ON BOUNDARY EXACT CONTROLLABILITY OF ONE-DIMENSIONAL WAVE EQUATIONS WITH WEAK AND STRONG INTERIOR DEGENERATION

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Abstract. In this paper we study exact boundary controllability for a linear wave equation with strong and weak interior degeneration of the coefficients in the principle part of the elliptic operator. The objective is to provide a well-posedness analysis of the corresponding system and derive conditions for its controllability through boundary actions. Passing to a relaxed version of the original problem, we discuss existence and uniqueness of solutions, and using the HUM method we derive conditions on the rate of degeneracy for both exact boundary controllability and the lack thereof.

Key words. Degenerate wave equation, boundary control, existence result, weighted Sobolev spaces, exact controllability.

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1. Introduction. In this paper we discuss exact boundary controllability for one-dimensional degenerate wave equations with a weak and strong interior degeneration in the principle part of the elliptic operator. Let $[0,T]$ be a given time interval. For simplicity, let c and d be a given pair of real numbers such that $0 \le c < 1 < d \le 2$. We set

$$\Omega_1 = (c,1), \quad \Omega_2 = (1,d), \quad \Omega = (c,d), \quad \text{and} \quad \Omega_0 = \Omega \setminus \{1\} = \Omega_1 \cup \Omega_2.$$

Let $a : \overline{\Omega} \to \mathbb{R}$ be a given weight function with properties

(i) $a(1) = 0$, $a(x) > 0$ for all $x \in \Omega_0$, and there exists subintervals $(x_1^*, 1) \subset \Omega_1$ and $(1, x_2^*) \subset \Omega_2$ such that $a(\cdot)$ is monotonically decreasing on $(x_1^*, 1)$ and monotonically increasing on $(1, x_2^*)$;

(ii) $a \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{1\})$;

(iii) $(\sqrt{a})_x \notin L^\infty(\Omega)$ whereas $(\sqrt{a})_x^{-1} \in L^\infty(\Omega)$.

We are concerned with the following controlled system

$$y_{tt} - (a(x) y_x)_x = 0 \quad \text{in} \ \ (0,T) \times \Omega, \tag{1.1}$$

$$y(t,c) = 0, \quad y(t,d) = f(t) \quad \text{on} \ \ (0,T), \tag{1.2}$$

$$y(0,\cdot) = y_0, \quad y_t(0,\cdot) = y_1 \quad \text{in} \ \ \Omega, \tag{1.3}$$

$$f \in \mathcal{F}_{ad} = L^2(0,T). \tag{1.4}$$

Here, y_0, and y_1 are given functions, and $\mathcal{F}_{ad}$ stands for the class of admissible controls.

(1.1)-(1.4) describes the dynamics of a linear elastic string with out-of-the-plane displacement under the action of a boundary source f acting on the system as a control through the Dirichlet boundary condition at $x = d$. The coefficient $a(x)$ can be interpreted as the spatially varying stiffness (modulus of elasitcity) of the elastic string. In contrast to the

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standard case that is widely studied in the literature (see, for instance, [18]), where the stiffness is assumed to be positive and bounded away from zero, we assume that the string $[c,d]$ has a defect at the internal point $x_0 = 1$. In case the defect occurs at the endpoint $x_0 = c$, the problem has been investigated in [1]. In the latter case, the spatial operator is related to a classical singular Sturm-Liouville-problem that has been treated already by Weyl in [22]. Degeneration in that context is related to the notions of limit-point and limit-cycle. In [1], the authors define

$$\mu_a := \sup_{0<x\le\ell} \frac{x|a'(x)|}{x} \tag{1.5}$$

for the problem on the interval $[0,\ell]$. The problem is called weakly damaged if $0 \le \mu_a < 1$ in which case $\frac{1}{a} \in L^1$, and strongly damaged in case $1 \le \mu_a < 2$. The authors show, among other things, one-sided boundary observability and consequently one-sided boundary exact controllability if $\mu_a < 2$ with an observability/controllability time approaching $+\infty$ as $\mu_a \to 2$. Thus, for $\mu_a \ge 2$, these properties are lost.

Using the Liouville transform (see [7] 1954), it is possible to transform the system above into a homogeneous wave equation with singular potential on an interval that tends to infinity if $\mu_a \ge 2$. See e.g. [8], where controllability properties are investigated based on this transformation. Working in the L^∞-framework, the author obtains similar results as in [1].

The authors of this article are not aware of any publication, where in-span degeneration of the wave equation is treated, in particular in the context of controllability or observability. For the parabolic case see [3]. The main question that we are going to discuss in this paper, therefore, is how the defect at the internal point $x_0 = 1$ affects the transmission conditions at the singular (damage) point and the corresponding solution of the system (1.1)–(1.4) as well as its its observability or controllability properties.

Such analysis could be important for applications, in particular when extended into higher dimensions, e.g. for the cloaking problem (building of devices that lead to invisibility, i.e. to lack of obeservability) [11], where a quadratic singular behaviour of the material properties - strong damage in our terminology -around the to be cloaked object occurs, the evolution of damage in materials, optimization problems for elastic bodies arising in contact mechanics, coupled systems, composite materials, where 'life-cycle-optimization' appears as a challenge.

The indicated type of degeneracy raises a number of new questions related to the well-posedness of the hyperbolic equations in suitable functional spaces as well as new estimates for their solutions. Hence, new tools are necessary for the analysis of the corresponding optimal control problems. It should be emphasized here that boundary value problems for degenerate elliptic and parabolic equations have received a lot of attention in the last years (see, for instance, [3, 4, 6, 16, 17, 19]). As for the control issue for degenerate wave equations, we already mentioned [1, 10] (see also [13] for the sensitivity analysis of OCPs for wave equations in domain with defects).

The purpose of this paper is to provide a qualitative analysis of system (1.1)–(1.4), prove an exact controllability result, and investigate how the degree of degeneracy in the principle coefficient $a(x)$ affects the system (1.1)–(1.4) and its solution. In contrast to the recent results [1], where the authors study controllability and observability for degenerate equation of the form (1.1) with the degeneracy of (1.1) at the boundary $x = c = 0$, we focus on the case where the 'damaged' point is internal. So, our core idea is to to pass from the original initial-boundary value problem (1.1)–(1.4) to a relaxed version, namely, to some transmission problem with appropriate compatibility conditions at the 'damaged' point. We show that these conditions play a crucial role and essentially depend on the 'degree of degeneracy'.

In Section 2 we introduce a special class of weighted Sobolev spaces that are associated with the original initial-boundary value problem. It allows us not only to to study in detail some properties of their elements in the regions which are in close vicinity to the 'damaged' point, but also propose an appropriate relaxation for to the initial-boundary value problems (1.1)–(1.3). In Section 3, we mainly focus on the well-posedness of the proposed relaxation for the original controlled system. It allows us to consider in Section 4 the issues related to the boundary observability of degenerate wave equations. In Section 5 we discuss the questions of exact and null boundary controllability of the original degenerate system and the lack of these properties for strong degeneration.

2. Preliminaries. To specify the original controlled system (1.1)–(1.4) and fix the main ideas, we begin with some preliminaries and assumptions. Let $a:\overline{\Omega}\to\mathbb{R}$ be a given weight function with properties (i)–(iii). We define the Banach spaces $W_0^{1,2}(\Omega;c)$ and $H_a^1(\Omega)$ as the closure of the set $C_0^\infty(\mathbb{R};c)=\{\varphi\in C_0^\infty(\mathbb{R})\ :\ \varphi(c)=0\}$ with respect to the norms

$$\|y\|_{W_0^{1,2}(\Omega;c)}=\left(\int_\Omega |y_x|^2\,dx\right)^{1/2}\quad\text{and}\quad \|y\|_{H_a^1(\Omega)}=\left(\int_\Omega \left(y^2+a|y_x|^2\right)\,dx\right)^{1/2},$$

respectively.

We also introduce the closed subspace $H_{a,0}^1(\Omega)$ of $H_a^1(\Omega)$ defined as

$$H_{a,0}^1(\Omega):=\left\{y\in H_a^1(\Omega)\ :\ y(c)=0\right\}.$$

We note that this subspace is correctly defined due to compactness of the embedding $H_{a,0}^1(c,\varepsilon)\subset W_0^{1,2}((c,\varepsilon);c)\hookrightarrow C([c,\varepsilon])$, for any $\varepsilon\in(c,1)$. So, if $y\in H_{a,0}^1(\Omega)$, then $y(\cdot)$ is a continuous function at $x=c$, and, therefore, the condition $y(c)=0$ is consistent.

