THE UNBOUNDEDNESS OF HAUSDORFF OPERATORS ON QUASI-BANACH SPACES

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Abstract. In this note, we show that the Hausdorff operator H_Φ is unbounded on a large family of Quasi-Banach spaces, unless H_Φ is a zero operator.

1. Introduction and Motivation

Hausdorff operator, in connection with some classical summation methods, has been studied for a long time in the field of real and complex analysis. For the historical background and recent developments of the boundedness on function spaces regarding Hausdorff operators, we refer the reader to survey papers by Chen-Fan-Wang [3] and Liflyand [13].

For a suitable function Φ, the corresponding Hausdorff operator H_Φ can be defined by

$$H_\Phi f(x) := \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy. \tag{1.1}$$

Particularly, when Φ is taken suitably, Hausdorff operator contains some important operators in the field of harmonic analysis. For instance, the Hardy operator, adjoint Hardy operator (see [1, 4, 5]), and the Cesàro operator [17, 20] in one dimension. The Hardy-Littlewood-Pólya operator and the Riemann-Liouville fractional integral operator can also be derived from the Hausdorff operator.

In recent years, there is an increasing interest on the boundedness of Hausdorff operators on function spaces, one can see [6, 7] for the the boundedness on Lebesgue spaces, [22] for the boundedness on modulation and Wiener amalgam spaces, and [5, 16, 18] for the boundedness on H^1 and h^1. However, since the argument of using Minkowski's inequality cannot be applied to Quasi-Banach spaces, there are only a few boundedness results considering the boundedness of Hausdorff operators on Quasi-Banach spaces.

Among the previous results, the study of Hausdorff operators on $H^p(\mathbb{R}^n)$ $(0 < p < 1)$ has its special status, since $H^p(\mathbb{R}^n)$ is a Quasi-Banach space and also an important function space in the field of harmonic analysis. The study of $H_\Phi f$ on $H^p(\mathbb{R})$ was first initiated by Kanjin [12] and continued in [14]. Very recently, Liflyand-Miyachi [15] establishes the multidimensional boundedness results on $H^p(\mathbb{R}^n)$. For the $H^p(\mathbb{R}^n)$ boundedness of another type of Hausdorff operator, we refer the reader to [2, 19].

Based on the previous results, an interesting problem is whether we can establish the boundedness result on local Hardy space $h^p(\mathbb{R}^n)$, or even on the general inhomogeneous

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frequency decomposition spaces such as Triebel-Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$, Besov spaces $B_s^{p,q}(\mathbb{R}^n)$ and modulation spaces $M_s^{p,q}(\mathbb{R}^n)$ with $p \in (0,1)$. In this paper, we will explore this problem and give an negative answer, showing that the Hausdorff operator is unboundedness on a large family of Quasi-Banach spaces. In particular, we prove that all the nonzero Hausdorff operators are unbounded on $F_s^{p,q}$, $B_s^{p,q}$ or $M_s^{p,q}$ with $p \in (0,1)$.

Our paper is organized as follows. In Section 2, we collect some notations and basic properties of function spaces. The unboundedness results and their proofs will be presented in Section 3.

Throughout this paper, we will adopt the following notations. We use $X \lesssim Y$ to denote the statement that $X \leq CY$, with a positive constant C that may depend on n, p, but it might be different from line to line. The notation $X \sim Y$ means the statement $X \lesssim Y \lesssim X$. We use $X \lesssim_\lambda Y$ to denote $X \leq C_\lambda Y$, meaning that the implied constant C_λ depends on the parameter λ. For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we denote $|x|_\infty := \max_{i=1,2,\ldots,n} |x_i|$ and $\langle x \rangle := (1+|x|^2)^{1/2}$.

2. Preliminary

Let $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ be the space of tempered distributions. We define the Fourier transform $\mathscr{F} f$ and the inverse Fourier transform $\mathscr{F}^{-1} f$ of $f \in \mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \ \mathscr{F}^{-1} f(x) = \check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi} d\xi.$$

The local Hardy space was introduced by Goldberg [8]. Let $0 < p < \infty$ and let $\psi \in \mathscr{S}$ satisfy $\int_{\mathbb{R}^n} \psi(x)dx \neq 0$. Define $\psi_t(x) := t^{-n}\psi(x/t)$. The *local Hardy space* is defined by

$$h^p := \{f \in \mathscr{S}' : \|f\|_{h^p} = \|\sup_{0<t<1} |\psi_t * f|\|_{L^p} < \infty\}.$$

We note that the definition of the local Hardy spaces is independent of the choice of $\psi \in \mathscr{S}$.

