.0209	4.8788e-003	1.0761

Example 4.2. *Consider the nonlinear fractional differential equation*

$$ {}_{c}^{\rho}D_{0+}^{\alpha}\left(y(t)\right) = y^2(t) + g(t), \quad 0 < t \leq 1, \tag{54}$$

with the initial condition $y(0) = 1$. The analytical solution of (54) is given by $y(t) = t^5 + 1$.

Table 3: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1$, $\rho = 0.4$.

h	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.9$	
	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order
1/10	1.2551e-01		1.4771e-01		1.9934e-01	
1/20	5.5593e-02	1.1748	6.6365e-02	1.1543	9.6522e-02	1.0463
1/40	2.5741e-02	1.1108	3.1064e-02	1.0952	4.7324e-02	1.0283
1/80	1.2173e-02	1.0804	1.4881e-02	1.0618	2.3408e-02	1.0156
1/160	5.8237e-03	1.0637	7.2296e-03	1.0415	1.1636e-02	1.0084

Table 4: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1$, $\alpha = 0.5$.

h	$\rho = 0.5$		$\rho = 2$		$\rho = 6$	
	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order	$\|e_h\|_\infty$	order
1/10	1.4456e-01		9.4993e-02		5.2856e-02	
1/20	6.4858e-02	1.1563	4.1232e-02	1.2041	2.9685e-02	0.8323
1/40	3.0331e-02	1.0965	1.8826e-02	1.1310	1.5942e-02	0.8969
1/80	1.4520e-02	1.0628	8.8493e-03	1.0891	8.3368e-03	0.9352
1/160	7.0508e-03	1.0422	4.2386e-03	1.0620	4.2906e-03	0.9583

4.2 Numerical method for the terminal value problem

This part is to test the numerical algorithm for the terminal value problem (1)-(2) and check the theoretical analysis presented in Section 3. Firstly, with the help of Theorem 3.1 of [9], we have that if y_1 and y_2 are two solutions of the differential equations

$$ {}_{c}^{\rho}D_{0+}^{\alpha}\left(y_j(t)\right) = f(t, y_j(t)), \quad j = 1, 2, \tag{55}$$

subject to the initial conditions $y_j(0) = y_{j0}$, $j = 1, 2$, respectively, where $y_{10} \neq y_{20}$. Then for all t where both $y_1(t)$ and $y_2(t)$ exist we have $y_1(t) \neq y_2(t)$. So the solution of a generalized Caputo fractional differential equation of order $\alpha \in (0, 1)$ is uniquely defined by a condition that can be specified at any point $t \geq 0$. It follows that for the solution of (1) that passes through the point (T, y_T) we are able to find at most one point $(0, y_0)$ that also lies on the same solution trajectory. To evaluate the value of $y(T)$, we need a numerical method to solve initial value problems. We use the algorithm given in [13, 10] to obtain the initial value $y(0)$. The selection of the initial value for the terminal value problem is implemented by the following steps. The precision $\epsilon_{stop} = 10^{-5}$ is used for computing $y(0)$. The next numerical example is given to show the dependence on the problem parameters which is analyzed in Theorems 3.1-3.4.

Example 4.3. *Consider the following problem*

$$ {}_{c}^{\rho}D_{0+}^{\alpha}\left(y(t)\right) = -y(t) + t^2 + \frac{\Gamma\left(1+\frac{2}{\rho}\right)\rho^{\alpha}}{\Gamma\left(1+\frac{2}{\rho}-\alpha\right)} t^{2-\rho\alpha} := f(t, y), \quad t > 0, \tag{56}$$

with the terminal condition $y(1) = 1$. The analytical solution of (56) is given by $y(t) = t^2$.

In this example, we denote $\|y - z\|_\infty = \max\limits_{1\leq k\leq K} |y_k - z_k|$. Note that the function f satisfies the assumptions of Theorems 3.1- 3.4. We consider the following perturbed problems

$$ \begin{aligned} &{}_{c}^{\rho}D_{0+}^{\alpha}\left(z(t)\right) = f(t, z), \quad t > 0, \\ &z(T) = y_T + \varepsilon_T, \end{aligned} \tag{57}$$

Algorithm 4.1 The procedure of selecting initial value.

1: Guess an approximation of $y(0)$ denoted by y_{10}, solve the initial value problem

$$
\begin{aligned}
&{}^{\rho}_{c}D^{\alpha}_{0+}\,(y(t)) = f(t, y(t)), \quad t \in (0, T],\\
&y(0) = y_{10},
\end{aligned}
$$

and get $y(T)$ denoted by y_{1T}.

2: **if** $|y(T) - y_{1T}| < \epsilon_{stop}$, **then** stop, $y(0) = y_{10}$.
else
Re-guess an approximation $y(0)$ as $y_k(0)$, solve the initial value problem

$$
\begin{aligned}
&{}^{\rho}_{c}D^{\alpha}_{0+}\,(y(t)) = f(t, y(t)), \quad t \in (0, T],\\
&y(0) = y_{k0},
\end{aligned}
$$

and get $y(T)$ denoted by y_{kT}.