The common characteristic of the weight functions $a:\overline{\Omega}\to\mathbb{R}$ with properties (i)–(iii) can be summarizing as follows (see [12] for the details).

Let $G_i:\Omega\to[0,\infty)$, $i=1,2$, be non-decreasing continuous functions such that $G_i(0)=0$, and let

$$A_{1,a}:=\sup_{x\in\Omega_1}\frac{G_1(1-x)\left|\left(\sqrt{a(x)}\right)_x\right|}{\sqrt{a(x)}}=\sup_{x\in\Omega_1}\frac{G_1(1-x)|a'(x)|}{2a(x)}<+\infty,\tag{2.1}$$

$$A_{2,a}:=\sup_{x\in\Omega_2}\frac{G_2(x-1)\left|\left(\sqrt{a(x)}\right)_x\right|}{\sqrt{a(x)}}=\sup_{x\in\Omega_2}\frac{G_2(x-1)|a'(x)|}{2a(x)}<+\infty.\tag{2.2}$$

By analogy with [1], we also set

$$\mu_{1,a}:=\sup_{x\in\Omega_1}\frac{(1-x)|a'(x)|}{a(x)},\quad \mu_{2,a}:=\sup_{x\in\Omega_2}\frac{(x-1)|a'(x)|}{a(x)}.\tag{2.3}$$

We make use of the following result (for the proof we refer to [12], see Theorems 3.1 and 3.2).

Proposition 2.1. *Let $a:\overline{\Omega}\to\mathbb{R}$ be a weight function satisfying properties (i)–(iii). Then the following assertions hold true:*

$$0\le\max\{2A_{1,a},\mu_{1,a}\}<2\quad\textit{and}\quad 0\le\max\{2A_{2,a},\mu_{2,a}\}<2,\tag{2.4}$$

$$a(x)\ge a(c)(1-x)^{\max\{\mu_{1,a},2A_{1,a}\}}\quad\forall\,x\in[c,1],\tag{2.5}$$

$$a(x)\ge a(d)(x-1)^{\max\{\mu_{2,a},2A_{2,a}\}}\quad\forall\,x\in[1,d],\tag{2.6}$$

$$\|y\|_{L^2(\Omega)}\le[C_1+C_2]\,\|y\|_{H_a^1(\Omega)},\quad\forall\,y\in H_{a,0}^1(\Omega),\tag{2.7}$$

where

$$C_1 = \frac{1}{\sqrt{a(c)}} \min\left\{ \frac{1}{\sqrt{[2-\max\{\mu_{1,a}, 2A_{1,a}\}]}}, 2 \right\}, \tag{2.8}$$

$$C_2 = \frac{1}{\sqrt{a(d)}} \left[\min\left\{ \frac{1}{\sqrt{[2-\max\{\mu_{2,a}, 2A_{2,a}\}]}}, 2 \right\} + C_{Sob}\sqrt{1+a(d)} \right], \tag{2.9}$$

and $C_{Sob} > 0$ is a constant coming from the continuous embedding $W^{1,2}(1,d) \hookrightarrow C^{0,1}([1,d])$,

$$\max_{x\in[1,d]} |y(x)| \le C_{Sob}\|y\|_{W^{1,2}(1,d)}, \quad \forall\, y \in W^{1,2}(1,d).$$

Remark 2.1. *In order to avoid any ambiguity coming from the choice of functions $G_i : \Omega \to [0,\infty)$, $i = 1,2$, we set $G_i(x) = 2x$, $i = 1,2$. Then $2A_{i,a} = \mu_{i,a}$ and, therefore the estimates (2.4)–(2.7) can be simplified. Hereinafter in this paper, we will follow this agreement.*

As a direct consequence, we have: Under the assumptions of Proposition 2.1, $H^1_{a,0}(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_{H^1_{a,0}(\Omega)} = \int_\Omega a(x)u'(x)v'(x)\,dx, \quad \forall\, u, v \in H^1_{a,0}(\Omega). \tag{2.10}$$

The proposition given below reveals some extra properties of the weight function $a(x)$ that can be interesting per se.

Proposition 2.2. *Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii). Then*

$$\frac{\sqrt{a}}{|x-1|} \notin L^q(\Omega) \qquad \forall\, q \ge q^*, \tag{2.11}$$

$$\left(\sqrt{a}\right)_x \in L^q(\Omega) \quad \textit{provided} \quad \frac{\sqrt{a}}{|x-1|} \in L^q(\Omega), \quad \textit{for some} \quad q \in [1, q^*), \tag{2.12}$$

with

$$q^* = 2\min\left\{ \frac{1}{2-\mu_{1,a}}, \frac{1}{2-\mu_{2,a}}, \right\}.$$

Proof. Fixing an arbitrary function $a : \overline{\Omega} \to \mathbb{R}$ satisfying properties (i)–(iii), we see that, for a given exponent $q \ge 1$, the following inequality

$$\int_\Omega \left| \frac{\sqrt{a}}{|x-1|} \right|^q dx \overset{\text{by (2.5)–(2.6)}}{\ge} a^q(c)\int_{\Omega_1} \left| |x-1|^{\mu_{1,a}-2} \right|^{\frac{q}{2}} dx + a^q(d)\int_{\Omega_2} \left| |x-1|^{\mu_{2,a}-2} \right|^{\frac{q}{2}} dx$$

holds. Since the right hand side of this inequality blows up provided $q \ge q^*$, the first assertion (2.11) follows. As for (2.12),

$$\int_\Omega \left| \left(\sqrt{a}\right)_x \right|^q dx = \int_\Omega \left| \frac{|x-1|a_x}{2a} \frac{\sqrt{a}}{|x-1|} \right|^q dx \overset{\text{by (2.3)}}{\le} \int_\Omega \left(\left| \frac{\mu_{1,a}}{2} \right|^q + \left| \frac{\mu_{1,a}}{2} \right|^q \right) \left| \frac{\sqrt{a}}{|x-1|} \right|^q dx \overset{\text{by (2.4)}}{\le} 2\int_\Omega \left| \frac{\sqrt{a}}{|x-1|} \right|^q dx.$$

□

EXAMPLE 2.1. *As an example of function $a : \overline{\Omega} \to \mathbb{R}_+$ with the above indicated properties (i)–(iii), we may consider:*

$$a(x) = \begin{cases} (1-x)^{2p_1}, & \text{if } x \in [0,1] \\ (x-1)^{2p_2}, & \text{if } x \in (1,2], \end{cases} \qquad \text{with } p_1, p_2 > 0. \tag{2.13}$$

It is easy to check that, in this case, properties (i)–(iii) hold and in addition

$$\mu(x) = \begin{cases} \dfrac{p_1(p_1-1)}{(1-x)^2}, & \text{if } x \in [0,1] \\ \dfrac{p_2(p_2-1)}{(1-x)^2}, & \text{if } x \in (1,2], \end{cases}, \quad \mu_{1,a} = 2p_1, \quad \mu_{2,a} = 2p_2.$$

Moreover, setting $G_1(x) = k_1 x$ and $G_2(x) = k_2 x$, where k_1, k_2 are some positive constants, we obtain

$$A_{1,a} = \frac{k_1}{2}\mu_{1,a} = k_1 p_1 \quad \text{and} \quad A_{2,a} = \frac{k_2}{2}\mu_{2,a} = k_2 p_2.$$

In addition, we have the following properties

$$\left(\sqrt{a}\right)_x \notin L^\infty(\Omega) \quad \text{and} \quad \left(\sqrt{a}\right)_x^{-1} \in L^\infty(\Omega) \quad \text{if } p_1 \text{ and } p_2 \text{ are less than } 1,$$
$$\frac{1}{a} \in L^1(\Omega) \quad \text{provided} \quad 0 < p_1, p_2 < \frac{1}{2}.$$

In what follows, we will distinguish two possible cases for the weight function $a : \overline{\Omega} \to \mathbb{R}$. Namely, we say that we deal with

- a weak degeneration in (1.1) if $a : \overline{\Omega} \to \mathbb{R}$ satisfies properties (i)–(iii) and $1/a \in L^1(\Omega)$;
- a strong degeneration in (1.1) if $a : \overline{\Omega} \to \mathbb{R}$ satisfies properties (i)–(iii) and $1/a \notin L^1(\Omega)$.

Starting with the weak degenerate case, we note that due to the continuous embedding $W^{1,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ and estimates

$$\int_\Omega |y|\,dx \le |\Omega|^{1/2}\left(\int_\Omega |y|^2\,dx\right)^{1/2} \le \sqrt{|\Omega|}\,\|y\|_{H^1_a(\Omega)},$$
$$\int_\Omega |y_x|\,dx \le \left(\int_\Omega |y_x|^2 a\,dx\right)^{1/2}\left(\int_\Omega a^{-1}\,dx\right)^{1/2} \le C\|y\|_{H^1_a(\Omega)},$$

we have the following result (we refer to [1, Proposition 2.5] for the details).