To define Besov space and Triebel-Lizorkin space, we introduce the dyadic decomposition of $\mathbb{R}^n$. Let φ be a smooth bump function supported in the ball $\{\xi : |\xi| < 3/2\}$ and be equal to 1 on the ball $\{\xi : |\xi| \leq 4/3\}$. Denote

$$\phi(\xi) = \varphi(\xi) - \varphi(2\xi), \tag{2.1}$$

and a function sequence

$$\begin{cases} \phi_j(\xi) = \phi(2^{-j}\xi), \ j \in \mathbb{Z}^+, \\ \phi_0(\xi) = 1 - \sum_{j\in\mathbb{Z}^+} \phi_j(\xi) = \varphi(\xi). \end{cases} \tag{2.2}$$

For integers $j \geq 0$, we define the Littlewood-Paley operators

$$\Delta_j = \mathscr{F}^{-1} \phi_j \mathscr{F}. \tag{2.3}$$

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$. For a tempered distribution f, we set the norm

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q}, \tag{2.4}$$

with the usual modifications when $q = \infty$. The (inhomogeneous) Besov space $B_s^{p,q}$ is the space of all tempered distributions f for which the quantity $\|f\|_{B_s^{p,q}}$ is finite. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$. For a tempered distribution f, we set the norm

$$\|f\|_{F_s^{p,q}} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p} \tag{2.5}$$

with the usual modifications when $q = \infty$. The Triebel-Lizorkin space $F_s^{p,q}$ is the space of all tempered distributions f for which the quantity $\|f\|_{F_s^{p,q}}$ is finite. We recall that the local Hardy space h^p is equivalent with the inhomogeneous Triebel-Lizorkin space $F_0^{p,2}$ for $p \in (0, \infty)$

Next, we introduce the modulation space. Denote by Q_k the unit cube with the center at k. Then the family $\{Q_k\}_{k \in \mathbb{Z}^n}$ constitutes a decomposition of $\mathbb{R}^n$. Let $\eta : \mathbb{R}^n \to [0, 1]$ be a smooth function satisfying that $\eta(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\eta(\xi) = 0$ for $|\xi|_\infty \geq 3/4$. Let

$$\eta_k(\xi) = \eta(\xi - k), k \in \mathbb{Z}^n$$

be a translation of η. Since $\eta_k(\xi) = 1$ in Q_k, we have that $\sum_{k \in \mathbb{Z}^n} \eta_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$\sigma_k(\xi) = \eta_k(\xi) \left(\sum_{l \in \mathbb{Z}^n} \eta_l(\xi) \right)^{-1}, \ k \in \mathbb{Z}^n.$$

It is easy to know that $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ constitutes a smooth partition of the unity, and $\sigma_k(\xi) = \sigma(\xi - k)$. The frequency-uniform decomposition operators can be defined by

$$\Box_k := \mathscr{F}^{-1} \sigma_k \mathscr{F}$$

for $k \in \mathbb{Z}^n$. Now, we give the (discrete) definition of modulation space $M_s^{p,q}$.

Definition 2.1. Let $s \in \mathbb{R}, 0 < p, q \leq \infty$. The modulation space $M_s^{p,q}$ consists of all $f \in \mathscr{S}'$ such that the (quasi-)norm

$$\|f\|_{M_s^{p,q}} := \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|_{L^p}^q \right)^{1/q}$$

is finite. Note that this definition is independent of the choice of $\{\sigma_k\}_{k \in \mathbb{Z}^n}$. We also recall a basic fact that $C_c^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset M_s^{p,q}$ for any $s \in \mathbb{R}, 0 < p, q \leq \infty$.

We say $X \hookrightarrow L_0^p(\mathbb{R}^n)$, if for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\|\varphi * f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_X$$

for all $f \in X$, where the constant C is only dependent on φ.