3: until $|y(T) - y_{kT}| < \epsilon_{stop}$, **then** stop, $y(0) = y_{k0}$.

$$
\begin{aligned}
&{}^{\rho}_{c}D^{\alpha}_{0+}\,(z(t)) = f(t, z) + \varepsilon_f, \quad t > 0,\\
&z(T) = y_T,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
&{}^{\rho}_{c}D^{\alpha+\varepsilon_\alpha}_{0+}\,(z(t)) = f(t, z), \quad t > 0,\\
&z(T) = y_T,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
&{}^{\rho+\varepsilon_\rho}_{c}D^{\alpha}_{0+}\,(z(t)) = f(t, z), \quad t > 0,\\
&z(T) = y_T.
\end{aligned}
\tag{60}
$$

Table 5: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1,\ \rho = 2$.

h	ε_T				
	0.3	0.1	0.03	0.01	0.0025
1/10	3.0000×10^{-1}	1.0000×10^{-1}	3.0000×10^{-2}	1.0000×10^{-2}	2.5000×10^{-3}
1/20	3.0000×10^{-1}	1.0000×10^{-1}	3.0000×10^{-2}	1.0000×10^{-2}	2.5000×10^{-3}
1/40	3.0000×10^{-1}	1.0000×10^{-1}	3.0000×10^{-2}	1.0000×10^{-2}	2.5000×10^{-3}
1/80	3.0000×10^{-1}	1.0000×10^{-1}	3.0000×10^{-2}	1.0000×10^{-2}	2.5000×10^{-3}
1/160	3.0000×10^{-1}	1.0000×10^{-1}	3.0000×10^{-2}	1.0000×10^{-2}	2.5000×10^{-3}

Table 6: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1,\ \alpha = 0.5$.

h	ε_f				
	0.5	0.2	0.1	0.05	0.02
1/10	2.4241×10^{-1}	9.6963×10^{-2}	4.8481×10^{-2}	2.4241×10^{-2}	9.6963×10^{-3}
1/20	2.4043×10^{-1}	9.6172×10^{-2}	4.8086×10^{-2}	2.4043×10^{-2}	9.6172×10^{-3}
1/40	2.3943×10^{-1}	9.5771×10^{-2}	4.7885×10^{-2}	2.3943×10^{-2}	9.5771×10^{-3}
1/80	2.3892×10^{-1}	9.5569×10^{-2}	4.7784×10^{-2}	2.3892×10^{-2}	9.5569×10^{-3}
1/160	2.3892×10^{-1}	9.5468×10^{-2}	4.7734×10^{-2}	2.3867×10^{-2}	9.5468×10^{-3}

Table 7: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1$, $\rho = 2$.

h	ε_α				
	0.05	0.1	0.15	0.2	0.5
1/10	5.1809×10^{-1}	4.8886×10^{-1}	4.5895×10^{-1}	4.2848×10^{-1}	2.4200×10^{-1}
1/20	5.6454×10^{-1}	5.3329×10^{-1}	5.0139×10^{-1}	4.6889×10^{-1}	2.6847×10^{-1}
1/40	5.8723×10^{-1}	5.5498×10^{-1}	5.2209×10^{-1}	4.8861×10^{-1}	2.8143×10^{-1}
1/80	5.9844×10^{-1}	5.6570×10^{-1}	5.3233×10^{-1}	4.9836×10^{-1}	2.8787×10^{-1}
1/160	6.0400×10^{-1}	5.7104×10^{-1}	5.3743×10^{-1}	5.0322×10^{-1}	2.9109×10^{-1}

Table 8: Maximum errors and convergence orders of Example 4.2 solved by the scheme (46) with $T = 1$, $\alpha = 0.5$.

h	ε_ρ				
	0.001	0.025	0.05	0.1	0.2
1/10	1.8950×10^{-5}	4.7294×10^{-4}	9.4418×10^{-4}	1.8816×10^{-3}	3.7360×10^{-3}
1/20	2.3905×10^{-5}	5.9656×10^{-4}	1.1909×10^{-3}	2.3730×10^{-3}	4.7107×10^{-3}
1/40	2.6686×10^{-5}	6.6594×10^{-4}	1.3293×10^{-3}	2.6486×10^{-3}	5.2568×10^{-3}
1/80	2.8156×10^{-5}	7.0260×10^{-4}	1.4025×10^{-3}	2.7941×10^{-3}	5.5452×10^{-3}
1/160	2.8912×10^{-5}	7.2144×10^{-4}	1.4401×10^{-3}	2.8690×10^{-3}	5.6935×10^{-3}

The numerical results of the perturbed problems (57)-(60) are presented in Tables 5, 6, 7 and 8, respectively, from which we see that $\|y - z\|_\infty \sim \epsilon_T$, $\|y - z\|_\infty \sim \epsilon_f$ $\|y - z\|_\infty \sim \epsilon_\alpha$ and $\|y - z\|_\infty \sim \epsilon_\rho$, which are consistent with the theoretical results proved in Theorems 3.1 - 3.4.

5 Conclusion

We have discussed a terminal value problem with a generalized Caputo fractional derivative. The existence and uniqueness of solutions are obtained by using the Banach fixed point theorem. The continuous dependence of the solution on the data of terminal value problem are considered. We have proposed a simple one step method to confirm the continuous dependence of the solution on the data for the corresponding perturbed problems. The numerical results agree with our theoretical findings. The convergence rate of the proposed numerical method is of the first order. Developing high-order numerical methods for the considered model will be our further work.

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