THEOREM 2.3. *Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii) and $1/a \in L^1(\Omega)$, i.e., $a(x)$ belongs to the class of Muckenhoupt weights $A_2(\Omega)$. Then $W^{1,2}(\Omega) \hookrightarrow H^1_a(\Omega)$, $H^1_a(\Omega) \hookrightarrow W^{1,1}(\Omega)$, $H^1_a(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ compactly, and $H^1_a(\Omega)$ is continuously embedded into the class of absolutely continuous functions on $\overline{\Omega}$, so*

$$\lim_{x\nearrow 1} y(x) = \lim_{x\searrow 1} y(x), \quad |y(1)| < +\infty, \quad \forall\, y \in H^1_a(\Omega), \tag{2.14}$$

$$\lim_{x\nearrow 1} \sqrt{a(x)}y(x) = \lim_{x\searrow 1} \sqrt{a(x)}y(x) = 0, \quad \forall\, y \in H^1_a(\Omega). \tag{2.15}$$

In addition, if y is an arbitrary element of the space

$$H^2_a(\Omega) := \left\{ y \in H^1_a(\Omega) \ : \ a y_x \in W^{1,2}(\Omega) \right\}, \tag{2.16}$$

then the following transmission condition

$$\lim_{x \nearrow 1} a(x) y_x(x) = \lim_{x \searrow 1} a(x) y_x(x) = L, \quad \textit{with} \ \ |L| < +\infty, \tag{2.17}$$

holds true.

However, the situation changes drastically if we deal with strong degeneration in (1.1). Indeed, let us consider the following example. Let $c = 0$, $d = 2$, and

$$y(x) = \begin{cases} |x-1|^{-\frac{1}{4}} - 1, & \text{if } \ x \in (0,1), \\ |x-1|^{\frac{1}{2}}, & \text{if } \ x \in [1,2). \end{cases}$$

Setting $a(x) = |x-1|^{7/4}$, we see that properties (i)–(iii) hold true. Moreover, in this case we have $1/a \notin L^1(\Omega)$. Then, in spite of the fact that the function $y : \Omega \to \mathbb{R}$ has a discontinuity of the second kind at $x^0 = 1$, a direct calculations show that $y \in H^1_{a,0}(\Omega)$ and

$$a(x) y_x(x) = \begin{cases} -\frac{1}{4} |x-1|^{\frac{1}{2}}, & \text{if } \ x \in (0,1), \\ +\frac{1}{2} |x-1|^{\frac{5}{4}}, & \text{if } \ x \in [1,2). \end{cases}$$

So, instead of transmission conditions (2.14)–(2.17), we have

$$\lim_{x \nearrow 1} a(x) y_x(x) = \lim_{x \searrow 1} a(x) y_x(x) = 0. \tag{2.18}$$

In fact, we have the following result (see [1, Proposition 2.5] for comparison).

Theorem 2.4. *Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii) and $1/a \notin L^1(\Omega)$. Then the following assertions hold true:*

$$\lim_{x \nearrow 1} |x-1| y^2(x) = \lim_{x \searrow 1} |x-1| y^2(x) = 0, \quad \forall \, y \in H^1_a(\Omega), \tag{2.19}$$

$$\exists \, x_i \in \Omega_i, \ i = 1,2, \ \textit{such that} \ \ y(x) = o\left(|x-1|^{-\frac{1}{2}} \right) \ \textit{for a.a.} \ \ x \in (x_1, x_2), \tag{2.20}$$

$$\lim_{x \nearrow 1} \sqrt{a(x)} y(x) = \lim_{x \searrow 1} \sqrt{a(x)} y(x) = 0, \quad \forall \, y \in H^1_a(\Omega), \tag{2.21}$$

$$\lim_{x \nearrow 1} a(x) y_x(x) = \lim_{x \searrow 1} a(x) y_x(x) = 0, \quad \forall \, y \in H^2_a(\Omega), \tag{2.22}$$

$$\lim_{x \nearrow 1} |x-1| a(x) y_x(x)^2 = \lim_{x \searrow 1} |x-1| a(x) y_x(x)^2 = 0, \quad \forall \, y \in H^2_a(\Omega), \tag{2.23}$$

$$\lim_{x \nearrow 1} a(x) \varphi_x(x) y(x) = \lim_{x \searrow 1} a(x) \varphi_x(x) y(x) = 0, \quad \forall \, y \in H^1_{a,0}(\Omega), \ \forall \, \varphi \in H^2_a(\Omega), \tag{2.24}$$

$$a(d) y_x^2(d) \le 3 \|y\|^2_{H^1_a(\Omega)} + \frac{2}{\sqrt{a(d)}} \|y\|_{H^1_a(\Omega)} \|a y_x\|_{W^{1,2}(\Omega)}, \quad \forall \, y \in H^2_a(\Omega), \tag{2.25}$$

where the small symbol o stands for the Bachmann-Landau asymptotic notation.

Proof. Let $y \in H^1_a(\Omega)$. To begin with, let us show that the function

$$v(x) = \begin{cases} (1-x) y^2(x), & c \le x < 1, \\ 0, & x = 1, \\ (x-1) y^2(x), & 1 < x \le d \end{cases}$$

is continuous on $\overline{\Omega}$. Indeed, v is locally absolutely continuous in $\overline{\Omega}\setminus\{1\}$ and

$$v_x = \operatorname{sign}(x-1)\,y^2(x) + 2|x-1|y(x)y_x(x), \quad \text{a.e. in } \overline{\Omega}.$$

Since $y \in L^2(\Omega)$ and

$$\begin{aligned}
\int_\Omega |x-1|^2 y_x(x)^2\,dx &\overset{\text{by (2.4)}}{\leq} \int_{\Omega_1} |x-1|^{\mu_{1,a}} y_x(x)^2\,dx + \int_{\Omega_2} |x-1|^{\mu_{2,a}} y_x(x)^2\,dx \\
&\overset{\text{by (2.5)–(2.6)}}{\leq} \frac{1}{a(c)}\int_{\Omega_1} a(x)y_x(x)^2\,dx + \frac{1}{a(d)}\int_{\Omega_2} a(x)y_x(x)^2\,dx \\
&\leq \max\left\{\frac{1}{a(c)}, \frac{1}{a(d)}\right\}\|y\|^2_{H^1_a(\Omega)}, \qquad (2.26)
\end{aligned}$$

it follows that $v_x \in L^1(\Omega)$. Hence, v is an absolutely continuous functions and, as a consequence, the limits $\lim_{x\nearrow 1}|x-1|y^2(x) = \lim_{x\searrow 1}|x-1|y^2(x) = L$ do exist and must vanish, for otherwise $y(x)^2 \sim L/|x-1|$ (near the point $x_0 = 1$) would be not integrable. So, we come into conflict with the initial condition: $y \in L^2(\Omega)$. From this and the fact that $a(x) = O(|x-1|)$ in some neighborhood of $x = 1$, we immediately deduce properties (2.20)–(2.21).

To prove the equality (2.22), it is enough to observe that the function $a(x)y_x(x)$ is absolutely continuous. Hence, the limits $\lim_{x\nearrow 1} a(x)y_x(x) = \lim_{x\searrow 1} a(x)y_x(x) = L$ do exist and must vanish, for otherwise $a(x)y_x(x)^2 \sim L^2/a(x)$ (near the point $x_0 = 1$) would be not integrable. So, we come into conflict with the initial condition: $y \in H^1_a(\Omega)$.