Following, we collect some basic embedding results of function spaces.

Lemma 2.2. *Let $0 < p \leq 1, 0 < q \leq \infty$, $s \in \mathbb{R}$.*

(1) *For* $X = h^p, M^{p,p}, B^{p,p}, F^{p,p}$, *we have* $\|g\|_{L^p} \lesssim \|g\|_X$ *for all measurable functions* $g \in X$.

(2) *For* $Y = M_s^{p,q}, \mathscr{F}M^{q,p}, B_s^{p,q}, F_s^{p,q}$, *we have* $Y \hookrightarrow L_0^p(\mathbb{R}^n)$.

Proof. We first verify that $\|g\|_{L^p} \lesssim \|g\|_{h^p}$ for measurable functions $g \in h^p$. Take a C_c^∞ function ψ with $\int_{\mathbb{R}^n} \psi(x)dx = 1$. We have

$$\lim_{t\to 0} \psi_t * g = g \quad a.e. \text{ on } \mathbb{R}^n.$$

Thus,

$$\|g\|_{L^p} = \|\lim_{t\to 0} \psi_t * g\|_{L^p} \le \|\sup_{0<t<1} |\psi_t * g|\|_{L^p} = \|g\|_{h^p},$$

where we use the definition of h^p in the last equality.

For $X = M^{p,p}, B^{p,p}, F^{p,p}$, the inequality $\|g\|_{L^p} \lesssim \|g\|_X$ follows directly by the definition of function space and the triangle inequality, we only show the details for $B^{p,p}$:

$$\|f\|_{L^p} = \left\|\sum_{j=0}^\infty \Delta_j f\right\|_{L^p} \le \left(\sum_{j=0}^\infty \|\Delta_j f\|_{L^p}^p\right)^{1/p} = \|f\|_{B^{p,p}} = \|f\|_{F^{p,p}}.$$

Next, we turn to the proof of statement (2). First, we deal with the case $Y = M_s^{p,q}$. Using the triangle inequality, we have

$$\|\varphi * f\|_{L^p} = \left\|\sum_{k\in\mathbb{Z}^n} \Box_k(\varphi * f)\right\|_{L^p} \le \left(\sum_{k\in\mathbb{Z}^n} \|\Box_k(\varphi * f)\|_{L^p}^p\right)^{1/p}.$$

Moreover, there exists a constant c_n such that $\Box_k = \sum_{|l|\le c_n} \Box_{k+l} \circ \Box_k$, then

$$\|\Box_k(\varphi * f)\|_{L^p} = \left\|\sum_{|l|\le c_n} \Box_{k+l}\varphi * \Box_k f\right\|_{L^p} \lesssim \sum_{|l|\le c_n} \|\Box_{k+l}\varphi\|_{L^p} \|\Box_k f\|_{L^p}.$$

The above two estimates then yield that

$$\begin{aligned}
\|\varphi * f\|_{L^p} &\lesssim \left(\sum_{k\in\mathbb{Z}^n} \left(\sum_{|l|\le c_n} \|\Box_{k+l}\varphi\|_{L^p} \|\Box_k f\|_{L^p}\right)^p\right)^{1/p} \\
&\le \sup_{k\in\mathbb{Z}^n} \langle k\rangle^s \|\Box_k f\|_{L^p} \left(\sum_{k\in\mathbb{Z}^n} \left(\langle k\rangle^{-s} \sum_{|l|\le c_n} \|\Box_{k+l}\varphi\|_{L^p}\right)^p\right)^{1/p} \\
&\lesssim \|f\|_{M_s^{p,\infty}} \|\varphi\|_{M_{-s}^{p,p}} \lesssim \|f\|_{M_s^{p,q}},
\end{aligned}$$

where we use the fact $\varphi \in M_{-s}^{p,q}$ and $M_s^{p,q} \subset M_s^{p,\infty}$.