It remains to establish properties (2.23)–(2.25). To do so, we set

$$v(x) = \begin{cases} (1-x)a(x)y_x(x)^2, & c \leq x < 1, \\ 0, & x = 1, \\ (x-1)a(x)y_x(x)^2, & 1 < x \leq d, \end{cases}$$

where y is an arbitrary element of $H^2_a(\Omega)$. Then $v(x)$ is continuous on Ω. Indeed, v is locally absolutely continuous in $\overline{\Omega}\setminus\{1\}$ and

$$\begin{aligned}
v_x(x) &= \operatorname{sign}(x-1)\,a(x)y_x(x)^2 + 2|x-1|y_x(x)\,(a(x)y_x(x))_x \\
&\quad - |x-1|a_x(x)y_x(x)^2 = I_1(x) + I_2(x) + I_3(x), \quad \text{a.e. in } \overline{\Omega}.
\end{aligned}$$

Since $y \in H^1_a(\Omega)$, it follows that $I_1 \in L^1(\Omega)$. The same conclusion is true for the second term I_2. Indeed, in view of estimate (2.26), we have

$$\begin{aligned}
\|I_2\|_{L^1(\Omega)} &\leq 2\left(\int_\Omega |x-1|^2 y_x(x)^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega (a(x)y_x(x))_x^2\,dx\right)^{\frac{1}{2}} \\
&\overset{\text{by (2.26)}}{\leq} 2\sqrt{\max\left\{\frac{1}{a(c)}, \frac{1}{a(d)}\right\}}\|y\|_{H^1_a(\Omega)}\|(ay_x)_x\|_{L^2(\Omega)} < +\infty.
\end{aligned}$$

As for the third term, we see that

$$\begin{aligned}
\|I_3\|_{L^1(\Omega)} &\overset{\text{by (2.3)}}{\leq} \mu_{1,a}\int_{\Omega_1} a(x)y_x(x)^2\,dx + \mu_{2,a}\int_{\Omega_2} a(x)y_x(x)^2\,dx \\
&\overset{\text{by (2.4)}}{\leq} 2\int_\Omega a(x)y_x(x)^2\,dx \leq 2\|y\|^2_{H^1_a(\Omega)} < +\infty.
\end{aligned}$$

So, $v(x)$ is absolutely continuous in $\overline{\Omega}$. As a consequence, we see that the limits $\lim_{x\nearrow 1}|x-1|a(x)y_x(x)^2 = \lim_{x\searrow 1}|x-1|a(x)y_x(x)^2 = L$ do exist and must vanish, for otherwise $a(x)y_x(x)^2 \sim L/|x-1|$ (near the point $x_0=1$) would be not integrable. Taking this property into account, we obtain

$$\begin{aligned} a(d)y_x^2(d) = v(d) &= \int_{\Omega_2} [I_1(x)+I_2(x)+I_3(x)]\,dx \\ &\le \|\sqrt{a}y_x\|^2_{L^2(\Omega)} + \frac{2}{\sqrt{a(d)}}\|y\|_{H^1_a(\Omega)}\|(ay_x)_x\|_{L^2(\Omega)} + 2\|y\|^2_{H^1_a(\Omega)} \\ &\le 3\|y\|^2_{H^1_a(\Omega)} + \frac{2}{\sqrt{a(d)}}\|y\|_{H^1_a(\Omega)}\|ay_x\|_{W^{1,2}(\Omega)}. \end{aligned}$$

It remains to prove relation (2.24). We do it by proving that the function

$$v(x) = \begin{cases} a(x)\varphi_x(x)y(x), & c\le x<1, \\ 0, & x=1, \\ a(x)\varphi_x(x)y(x), & 1<x\le d \end{cases}$$

is continuous on $\overline{\Omega}$. This follows by the arguments as above, because

$$\begin{aligned} \|v_x\|_{L^1(\Omega)} &\le \int_\Omega |\sqrt{a}y_x||\sqrt{a}\varphi_x|\,dx + \int_\Omega |y||(a\varphi_x)_x|\,dx \\ &\le \|y\|_{H^1_a(\Omega)}\|\varphi\|_{H^1_a(\Omega)} + \|y\|_{L^2(\Omega)}\|a\varphi_x\|_{L^2(\Omega)} < +\infty, \end{aligned}$$

end, therefore, v is absolutely continuous in $\overline{\Omega}$. Thus, we see that the limits $\lim_{x\nearrow 1}|x-1|a(x)y_x(x)^2 = \lim_{x\searrow 1}|x-1|a(x)y_x(x)^2 = L$ do exist. To conclude the proof, we show that $L=0$. Indeed, in view of the property (2.21), we have

$$a(x)|\varphi_x(x)| = \left|\int_1^x (a\varphi_x)_x\,dx\right| \le \sqrt{|x-1|}\|a\varphi_x\|_{L^2(\Omega)},\ \forall\, x\in\Omega_0,\ \forall\,\varphi\in H^2_a(\Omega).$$

Hence, if we assume that $L\neq 0$, then, in a neighborhood of $x=1$, for any functions $y\in H^1_a(\Omega)$ and $\varphi\in H^2_a(\Omega)$, we have the inequality

$$\frac{L}{2} \le a(x)|\varphi_x(x)||y(x)| \le \sqrt{|x-1|}|y(x)|\|a\varphi_x\|_{L^2(\Omega)},\ \forall\, x\in\Omega_0.$$

However, since y is an $L^2(\Omega)$-function, this relations becomes inconsistent. Thus, $L=0$. □

The main technical difficulty related to the problem (1.1)–(1.3) comes from the degeneration effect at the point $x_0=1$. Therefore, taking now into account Theorems 2.3 and 2.4, we specify the original initial-boundary value problem (1.1)–(1.3) in the form of the following transmission problem:

$$y_{tt} - (a(x)y_x)_x = 0 \quad \text{in} \ \ (0,T)\times\Omega_1 \text{ and } \ (0,T)\times\Omega_2, \tag{2.27}$$

with the initial conditions

$$y(0,\cdot) = y_0, \quad y_t(0,\cdot) = y_1 \quad \text{in} \ \ \Omega, \tag{2.28}$$

the boundary conditions

$$y(t,c) = 0, \quad y(t,d) = f(t) \quad \text{on} \ \ (0,T), \tag{2.29}$$

and the transmission conditions:

(I) For the case $1/a \in L^1(\Omega)$

$$\lim_{x\nearrow 1} y(t,x) = \lim_{x\searrow 1} y(t,x), \quad 0<t<T, \tag{2.30}$$

$$\lim_{x\nearrow 1} a(x)y_x(t,x) = \lim_{x\searrow 1} a(x)y_x(t,x), \quad 0<t<T; \tag{2.31}$$

(II) For the case $1/a \notin L^1(\Omega)$

$$\lim_{x\nearrow 1} a(x)\varphi_x(x)y(t,x) = 0 = \lim_{x\searrow 1} a(x)\varphi_x(x)y(t,x),\ \forall\,\varphi\in H^1_{a,0}(\Omega),\ 0<t<T, \tag{2.32}$$

$$\lim_{x\nearrow 1} a(x)y_x(t,x) = 0 = \lim_{x\searrow 1} a(x)y_x(t,x), \quad 0<t<T. \tag{2.33}$$

Since transmission conditions (2.31)–(2.33) were substantiated in Theorems 2.3 and 2.4 if only $y(t,\cdot)\in H^2_a(\Omega)$ and $\varphi\in H^2_a(\Omega)$ (which mainly corresponds to the case of classical solutions), it is reasonable to consider the transmission problems (2.27)–(2.33) as a relaxed version of the original problem (1.1)–(1.3).

3. On well-posedness of the degenerate transmission problems. In this section we recall the main results of semi-group theory concerning weak and classical notions of solutions for differential operator equation. By analogy with [1], we consider the Hilbert space $\mathcal{H}_a := H^1_{a,0}(\Omega)\times L^2(\Omega)$ and endow it with the scalar product

$$\left\langle \begin{bmatrix} u\\ v\end{bmatrix}, \begin{bmatrix} \widetilde{u}\\ \widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a} = \int_\Omega v(x)\widetilde{v}(x)\,dx + \int_\Omega a(x)u_x(x)\widetilde{u}_x(x)\,dx.$$

We define the unbounded operator $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}_a\to\mathcal{H}_a$, associated with the problem (2.27)–(2.33) provided $f(t)\equiv 0$, as follows

$$\mathcal{A}\begin{bmatrix} u\\ v\end{bmatrix} = \begin{bmatrix} v\\ (au_x)_x\end{bmatrix}. \tag{3.1}$$

and either

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} u\\ v\end{bmatrix} \in H^2_a(\Omega)\times H^1_{a,0}(\Omega)\ :\ \begin{array}{c} \lim\limits_{x\nearrow 1} u(x) = \lim\limits_{x\searrow 1} u(x),\\ \lim\limits_{x\nearrow 1} a(x)u_x(x) = \lim\limits_{x\searrow 1} a(x)u_x(x),\\ u(d)=0 \end{array}\right\} \tag{3.2}$$

if $1/a\in L^1(\Omega)$, or

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} u\\ v\end{bmatrix} \in H^2_a(\Omega)\times H^1_{a,0}(\Omega)\ :\ \begin{array}{c} \lim\limits_{x\nearrow 1} a\varphi_x u = 0 = \lim\limits_{x\searrow 1} a\varphi_x u,\ \forall\,\varphi\in H^2_a(\Omega),\\ \lim\limits_{x\nearrow 1} a(x)u_x(x) = 0 = \lim\limits_{x\searrow 1} a(x)u_x(x),\\ u(d)=0 \end{array}\right\} \tag{3.3}$$

provided $1/a\notin L^1(\Omega)$.

Arguing as in [9, Section II.2], it can be shown that $D(\mathcal{A})$ is a dense subset of $\mathcal{H}_a$.

Lemma 3.1. *$\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}_a\to\mathcal{H}_a$ is the generator of a contraction semi-group in $\mathcal{H}_a$.*

Proof. It is well-known that if H is a Hilbert space and $B: D(D)\subset H\to H$ is a densely defined linear operator such that both B and B^* are dissipative, i.e.,

$$\langle Bu,u\rangle_H \le 0 \quad\text{and}\quad \langle u,B^*u\rangle_H\le 0 \quad \forall\,u\in D(B),$$

then B generates a strongly continuous semi-group of contraction operators [15, p. 686]. Let us show that $\mathcal{A}\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in \mathcal{H}_a$ for all $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in D(\mathcal{A})$, and this operator satisfies the above mentioned properties.

Since the inclusion $\mathcal{A}\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in \mathcal{H}_a$ is obvious for each $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in D(\mathcal{A})$, it remains to check the properties

$$\left\langle \mathcal{A}\begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}u\\v\end{bmatrix}\right\rangle_{\mathcal{H}_a} \le 0, \quad \text{and} \quad \left\langle \begin{bmatrix}u\\v\end{bmatrix}, (\mathcal{A})^* \begin{bmatrix}u\\v\end{bmatrix}\right\rangle_{\mathcal{H}_a} \le 0 \quad \forall \begin{bmatrix}u\\v\end{bmatrix} \in D(\mathcal{A}). \tag{3.4}$$

We do it for the case (II), $1/a \notin L^1(\Omega)$, because the case (I) can be considered in a similar manner. Then the first inequality in (3.4) immediately follows from the definition of the set $D(\mathcal{A})$ and the following relations

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}u\\v\end{bmatrix}\right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix}v\\(au_x)_x\end{bmatrix}, \begin{bmatrix}u\\v\end{bmatrix}\right\rangle_{\mathcal{H}_a} = \sum_{i=1}^{2}\int_{\Omega_i} (au_x)_x\, v\, dx + \sum_{i=1}^{2}\int_{\Omega_i} a v_x u_x\, dx \\
&= \lim_{x\nearrow 1}\left[\int_c^x (au_s)_s\, v\, ds + \int_c^x a v_s u_s\, ds\right] + \lim_{x\searrow 1}\left[\int_x^d (au_s)_s\, v\, ds + \int_x^d a v_s u_s\, ds\right] \\
&= \left[\lim_{x\nearrow 1} a(x)u_x(x)v(x)\right] - \left[\lim_{x\searrow 1} a(x)u_x(x)v(x)\right] = 0, \quad \text{for all } \begin{bmatrix}u\\v\end{bmatrix} \in D(\mathcal{A})
\end{aligned} \tag{3.5}$$

by the transmission conditions.

Taking into account the equality

$$\left\langle \mathcal{A}\begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a} = \left\langle \begin{bmatrix}u\\v\end{bmatrix}, \mathcal{A}^* \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a}, \quad \begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix} \in D(\mathcal{A}),$$

we see that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a} &= \left\langle \begin{bmatrix}v\\(au_x)_x\end{bmatrix}, \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a} = \sum_{i=1}^{2}\int_{\Omega_i} (au_x)_x\, \widetilde{v}\, dx + \sum_{i=1}^{2}\int_{\Omega_i} a v_x \widetilde{u}_x\, dx \\
&= \lim_{x\nearrow 1}\left[\int_c^x (au_s)_s\, \widetilde{v}\, ds + \int_c^x a v_s \widetilde{u}_s\, ds\right] + \lim_{x\searrow 1}\left[\int_x^d (au_s)_s\, \widetilde{v}\, ds + \int_x^d a v_s \widetilde{u}_s\, ds\right] \\
&= \lim_{x\nearrow 1}\left[-\int_c^x a u_s \widetilde{v}_s\, ds - \int_c^x v\,(a\widetilde{u}_s)_s\, ds\right] + \lim_{x\searrow 1}\left[-\int_x^d a u_s \widetilde{v}_s\, ds - \int_x^d v\,(a\widetilde{u}_s)_s\, ds\right] \\
&\quad + \left[\lim_{x\nearrow 1} a(x)u_x(x)\widetilde{v}(x) - \lim_{x\searrow 1} a(x)u_x(x)\widetilde{v}(x)\right] \\
&\quad + \left[\lim_{x\nearrow 1} a(x)\widetilde{u}_x(x)v(x) - \lim_{x\searrow 1} a(x)\widetilde{u}_x(x)v(x)\right] \\
&\overset{\text{by t.c.}}{=} -\int_\Omega (a\widetilde{u}_x)_x\, v\, dx - \int_\Omega a\widetilde{v}_x u_x\, dx = \left\langle \begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}-\widetilde{v}\\-(a\widetilde{u}_x)_x\end{bmatrix}\right\rangle_{\mathcal{H}_a}.
\end{aligned}$$

Hence, $\mathcal{A}^*\begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix} = \begin{bmatrix}-\widetilde{v}\\-(a\widetilde{u}_x)_x\end{bmatrix}$, and arguing as in (3.5), we see that $\mathcal{A}^*$ is a dissipative operator as well. Thus, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_a \to \mathcal{H}_a$ generates a strongly continuous semi-group of contraction operators. $\square$

For further convenience, let us denote this semi-group by $e^{\mathcal{A}t}$. Then for any $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{H}_a$, the representation $U(t) = e^{\mathcal{A}t}U_0$ gives the so-called mild solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}U(t) &=& \mathcal{A}U(t), \quad t > 0, \\ U(0) &=& U_0. \end{cases} \tag{3.6}$$

When $U_0 \in D(\mathcal{A})$, the solution $U(t) = e^{\mathcal{A}t}U_0$ is classical in the sense that

$$U(\cdot) \in C^1([0,\infty);\mathcal{H}_a) \cap C([0,\infty);D(\mathcal{A}))$$

and equation (3.6) holds on $[0,\infty)$.

Thus, in view of the above consideration, we say that, for given $y_0 \in H^1_{a,0}(\Omega)$ and $y_1 \in L^2(\Omega)$, the function

$$y \in C^1([0,T];L^2(\Omega)) \cap C([0,T];H^1_{a,0}(\Omega))$$

is the weak solution of problem

$$y_{tt} - (a(x)y_x)_x = 0 \quad \text{in} \quad (0,T)\times\Omega_i,\ i = 1,2, \tag{3.7}$$

$$y(t,c) = 0, \quad y(t,d) = 0, \quad t \in (0,T), \tag{3.8}$$

$$y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \tag{3.9}$$

$$\text{with the transmission conditions (2.30)–(2.31) or (2.32)–(2.33),} \tag{3.10}$$

if $\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ for all $t \in [0,T]$. By the aforementioned regularity result for $e^{\mathcal{A}t}$, if

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in H^2_a(\Omega) \times H^1_{a,0}(\Omega),$$

then y is the classical solution of (3.7)–(3.10) meaning that

$$y \in C^2([0,T];L^2(\Omega)) \cap C^1([0,T];H^1_{a,0}(\Omega)) \cap C([0,T];H^2_a(\Omega))$$

and the equation (3.7) is satisfied for all $t \in [0,T]$ and a.e. $x \in \Omega_0$.