For $Y = \mathscr{F}M^{q,p}$. By the conclusion for $Y = M^{p,\infty}$, we have $\|\varphi * f\|_{L^p} \lesssim \|f\|_{M^{p,\infty}}$. Note that $\mathscr{F}M^{q,p}$ is equal to the Wiener amalgam space $W^{p,q}$ (see[11, pp.10]), and recall the embedding relations (see [10, Lemma 2.5]):

$$\mathscr{F}M^{q,p} \subset \mathscr{F}M^{\infty,p} = W^{p,\infty} \subset M^{p,\infty}.$$

The conclusion follows by

$$\|\varphi * f\|_{L^p} \lesssim \|f\|_{M^{p,\infty}} \lesssim \|f\|_{\mathscr{F}M^{q,p}}.$$

For $Y = B^{p,q}_s$, there exists a constant $\delta(p,q) > 0$ such that $B^{p,q}_s \subset M^{p,q}_{s-\delta(p,q)}$ (see [9, Theorem 1.2]). This and the conclusion for $Y = M^{p,q}_{s-\delta(p,q)}$ imply that

$$\|\varphi * f\|_{L^p} \lesssim \|f\|_{M^{p,q}_{s-\delta(p,q)}} \lesssim \|f\|_{B^{p,q}_s}.$$

For $Y = F^{p,q}_s$, take a constant $\delta > 0$, then $F^{p,q}_s \subset B^{p,q}_{s-\delta}$ (see [21, pp.47]). This and the conclusion for $Y = B^{p,q}_{s-\delta}$ imply that

$$\|\varphi * f\|_{L^p} \lesssim \|f\|_{B^{p,q}_{s-\delta}} \lesssim \|f\|_{F^{p,q}_s}.$$

□

3. Main Theorems

In this section, we give our main theorems and their proofs. Suppose X is a (Quasi-)Banach space with translation invariant: $\|T_y f\|_X = \|f\|_X$, where $T_y f(x) := f(x-y)$ is the translation of f.

Theorem 3.1. *Let $0 < p < 1$, $\Phi \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$. Suppose $C^\infty_c(\mathbb{R}^n) \subset X$ and $\|g\|_{L^p} \lesssim \|g\|_X$ for all measurable function $g \in X$. We have*

$$H_\Phi \text{ is bounded on } X \Longleftrightarrow H_\Phi = 0.$$

Proof. The "if" part is trivial. We only present the proof for the "only if" part.

We first point out that $H_\Phi f$ is pointwise well-defined for any smooth function f supported away from the origin. In fact, in this case we denote $E_x := \{y : f\big(\frac{x}{|y|}\big) \neq 0\}$. Observe that for any fixed $x \in \mathbb{R}^n$, E_x is a bounded measurable set away from the origin. Recalling that $\Phi \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$, then the following integral is convergent:

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy = \int_{E_x} \Phi(y) f\left(\frac{x}{|y|}\right) dy.$$

Moreover, $H_\Phi f$ is a measurable function on $\mathbb{R}^n$. Using the polar coordinates, we write

$$\begin{aligned} H_\Phi f(x) &= \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy \\ &= \int_0^\infty \int_{\mathbb{S}^{n-1}} \Phi(ry') f(x/r) d\sigma(y') r^{n-1} dr =: \int_0^\infty \phi(r) f(x/r) dr, \end{aligned}$$

where

$$\phi(r) := \int_{\mathbb{S}^{n-1}} \Phi(ry') r^{n-1} d\sigma(y').$$

It follows by $\Phi \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$ that $\phi \in L^1_{loc}(\mathbb{R}^+)$. Since ϕ is a measurable function on $(0,\infty)$, almost every point in $(0,\infty)$ is a Lebesgue point. Hence, it suffices to verify that if there exists a Lebesgue point $r_0 > 0$ such that $\phi(r_0) \neq 0$, then H_Φ is unbounded.

Without loss of generality, we assume that $r_0 = 1$ is a Lebesgue point of ϕ, satisfying $\phi(1) = 1$. The proof for other cases is similar.

Taking $\theta = \frac{1-p}{2}$, we set

$$A_j = [1 - 2^{-\theta j}, 1 + 2^{-\theta j}].$$

Take g to be a nonnegative smooth function supported on $B(0,2)$, satisfying $g=1$ on $B(0,1)$. Denote by $g_j(x):=g(x-2^je_0)$ the translation of g, where $e_0=(1,0,0,\cdots,0)$ be the unit vector on $\mathbb{R}^n$. For sufficiently large j, we have

$$(2^j+2)(1-2^{-\theta j})+1<(2^j-2)(1+2^{-\theta j})-1.$$

Denote

$$E_j:=\bigcup_{a\in\mathbb{R}:(2^j+2)(1-2^{-\theta j})+1\le a\le(2^j-2)(1+2^{-\theta j})-1}B(ae_0,1/2).$$

We have $|E_j|\sim 2^{(1-\theta)j}$. For $x\in E_j$, we have

$$(2^j+2)(1-2^{-\theta j})\le|x|\le(2^j-2)(1+2^{-\theta j}),\quad |x|\sim 2^j.$$

This implies that

$$\frac{|x|}{2^j-2}\le 1+2^{-\theta j},\quad \frac{|x|}{2^j+2}\ge 1-2^{-\theta j}. \tag{3.1}$$

Recall $\operatorname{supp} g\subset B(0,2)$. For every $x\in E_j$, set

$$E^0_{j,x}:=\{r:g_j\big(\frac{x}{r}\big)\ne 0\},\qquad E^1_{j,x}:=\{r:g_j\big(\frac{x}{r}\big)=1\}.$$

By a direct calculation and (3.1), we deduce that for $x\in E_j$,

$$E^1_{j,x}\subset E^0_{j,x}\subset\left(\frac{|x|}{2^j+2},\frac{|x|}{2^j-2}\right)\subset A_j. \tag{3.2}$$

Since $|x|\sim 2^j$ for $x\in E_j$, we have the upper estimate of $|E^0_{j,x}|$:

$$|E^0_{j,x}|\le\left|\frac{|x|}{2^j-2}-\frac{|x|}{2^j+2}\right|\sim 2^{-j}.$$

Next, we turn to the lower estimate of $|E^1_{j,x}|$. For $x\in E_j$, write $x=ae_0+y$ for $y\in B(0,1/2)$, we have

$$\begin{aligned} r\in E^1_{j,x}\Longleftarrow\left|\frac{x}{r}-2^je_0\right|\le 1 &\Longleftrightarrow\left|\frac{ae_0+y}{r}-2^je_0\right|\le 1\\ &\Longleftarrow\left|\frac{ae_0}{r}-2^je_0\right|+\left|\frac{y}{r}\right|\le 1\\ &\Longleftarrow\left|\frac{a}{r}-2^j\right|\le\frac{1}{4},\quad(\text{if } r\ge\frac{2}{3})\\ &\Longleftrightarrow r\in\left[\frac{a}{2^j+1/4},\frac{a}{2^j-1/4}\right]. \end{aligned}$$

Observe that

$$\lim_{j\to\infty}\frac{a}{2^j+1/4}=\lim_{j\to\infty}\frac{a}{2^j-1/4}=1$$

for $a\in[(2^j+2)(1-2^{-\theta j})+1,(2^j-2)(1+2^{-\theta j})-1]$. For sufficiently large j, we deduce that $\left[\frac{a}{2^j+1/4},\frac{a}{2^j-1/4}\right]\subset[2/3,\infty)$. Hence,

$$\left[\frac{a}{2^j+1/4},\frac{a}{2^j-1/4}\right]\subset E^1_{j,x}.$$

The lower estimate of $|E^1_{j,x}|$ follows by

$$|E^1_{j,x}| \geq \left|\frac{a}{2^j - 1/4} - \frac{a}{2^j + 1/4}\right| \sim 2^{-j}.$$

The combination of lower estimate of $|E^1_{j,x}|$ and upper estimate of $|E^0_{j,x}|$ yields that

$$2^{-j} \lesssim |E^1_{j,x}| \leq |E^0_{j,x}| \lesssim 2^{-j}, \quad \text{or equivalent: } |E^1_{j,x}| \sim |E^0_{j,x}| \sim 2^{-j}.$$

Next, we divide the Hausdorff operator into main term and error term by

$$\begin{aligned} H_\Phi g_j(x) &= \int_0^\infty \phi(r) g_j(x/r) dr = \int_0^\infty \phi(1) g_j(x/r) dr + \int_0^\infty (\phi(r) - \phi(1)) g_j(x/r) dr \\ &=: H^M_\Phi g_j(x) + H^E_\Phi g_j(x). \end{aligned}$$