The energy of a mild solution y of (3.7)–(3.10) is the continuous function defined by

$$E_y(t) = \frac{1}{2}\int_{\Omega_0} \left[y_t^2(t,x) + a(x)y_x^2(t,x)\right] dx, \quad \forall\, t \geq 0.$$

Proposition 3.2. *Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii), and let y be a mild solution of* (3.7)–(3.10). *Then*

$$E_y(t) = E_y(0), \quad \forall\, t \geq 0. \tag{3.11}$$

Proof. Suppose, first, that y is a classical solution of (3.7)–(3.10). Then, multiplying the equation by y_t and integrating by parts, in view of the transmission conditions (2.30)–(2.31) or (2.32)–(2.33), we obtain

$$\begin{aligned} 0 &= \int_{\Omega_0} y_t(t,x)y_{tt}(t,x)\,dx - \sum_{i=1}^{2}\int_{\Omega_i} y_t(t,x)\left(a(x)y_x(t,x)\right)_x\,dx \\ &= \int_{\Omega_0} \left[y_t(t,x)y_{tt}(t,x) + a(x)y_x(t,x)y_{xt}(t,x)\right] dx \\ &\quad - \left[y_t(t,x)a(x)y_x(t,x)\right]_{x=c}^{x=1} - \left[y_t(t,x)a(x)y_x(t,x)\right]_{x=1}^{x=d} \\ &= \frac{d}{dt}E_y(t) - y_t(t,1)\left(\lim_{x\nearrow 1}\left[a(x)y_x(t,x)\right] - \lim_{x\searrow 1}\left[a(x)y_x(t,x)\right]\right), \end{aligned}$$

where the last term vanishes because of the transmission conditions. Thus, we conclude that the energy of the classical solution y is constant. The same conclusion can be extended to any mild solution by approximation arguments. □

4. On Boundary Observability. We say that the system (3.7)–(3.10) is boundary observable (via the normal derivative at $x = c$ and $x = d$) in time $T > 0$ if there exists a constant $C_T > 0$ such that for any $y_0 \in H^1_{a,0}(\Omega)$ and $y_1 \in L^2(\Omega)$ the mild solution of (3.7)–(3.10) satisfies the estimate

$$\int_0^T y_x^2(t,c)\,dt + \int_0^T y_x^2(t,d)\,dt \geq C_T\, E_y(0). \tag{4.1}$$

Any constant satisfying (4.1) is called an observability constant for (3.7)–(3.10) in time T. We denoted the supremum of all observability constants for (3.7)–(3.10) by C_T.

LEMMA 4.1. *For any mild solution $y(t,x)$ of* (3.7)–(3.10) *we have that $y_x(\cdot,c) \in L^2(0,T)$ and $y_x(\cdot,d) \in L^2(0,T)$ for any $T > 0$, and*

$$a(c)\int_0^T y_x^2(t,c)\,dt \leq \frac{1}{1-c}\left[6T + \frac{4}{\min\{1,a(c),a(d)\}}\right] E_y(0), \tag{4.2}$$

$$a(d)\int_0^T y_x^2(t,d)\,dt \leq \frac{1}{d-1}\left[6T + \frac{4}{\min\{1,a(c),a(d)\}}\right] E_y(0). \tag{4.3}$$

Moreover,

$$\begin{aligned}(1-c)a(c)\int_0^T y_x^2(t,c)\,dt + (d-1)a(d)\int_0^T y_x^2(t,d)\,dt \\ = 2\left[\int_{\Omega_0}(x-1)y_x(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} \\ + \int_0^T\int_{\Omega_0}\left(y_t^2(t,x) + \left[1 - \frac{(x-1)a_x(x)}{a(x)}\right]a(x)y_x^2(t,x)\right)dx\,dt.\end{aligned} \tag{4.4}$$

Proof. To begin with, we assume that $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in H^2_a(\Omega) \times H^1_{a,0}(\Omega)$, that is, y given by the formula $\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ is a classical solution of the problem (3.7)–(3.10). Following in many aspects [1, Lemma 3.2], we multiply equation (3.7) by $(x-1)y_x$. Integrating over $(0,T)\times\Omega_0$, we obtain

$$\begin{aligned}0 &= \int_0^T\int_{\Omega_0}(x-1)y_x(t,x)\left(y_{tt}(t,x) - (a(x)y_x(t,x))_x\right)dx\,dt \\ &= \left[\int_{\Omega_0}(x-1)y_x(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} - \int_0^T\int_{\Omega_0}(x-1)y_{tx}(t,x)y_t(t,x)\,dx\,dt \\ &\quad - \int_0^T\int_{\Omega_0}\Big((x-1)a_x(x)y_x^2(t,x) + (x-1)a(x)y_x(t,x)y_{xx}(t,x)\Big)dx\,dt \\ &= \left[\int_{\Omega_0}(x-1)y_x(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} - \int_0^T\int_{\Omega_0}(x-1)a_x(x)y_x^2(t,x)\,dx\,dt \\ &\quad - \int_0^T\int_{\Omega_0}\left((x-1)\left[\frac{y_t^2(t,x)}{2}\right]_x + (x-1)a(x)\left[\frac{y_x^2(t,x)}{2}\right]_x\right)dx\,dt\end{aligned} \tag{4.5}$$

After integration of the last two term above, we have

$$
\begin{aligned}
\int_0^T \int_{\Omega_0} (x-1) \left[\frac{y_t^2(t,x)}{2}\right]_x dx\, dt &= -\frac{1}{2}\int_0^T \int_{\Omega_0} y_t^2(t,x)\, dx\, dt \\
&\quad + \frac{1}{2}\int_0^T \left[(x-1)y_t^2(t,x)\right]_{x=c}^{x=1} dt + \frac{1}{2}\int_0^T \left[(x-1)y_t^2(t,x)\right]_{x=1}^{x=d} dt \\
&\overset{\text{by (2.19), (3.8)}}{=} -\frac{1}{2}\int_0^T \int_{\Omega_0} y_t^2(t,x)\, dx\, dt, \qquad (4.6)\\
\int_0^T \int_{\Omega_0} (x-1)a(x) \left[\frac{y_x^2(t,x)}{2}\right]_x dx\, dt &= -\frac{1}{2}\int_0^T \int_{\Omega_0} \left[(x-1)a(x)\right]_x y_x^2(t,x)\, dx\, dt \\
&\quad + \frac{1}{2}\int_0^T \left[(x-1)a(x)y_x^2(t,x)\right]_{x=c}^{x=1} dt + \frac{1}{2}\int_0^T \left[(x-1)a(x)y_x^2(t,x)\right]_{x=1}^{x=d} dt \\
&\overset{\text{by (2.23), (3.8)}}{=} \frac{(1-c)a(c)}{2}\int_0^T y_x^2(t,c)\, dt + \frac{(d-1)a(d)}{2}\int_0^T y_x^2(t,d)\, dt \\
&\quad - \frac{1}{2}\int_0^T \int_{\Omega_0} \left[(x-1)a(x)\right]_x y_x^2(t,x)\, dx\, dt. \qquad (4.7)
\end{aligned}
$$

As a result, the identity (4.4) follows by inserting (4.6) and (4.7) into (4.5). To deduce the estimate (4.2), it is enough to notice that

$$
\begin{aligned}
\left|\int_{\Omega_0} (x-1)y_x(t,x)y_t(t,x)\, dx\right| &\le \frac{1}{2}\int_{\Omega_0} \left[y_t^2(t,x) + \frac{(x-1)^2}{a(x)}a(x)y_x^2(t,x)\right] dx \\
&\overset{\text{by Proposition 2.1}}{\le} \frac{E_y(0)}{\min\{1, a(c), a(d)\}}, \qquad (4.8)\\
\left[1 - \frac{(x-1)a_x(x)}{a(x)}\right] &\overset{\text{by (2.4)}}{\le} 3, \quad \text{in } \Omega,
\end{aligned}
$$

and the energy $E_y(t)$ is constant.

In order to extend relations (4.2) and (4.4) to the mild solution associated with the initial data $y_0 \in H^1_{a,0}(\Omega)$ and $y_1 \in L^2(\Omega)$, it suffices to approximate such data by $\left[\begin{smallmatrix} y_0^k \\ y_1^k \end{smallmatrix}\right] \in H^2_a(\Omega) \times H^1_{a,0}(\Omega)$ and use estimate (4.2) to show that the normal derivatives of the corresponding classical solutions give a Cauchy sequence in $L^2(0,T)$. □

LEMMA 4.2. *For any mild solution $y(t,x)$ of (3.7)–(3.10) we have that, for each $T > 0$,*

$$
\int_0^T \int_{\Omega_0} \left[a(x)y_x^2(t,x) - y_t^2(t,x)\right] dx\, dt + \left[\int_{\Omega_0} y(t,x)y_t(t,x)\, dx\right]_{t=0}^{t=T} = 0. \qquad (4.9)
$$

Proof. Let y be a classical solution of (3.7)–(3.10). Then, multiplying equation (3.7) by y and integrating over $(0,T)\times\Omega_0$, we obtain