Let us first turn to the estimate of main term. For $x \in E_j$, we have

$$H^M_\Phi g_j(x) = \int_0^\infty g_j(x/r) dr \geq \int_{E^1_{j,x}} g_j(x/r) dr = |E^1_{j,x}| \sim 2^{-j}.$$

Recalling $|E_j| \sim 2^{(1-\theta)j}$ and $\theta = \frac{1-p}{2}$, we have following estimate of the main term:

$$\|H^M_\Phi g_j\|^p_{L^p(E_j)} \gtrsim 2^{-jp} |E_j| \sim 2^{-jp} 2^{(1-\theta)j} = 2^{\frac{(1-p)j}{2}}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} \|H^E_\Phi g_j\|^p_{L^p(E_j)} \leq& \|H^E_\Phi g_j\|^p_{L^1(E_j)} |E_j|^{1-p} \\ \leq& \left(\int_{E_j} \int_0^\infty |\phi(r) - \phi(1)| g_j(x/r) dr dx\right)^p |E_j|^{1-p} \\ =& \left(\int_{E_j} \int_{E^0_{j,x}} |\phi(r) - \phi(1)| g_j(x/r) dr dx\right)^p |E_j|^{1-p}. \end{aligned} \tag{3.4}$$

Note that

$$\{(x,r) : x \in E_j, r \in E^0_{j,x}\} \subset \{(x,r) : r \in A_j, x \in \mathbb{R}^n\}.$$

We deduce that

$$\begin{aligned} &\int_{E_j} \int_{E^0_{j,x}} |\phi(r) - \phi(1)| g_j(x/r) dr dx \\ \leq& \int_{A_j} |\phi(r) - \phi(1)| \int_{\mathbb{R}^n} g_j(x/r) dx dr \\ =& \|g\|_{L^1} \int_{A_j} |\phi(r) - \phi(1)| r^n dr \lesssim \int_{A_j} |\phi(r) - \phi(1)| dr = \epsilon_j |A_j| \lesssim \epsilon_j 2^{-\theta j}, \end{aligned}$$

where $\epsilon_j \to 0^+$ as $j \to \infty$. Combining this with (3.4), we have

$$\|H^E_\Phi g_j\|^p_{L^p(E_j)} \lesssim \epsilon^p_j 2^{-\theta p j} |E_j|^{1-p} \lesssim \epsilon^p_j 2^{-\theta p j} 2^{(1-\theta)(1-p)j} = \epsilon^p_j 2^{\frac{(1-p)j}{2}}. \tag{3.5}$$

By (3.3) and (3.5), there exist two constants C_1 and C_2 such that

$$\|H^M_\Phi g_j\|^p_{L^p(E_j)} \geq C_1 2^{\frac{(1-p)j}{2}}, \quad \|H^E_\Phi g_j\|^p_{L^p(E_j)} \leq C_2 \epsilon^p_j 2^{\frac{(1-p)j}{2}}.$$

For sufficiently large j such that $C_2\epsilon_j^p \leq C_1/2$, we have

$$\begin{aligned}\|H_\Phi g_j\|_{L^p}^p \geq & \|H_\Phi g_j\|_{L^p(E_j)}^p \\ \geq & \|H_\Phi^M g_j\|_{L^p(E_j)}^p - \|H_\Phi^E g_j\|_{L^p(E_j)}^p \\ \geq & C_1 2^{\frac{(1-p)j}{2}} - C_2\epsilon_j^p 2^{\frac{(1-p)j}{2}} \geq (C_1/2)2^{\frac{(1-p)j}{2}}.\end{aligned}$$

However, the boundedness of H_Φ yields that

$$\|H_\Phi g_j\|_{L^p}^p \lesssim \|H_\Phi g_j\|_X^p \lesssim \|g_j\|_X^p = \|g\|_X^p,$$

which leads to a contradiction. □

Recall that all the spaces $L^p, h^p, M^{p,p}, B^{p,p}, F^{p,p}$ are translation invariant. The following corollary is a direct conclusion of Lemma 2.2 and Theorem 3.1.

Corollary 3.2. *Let $0 < p < 1$, $\Phi \in L^1_{loc}(\mathbb{R}^n\backslash\{0\})$. We have*

$$H_\Phi \text{ is bounded on } X \Longleftrightarrow H_\Phi = 0,$$

where $X = L^p, h^p, M^{p,p}, B^{p,p}, F^{p,p}$.

For more general frequency decomposition spaces such as $X = B_s^{p,q}$, $F_s^{p,q}$ or $M_s^{p,q}$, the embedding condition $\|g\|_{L^p} \lesssim \|g\|_X$ is no longer valid. We establish following theorem with the help of the modified embedding (see Lemma 2.2).

Theorem 3.3. *Let $0 < p < 1$, $\Phi \in L^1_{loc}(\mathbb{R}^n\backslash\{0\})$. Suppose $C_c^\infty(\mathbb{R}^n) \subset X$ and $X \hookrightarrow L_0^p(\mathbb{R}^n)$. We have*

$$H_\Phi \text{ is bounded on } X \Longleftrightarrow H_\Phi = 0.$$

Proof. The "if" part is trivial. We focus on the proof for the "only if" part. As the proof of Theorem 3.1, we write $H_\Phi f(x) = \int_0^\infty \phi(r)f(x/r)dr$, and assume $r_0 = 1$ is the Lebesgue point of ϕ with $\phi(1) = 1$. Let $g_j, A_j, E_j, E_{j,x}^0, E_{j,x}^1, H_\Phi^M g_j, H_\Phi^E g_j$ be as in the proof of Theorem 3.1. Take a $C_c^\infty(\mathbb{R}^n)$ nonnegative function φ satisfying $\varphi(x) = 1$ on $B(0,1)$ and $\mathrm{supp}\varphi \subset B(0,2)$. By the assumption we have

$$\|\varphi * H_\Phi g_j\|_{L^p(\mathbb{R}^n)} \lesssim \|H_\Phi g_j\|_X \lesssim \|g_j\|_X = \|g\|_X.$$

Set

$$\Xi_j = \{x : B(x,2) \subset E_j\}.$$

We have

$$|\Xi_j| \sim |E_j| \sim 2^{(1-\theta)j} \quad (j \to \infty).$$

Recall that $H_\Phi^M g_j(x) \geq 0$ for $x \in \mathbb{R}^n$ and $H_\Phi^M g_j(x) \gtrsim 2^{-j}$ for $x \in E_j$. For $x \in \Xi_j$ we have

$$\begin{aligned}\varphi * H_\Phi^M g_j(x) = & \int_{\mathbb{R}^n} \varphi(x-z)H_\Phi^M g_j(z)dz \\ \geq & \int_{B(x,1)} H_\Phi^M g_j(z)dz \gtrsim \int_{B(x,1)} 2^{-j}dz \sim 2^{-j}.\end{aligned}$$

From this and the fact $|\Xi_j| \sim 2^{(1-\theta)j}$, we deduce that

$$\|\varphi * H_\Phi^M g_j\|_{L^p(\Xi_j)}^p \gtrsim 2^{-jp}|\Xi_j| \gtrsim 2^{-jp}2^{(1-\theta)j} = 2^{\frac{(1-p)j}{2}}.$$

On the other hand, observing that $E^0_{j,z} \subset A_j$ for $z \in B(x,2)$ with $x \in \Xi_j$,

$$\begin{aligned}
\|\varphi * H^E_\Phi g_j\|^p_{L^p(\Xi_j)} \leq & \|\varphi * H^E_\Phi g_j\|^p_{L^1(\Xi_j)} |\Xi_j|^{1-p} \\
\leq & \left(\int_{\Xi_j} \int_{B(x,2)} \varphi(x-z) \int_0^\infty |\phi(r) - \phi(1)| g_j(z/r) dr dz dx \right)^p |\Xi_j|^{1-p} \\
= & \left(\int_{\Xi_j} \int_{B(x,2)} \varphi(x-z) \int_{E^0_{j,z}} |\phi(r) - \phi(1)| g_j(z/r) dr dz dx \right)^p |\Xi_j|^{1-p} \\
\leq & \left(\int_{A_j} |\phi(r) - \phi(1)| \int_{\mathbb{R}^n} g_j(z/r) \int_{\mathbb{R}^n} \varphi(x-z) dx dz dr \right)^p |\Xi_j|^{1-p} \\
\lesssim & \left(\int_{A_j} |\phi(r) - \phi(1)| dr \right)^p |\Xi_j|^{1-p} \sim \epsilon_j^p |A_j|^p |\Xi_j|^{1-p} \sim \epsilon_j^p 2^{\frac{(1-p)j}{2}},
\end{aligned}$$