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega_0} y(t,x)\left[y_{tt}(t,x) - (a(x)y_x(t,x))_x\right] dx\,dt \\
&= \left[\int_{\Omega_0} y(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} - \int_0^T \int_{\Omega_0} y_t^2(t,x)\,dx\,dt \\
&\quad - \int_0^T \left[a(x)y_x(t,x)y(t,x)\right]_{x=c}^{x=1} dt - \int_0^T \left[a(x)y_x(t,x)y(t,x)\right]_{x=1}^{x=d} dt \\
&\quad + \int_0^T \int_{\Omega_0} a(x)y_x^2(t,x)\,dx\,dt.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^T \left[a(x)y_x(t,x)y(t,x)\right]_{x=c}^{x=1} dt + \int_0^T \left[a(x)y_x(t,x)y(t,x)\right]_{x=1}^{x=d} dt \\
&\qquad \overset{\text{by (3.8)}}{=} \int_0^T \left[\lim_{x\nearrow 1} a(x)y_x(t,x)y(t,x) - \lim_{x\searrow 1} a(x)y_x(t,x)y(t,x)\right] dt = 0
\end{aligned}$$

by the transmission conditions (2.30)–(2.31) or (2.32)–(2.33), the announced equality (4.9) follows from the above identity. An approximation argument allows to extend this conclusion to mild solutions. □

Theorem 4.3. *Let $a:\overline{\Omega}\to\mathbb{R}$ be a weight function satisfying properties (i)–(iii), and let y be a mild solution of* (3.7)–(3.10). *Then, for every $T>0$, the estimate*

$$\begin{aligned}
&(1-c)a(c)\int_0^T y_x^2(t,c)\,dt + (d-1)a(d)\int_0^T y_x^2(t,d)\,dt \\
&\qquad \geq \left[(2-\max\{\mu_{1,a},\mu_{2,a}\})\,T - \frac{4}{\min\{1,a(c),a(d)\}} - 2(C_1+C_2)\right] E_y(0) \qquad (4.10)
\end{aligned}$$

holds true with C_1 and C_2 given by relations (2.8)–(2.9).

Proof. Since the case of mild solutions can be recovered by an approximation arguments, we restrict ourself by assumptions that y is a classical solution of the problem (3.7)–(3.10). Then adding to the right hand side of (4.4) the left side of (4.9) multiplied by , we obtain

$$\begin{aligned}
&(1-c)a(c)\int_0^T y_x^2(t,c)\,dt + (d-1)a(d)\int_0^T y_x^2(t,d)\,dt \\
&= 2\left[\int_{\Omega_0}(x-1)y_x(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} + \frac{\max\{\mu_{1,a},\mu_{2,a}\}}{2}\left[\int_{\Omega_0} y(t,x)y_t(t,x)\,dx\right]_{t=0}^{t=T} \\
&\qquad + \int_0^T\int_{\Omega_0}\left(1-\frac{\max\{\mu_{1,a},\mu_{2,a}\}}{2}\right) y_t^2(t,x)\,dx\,dt \\
&+ \int_0^T\int_{\Omega_0}\left(\left[1+\frac{\max\{\mu_{1,a},\mu_{2,a}\}}{2} - \frac{(x-1)a_x(x)}{a(x)}\right] a(x)y_x^2(t,x)\right) dx\,dt = I_1+I_2+I_3+I_4.
\end{aligned}$$

Since

$$-\frac{(x-1)a_x(x)}{a(x)} \geq -\frac{|x-1||a_x(x)|}{a(x)} \geq -\max\{\mu_{1,a},\mu_{2,a}\},$$

it follows that

$$I_3 + I_4 \geq (2 - \max\{\mu_{1,a}, \mu_{2,a}\}) \int_0^T E_y(0)\, dt = (2 - \max\{\mu_{1,a}, \mu_{2,a}\})\, T E_y(0).$$

Taking into account that

$$I_1 = 2 \left[\int_{\Omega_0} (x-1) y_x(t,x) y_t(t,x)\, dx \right]_{t=0}^{t=T} \overset{\text{by (4.8)}}{\geq} -\frac{4E_y(0)}{\min\{1, a(c), a(d)\}},$$

and

$$\frac{1}{2} \left| \int_{\Omega_0} y(t,x) y_t(t,x)\, dx \right| \leq \frac{1}{2} \int_{\Omega_0} \left(\frac{1}{C_a} y^2(t,x) + C_a y_t^2(t,x) \right) dx \leq C_a E_y(0),$$

where $C_a = C_1 + C_2$ is Poincaré's constant in (2.7) and C_i are defined in (2.8)–(2.9), we see that

$$I_2 \geq -2(C_1 + C_2) E_y(0).$$

Thus, the announced estimate (4.10) is proven. □

Due to Theorem 4.3, the observability constant C_T (see inequality (4.1)) for the problem (3.7)–(3.10) in time T can be derived from (4.10). Namely,

$$C_T = \frac{1}{\max\{(1-c)a(c), (d-1)a(d)\}} \times \left[(2 - \max\{\mu_{1,a}, \mu_{2,a}\})\, T - \frac{4}{\min\{1, a(c), a(d)\}} - 2(C_1 + C_2) \right].$$

As for the minimal time $T_a > 0$ when the system (3.7)–(3.10) becomes observable in time $T > T_a$, it can be defined as follows

$$T_a := \frac{1}{(2 - \max\{\mu_{1,a}, \mu_{2,a}\})} \left[\frac{4}{\min\{1, a(c), a(d)\}} + 2(C_1 + C_2) \right]. \tag{4.11}$$

5. On Boundary Null-Controllability. In this section the problem of boundary controllability of the degenerate wave equation is studied. The control is assumed to act at the boundary point $x = d$ through the Dirichlet condition. So, we consider the following degenerate control system

$$y_{tt} - (a(x) y_x)_x = 0 \quad \text{in} \quad (0, +\infty) \times \Omega_i, \; i = 1, 2, \tag{5.1}$$

$$y(t,c) = 0, \quad y(t,d) = f(t), \quad t \in (0, +\infty), \tag{5.2}$$

$$y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \tag{5.3}$$

$$\text{with the transmission conditions (2.30)–(2.31) or (2.32)–(2.33)}, \tag{5.4}$$

where $f \in L^2(0,T)$ is the control.

Let $H_a^{-1}(\Omega)$ be the dual space to $H^1_{a,0}(\Omega)$ with respect to the pivot space $L^2(\Omega)$. Since $y = y(t,x)$ can be considered as a function of t with values into a suitable space, in the sequel we will write $y(t)$ instead of $y(t,x)$, $\dot{y}$ instead of y_t, and $\ddot{y}$ instead of y_{tt}.

In order to make a precise definition of the solution to the boundary value problem (5.1)–(5.4), where $f \in L^2(0,T)$ is the control, and indicate its characteristic properties, we notice that Proposition 2.1 (see also (2.10)) implies that the operator $A_a : D(A_a) \subset L^2(\Omega) \to$

$L^2(\Omega)$, where $A_a(y) = -(ay_x)_x$ and $D(A_a) = H_a^2(\Omega)$, is an isomorphism from $H^1_{a,0}(\Omega)$ onto $H_a^{-1}(\Omega)$. In particular, $H_a^{-1}(\Omega) = A_a\left(H^1_{a,0}(\Omega)\right)$.

Definition 5.1. *System* (5.1)–(5.4) *is boundary null controllable in time* $T>0$ *if, for every initial data* $y_0 \in L^2(\Omega)$*, and* $y_1 \in H_a^{-1}(\Omega)$*, the set of reachable states* $(y(T),\dot{y}(T))$*, where* y *is a solution of* (5.1)–(5.4) *with* $f \in L^2(0,T)$*, contains the element* $(0,0)$*.*

Definition 5.2. *System* (5.1)–(5.4) *is boundary exactly controllable in time* $T>0$ *if, for every initial data* $y_0 \in L^2(\Omega)$*, and* $y_1 \in H_a^{-1}(\Omega)$*, the set of reachable states* $(y(T),\dot{y}(T))$*, coincides with* $L^2(\Omega)\times H_a^{-1}(\Omega)$*.*

Remark 5.1. *Arguing as in Proposition 2.2.1 in [20], and utilizing the linearity and reversibility properties of system* (5.1)–(5.4)*, it can be shown that this system is exactly controllable through the boundary Dirichlet condition at* $x=d$ *if and only if it is null controllable.*

Following the standard approach and utilizing the transmission conditions, we define the solution of controlled system (5.1)–(5.4) by transposition.

Definition 5.3. *Let* $f \in L^2(0,T)$*,* $y_0 \in L^2(\Omega)$*, and* $y_1 \in H_a^{-1}(\Omega)$ *be given distributions. We say that* y *is a solution by transposition of the problem* (5.1)–(5.4) *if*

$$y \in C^1\left([0,\infty); H_a^{-1}(\Omega)]\right) \cap C\left([0,\infty); L^2(\Omega)\right)$$

satisfies for all $T>0$ *and all* $w_T^0 \in H^1_{a,0}(\Omega)$ *and* $w_T^1 \in L^2(\Omega)$ *the following equality*

$$\begin{aligned}\left\langle \dot{y}(T), w_T^0\right\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} &- \int_\Omega y(T) w_T^1\,dx \\ &= \langle y_1, w(0)\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} - \int_\Omega y_0 \dot{w}(0)\,dx - a(d)\int_0^T f(t) w_x(t,d)\,dt, \end{aligned} \tag{5.5}$$

where w *is the solution of the backward homogeneous equation*

$$w_{tt} - (a(x)w_x)_x = 0 \quad \text{in} \quad (0,+\infty)\times\Omega_i,\ i=1,2 \tag{5.6}$$

with the final conditions

$$w(T) = w_T^0, \quad w_t(T) = w_T^1 \quad \text{in} \quad \Omega, \tag{5.7}$$

the boundary conditions

$$w(t,c) = 0, \quad w(t,d) = 0 \quad \text{on} \quad (0,T), \tag{5.8}$$

and the transmission conditions:

(I) For the case $1/a \in L^1(\Omega)$

$$\lim_{x\nearrow 1} w(t) = \lim_{x\searrow 1} w(t), \quad 0<t<T, \tag{5.9}$$

$$\lim_{x\nearrow 1} a w_x(t) = \lim_{x\searrow 1} a w_x(t), \quad 0<t<T; \tag{5.10}$$

(II) For the case $1/a \notin L^1(\Omega)$

$$\lim_{x\nearrow 1} a\varphi_x w(t) = 0 = \lim_{x\searrow 1} a\varphi_x w(t),\ \forall\,\varphi\in H^1_{a,0}(\Omega),\ 0<t<T, \tag{5.11}$$

$$\lim_{x\nearrow 1} a w_x(t) = 0 = \lim_{x\searrow 1} a w_x(t), \quad 0<t<T. \tag{5.12}$$

Following the results of Section 3 and thanks to the change of variable $u(t,x) = w(T - t, x)$, we see that the backward problem (5.6)–(5.12) admits a unique mild solution $w \in C^1\left([0,T]; L^2(\Omega)]\right) \cap C\left([0,T]; H^1_{a,0}(\Omega)\right)$ for each $T > 0$. Moreover, arguing as in Lemma 4.1, it can be shown that there exists a constant $C > 0$ such that

$$\int_0^T w_x^2(t,c)\,dt + \int_0^T w_x^2(t,d)\,dt \le C\,E_w(T), \tag{5.13}$$

where

$$E_w(t) = \frac{1}{2}\int_{\Omega_0}\left[w_t^2(t,x) + a(x)w_x^2(t,x)\right]\,dx = E_w(T), \quad \forall\, t \in [0,T],$$

is the energy of a mild solution w and it is conserved through time. Since

$$E_w(T) = \frac{1}{2}\left[\|w_T^1\|^2_{L^2(\Omega)} + \|w_T^0\|^2_{H^1_{a,o}(\Omega)}\right], \tag{5.14}$$

it follows that a mild solution w of (5.6)–(5.12) depends continuously on the data $(w_T^0, w_T^1) \in H^1_{a,0}(\Omega) \times L^2(\Omega)$, and, therefore, the right hand side of (5.5) defines a continuous linear form with respect to $(w_T^0, w_T^1) \in H^1_{a,0}(\Omega) \times L^2(\Omega)$ $T > 0$. Thus, a solution y by transposition of (5.1)–(5.4) is unique in $C^1\left([0,\infty); H_a^{-1}(\Omega)]\right) \cap C\left([0,\infty); L^2(\Omega)\right)$. The following theorem is a consequence of the classical results of existence and uniquencess of solutions of nonhomogeneous evolution equations. Full details can be found in [14] and [21].

THEOREM 5.4. *For any $f \in L^2(0,T)$ and $(y_0, y_1) \in L^2(\Omega) \times H_a^{-1}(\Omega)$ transmission problem* (5.1)–(5.4) *has a unique solution defined by transposition*

$$(y, \dot y) \in C\left([0,T]; L^2(\Omega) \times H_a^{-1}(\Omega)\right).$$

Moreover, the map $(y_0, y_1, f) \mapsto \{y, \dot y\}$ is linear and there exists a constant $C(T) > 0$ such that

$$\|y\|_{L^\infty(0,T;L^2(\Omega))} + \|\dot y\|_{L^\infty(0,T;H_a^{-1}(\Omega))} \le C(T)\left[\|y_0\|_{L^2(\Omega)} + \|y_1\|_{H_a^{-1}(\Omega)} + \|f\|_{L^2(0,T)}\right].$$

We are now in a position to prove the main result of this section.

THEOREM 5.5. *Let $a : \overline{\Omega} \to \mathbb{R}$ be a weight function satisfying properties (i)–(iii), and let T_a be a value defined as in* (4.11). *Then, for every $T > T_a$ and for any $(y_0, y_1) \in L^2(\Omega) \times H_a^{-1}(\Omega)$, there exists a control $f \in L^2(0,T)$ such that the solution of* (5.1)–(5.4) *(in the sense of transposition) satisfies condition $(y(T), \dot y(T)) \equiv (0,0)$, i.e. the system* (5.1)–(5.4) *is boundary null controllable in time $T > T_a$.*

Proof. Let

$$\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in L^2(\Omega) \times H_a^{-1}(\Omega), \qquad \begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat w_T^0 \\ \widehat w_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega)$$

be arbitrary pairs. Let w and $\widehat w$ be mild solutions of the backward problem (5.6)–(5.12) with final conditions $\left[\begin{smallmatrix} w_T^0 \\ w_T^1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \widehat w_T^0 \\ \widehat w_T^1 \end{smallmatrix}\right]$, respectively. Let us define the bilinear form Λ on $H^1_{a,0}(\Omega) \times L^2(\Omega)$ as follows

$$\Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat w_T^0 \\ \widehat w_T^1 \end{bmatrix}\right) := a(d)\int_0^T w_x(t,d)\widehat w_x(t,d)\,dt, \quad \forall \begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat w_T^0 \\ \widehat w_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega).$$

Then, in view of estimate (5.13) and representation (5.14), we deduce that the bilinear form $\Lambda : \left[H^1_{a,0}(\Omega) \times L^2(\Omega)\right]^2 \to \mathbb{R}$ is continuous. Moreover, due to Theorem 4.3 and observability inequality (4.10), this form is coercive on $H^1_{a,0}(\Omega) \times L^2(\Omega)$ provided $T > T_a$. Thus, by the Lax-Milgram Lemma, variational problem

$$\Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}\right) = \langle y_1, \widehat{w}(0)\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} - \int_\Omega y_0 \dot{\widehat{w}}(0)\,dx,\ \forall \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega)$$

has a unique solution $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega)$. Then setting $f = w_x(t,d)$ and $T > T_a$, where $w \in C^1\left([0,T]; L^2(\Omega)]\right) \cap C\left([0,T]; H^1_{a,0}(\Omega)\right)$ is a mild solution of the backward problem (5.6)–(5.12) with $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}$ as the final data, we see that

$$\begin{aligned} a(d)\int_0^T f(t)\widehat{w}_x(t,d)\,dt &= a(d)\int_0^T w_x(t,d)\widehat{w}_x(t,d)\,dt = \Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}\right) \\ &= \langle y_1, \widehat{w}(0)\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} - \int_\Omega y_0 \dot{\widehat{w}}(0)\,dx, \quad \forall \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega). \end{aligned} \tag{5.15}$$

On the other hand, if y is the solution by transposition of the problem (5.1)–(5.4), then equality (5.5) implies that, for all $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega)$, we have

$$\begin{aligned} a(d)\int_0^T f(t)\widehat{w}_x(t,d)\,dt &= \langle y_1, \widehat{w}(0)\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} - \int_\Omega y_0 \dot{\widehat{w}}(0)\,dx \\ &\quad - \left\langle \dot{y}(T), \widehat{w}_T^0\right\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} + \int_\Omega y(T)\widehat{w}_T^1\,dx. \end{aligned} \tag{5.16}$$

Comparing the last relations (5.15)–(5.16), we obtain

$$-\left\langle \dot{y}(T), \widehat{w}_T^0\right\rangle_{H_a^{-1}(\Omega);H^1_{a,0}(\Omega)} + \int_\Omega y(T)\widehat{w}_T^1\,dx = 0, \quad \forall \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in H^1_{a,0}(\Omega) \times L^2(\Omega).$$

From this we finally deduce that $(y(T), \dot{y}(T)) \equiv (0,0)$, i.e. the system (5.1)–(5.4) is boundary null controllable in time $T > T_a$. □

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