where $\epsilon_j \to 0^+$ as $j \to \infty$. Now, we have finished the estimates of main term $\|\varphi * H^M_\Phi g_j\|^p_{L^p(\Xi_j)}$ and error term $\|\varphi * H^E_\Phi g_j\|^p_{L^p(\Xi_j)}$, the remainder of this proof is the same as that of Theorem 3.1. $\square$

Using Theorem 3.3 and Lemma 2.2, we have following corollary.

Corollary 3.4. *Let $0 < q \leq \infty, s \in \mathbb{R}$, $\Phi \in L^1_{loc}(\mathbb{R}^n \backslash \{0\})$. If $0 < p < 1$, we have*

$$H_\Phi \text{ is bounded on } X \Longleftrightarrow H_\Phi = 0,$$

where $X = F^{p,q}_s$, $B^{p,q}_s$ or $M^{p,q}_s$.

In particular, due to the time-frequency symmetry of modulation space, we have following corollary.

Corollary 3.5. *Let $0 < p, q \leq \infty$. Let Φ be a measurable function satisfying*

$$\int_{B(0,1)} |y|^n \Phi(y) dy < \infty, \qquad \int_{B(0,1)^c} \Phi(y) dy < \infty.$$

If $0 < p < 1$ or $0 < q < 1$, we have

$$H_\Phi \text{ is bounded on } M^{p,q} \Longleftrightarrow H_\Phi = 0.$$

Proof. The "if" part is trivial. We focus on the proof for the "only if" part.

If $0 < p < 1$, we have $M^{p,q} \hookrightarrow L^p_0(\mathbb{R}^n)$, then the conclusion follows by Theorem 3.3.

If $p \geq 1$, $0 < q < 1$, we will use the Fourier transform to exchange the time and frequency space. It follows by Remark 1.4 in [22] that

$$\widehat{H_\Phi f} = \widetilde{H_\Phi} \widehat{f},$$

where

$$\widetilde{H_\Phi} f(x) = \int_{\mathbb{R}^n} \Phi(y) |y|^n f(|y|x) dy.$$

Thus,

$$\|H_\Phi f\|_{M^{p,q}} = \|\widehat{H_\Phi f}\|_{\mathscr{F}M^{p,q}} = \|\widetilde{H_\Phi} \widehat{f}\|_{\mathscr{F}M^{p,q}}.$$

If H_Φ is bounded on $M_{p,q}$, we have

$$\|\widetilde{H_\Phi} f\|_{\mathscr{F}M^{p,q}} = \|H_\Phi \check{f}\|_{M^{p,q}} \lesssim \|\check{f}\|_{M^{p,q}} = \|f\|_{\mathscr{F}M^{p,q}}.$$

Write

$$\begin{aligned}
\widetilde{H_\Phi} f(x) &= \int_{\mathbb{R}^n} \Phi(y)|y|^n f(|y|x)dy \\
&= \int_0^\infty \int_{\mathbb{S}^{n-1}} \Phi(ry')r^n f(rx)d\sigma(y')r^{n-1}dr \\
&= \int_0^\infty \int_{\mathbb{S}^{n-1}} \Phi(y'/r)r^{-1-2n} f(x/r)d\sigma(y')dr =: \int_0^\infty \widetilde{\phi}(r)f(x/r)dr,
\end{aligned}$$

where

$$\widetilde{\phi}(r) = \int_{\mathbb{S}^{n-1}} \Phi(y'/r)r^{-1-2n}d\sigma(y').$$

Observe $\widetilde{\phi} \in L^1_{loc}(\mathbb{R}^+)$ and recall $\mathscr{F}M^{p,q} \hookrightarrow L^q_0(\mathbb{R}^n)$ with $q \in (0,1)$. By the same argument as in the proofs of Theorem 3.1 and 3.3, we conclude that $\widetilde{\phi} = 0$ and complete this proof. $\square$

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