The existence and nonexistence of global L^2-constrained minimizers for Kirchhoff equations with L^2-subcritical general nonlinearity

Hongyu Ye,* Lina Zhang

College of Science, Wuhan University of Science and Technology,

Wuhan 430065, P. R. China

Abstract

In this paper, we study the existence of global L^2-constrained minimizers related to the following Kirchhoff type equation:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u-f(u)=\lambda u,\quad x\in\mathbb{R}^N,\ \lambda\in\mathbb{R},$$

where $N\leq 3$, a, $b>0$ are constants, $f(u)$ is a general L^2-subcritical nonlinearity. By using the concentration compactness principle, we prove the sharp existence and nonexistence of global L^2-constraint minimizers.

Keywords: Constrained minimization; Subadditivity inequality; Global L^2-constraint minimizers; L^2-subcritical general nonlinearity.

Mathematics Subject Classification(2010): 35J60, 35A15

1 Introduction and main result

In this paper, we study the existence and nonexistence of normalized solutions to the following Kirchhoff type problem with L^2-subcritical general nonlinearity:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\right)\Delta u-f(u)=\lambda u,\quad x\in\mathbb{R}^N,\ \lambda\in\mathbb{R},\tag{1.1}$$

where $N\leq 3$, a, $b>0$ are constants. We assume that the nonlinear term $f\in C(\mathbb{R},\mathbb{R})$ satisfies the following conditions:

(f_1) $\lim\limits_{t\to 0}\frac{f(t)}{t}=0$;

*E-mail address: yyeehongyu@163.com

(f_2) $\lim\limits_{t\to\infty} \frac{f(t)}{|t|^{1+\frac{8}{N}}} = 0$;

(f_3) There exists $t_0 > 0$ such that $F(t_0) > 0$, where $F(t) = \int_0^t f(s)ds$ for $t \in \mathbb{R}$.

Kirchhoff equation (1.1) is a nonlocal one as the appearance of the term $\int_{\mathbb{R}^N} |\nabla u|^2$ implies that (1.1) is not a pointwise identity. This causes some mathematical difficulties and makes the study of (1.1) particularly interesting. Problem (1.1) arises in a physical model presented by Kirchhoff in [7]. After Lions [13] introduced an abstract framework to the problem, it received much attention and many existence results can be found, see e.g. [1, 2, 3, 4, 5, 13, 15].

In the past years, a first line to study (1.1) is to consider the case where λ is fixed and assigned, or even with an additional external and fixed potential, see [6, 8, 9, 12, 16] and the references therein. In such direction, the critical point theory and variational methods are mainly used to prove the existence of nontrivial solutions, but nothing can be given a priori on the L^2-norm of the solutions. Since the physicists are interested in "normalized solutions", i.e. solutions with prescribed L^2-norm, it is interesting for us to study whether problem (1.1) has solutions with prescribed L^2-norm. By the critical point theory, solutions with prescribed L^2-norm are corresponding to critical points of the following C^1 functional

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} F(u) \tag{1.2}$$

constrained on the following L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S_c = \{u \in H^1(\mathbb{R}^N)| \ \left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{1}{2}} = c > 0\}.$$

For any fixed $c > 0$, we call $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ a couple of solution to (1.1) if u_c is a critical point of $I|_{S_c}$ and λ_c is the associated Lagrange multiplier.

For any $c > 0$, we define the following minimization problem

$$i_c := \inf_{u\in S_c} I(u). \tag{1.3}$$

It is standard that minimizers of i_c are critical points of $I(u)$ constrained on S_c. This minimization problem was first studied in [19], where the nonlinearity is a pure power nonlinear term, i.e. $f(u) = |u|^{p-2}u$. By using the well-known Gagliardo-Nirenberg inequality and L^2-preserving scaling arguments, in [19] the author proved that $p = 2 + \frac{8}{N}$ is L^2-critical exponent for problem (1.3), namely for all $c > 0$, $i_c > -\infty$ if $p \in (2, 2 + \frac{8}{N})$ (L^2-subcritical case) and $i_c = -\infty$ if $p \in (2 + \frac{8}{N}, \frac{2N}{N-2})$ (L^2-supcritical case), however, if $p = 2 + \frac{8}{N}$, there exists $c_0 > 0$ such that $i_c > -\infty$ for $c \in (0, c_0]$ and $i_c = -\infty$ for $c > c_0$. The author proved the existence, nonexistence and uniqueness of minimizers for i_c when $p \in (2, 2+\frac{8}{N})$ and also proved the existence of normalized solutions (local minimizers) when $p \in [2 + \frac{8}{N}, \frac{2N}{N-2})$, see [10, 19, 20]. For a L^2-supcritical general nonlinearity f, by raising a series of assumptions on f, it is proved in [11] that (1.1) has a normalized solution in $H_r^1(\mathbb{R}^N)$. However, there seems few work about (1.1) with a L^2-subcritical general nonlinearity. In this paper, we try to do it.

Recall in [17] the Gagliardo-Nirenberg inequalitywith the best constant: Let $p \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $p \geq 2$ if $N = 1, 2$, then

$$\int_{\mathbb{R}^N} |u|^p \leq \frac{p}{2(\int_{\mathbb{R}^N} |Q_p|^2)^{\frac{p-2}{2}}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{N(p-2)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 \right)^{\frac{2p-N(p-2)}{4}}, \tag{1.4}$$

with equality only for $u = Q_p$, where up to translations, Q_p is the unique ground state solution of

$$-\frac{N(p-2)}{4}\Delta Q + \left(1 + \frac{p-2}{4}(2-N)\right) Q = |Q|^{p-2}Q, \quad x \in \mathbb{R}^N. \tag{1.5}$$

Our main result is as follows:

Theorem 1.1. *Assume that $(f_1)-(f_3)$ hold. $N \leq 3$. There exists $c_* \geq 0$ such that $i_c = 0$ for all $0 < c \leq c_*$ and $i_c < 0$ for all $c > c_*$. Moreover,*

(i) i_c has a minimizer for each $c > c_*$.

(ii) If $c_* > 0$, i_c has no minimizer for each $0 < c < c_*$.

It is still unknown when $c_* > 0$ holds and whether i_{c_*} has a minimizer or not if $c_* > 0$. We found that it is dependent of the limit of the function $\frac{F(t)}{|t|^{1+\frac{4}{N}}}$ when $t \to 0$. The answer is as follows.

Theorem 1.2. *Assume that $(f_1)-(f_3)$ hold. $N \leq 3$.*

(1) If $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = +\infty$, then $c_* = 0$.

(2) If $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} < +\infty$, then $c_* > 0$.

(3) If $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = 0$, then i_{c_*} has a minimizer.

Remark 1.3. *Assume that $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = l > 0$ hold, we can not deduce whether i_{c_*} has a minimizer or not.*

For example, if $f(t) = l(2 + \frac{4}{N})|t|^{1+\frac{4}{N}}t$, it has been proved that $c_* = \left(\frac{a}{l(2+\frac{4}{N})}\right)^{\frac{N}{4}}$ $(\int_{\mathbb{R}^N} |Q_{2+\frac{4}{N}}|^2)^{\frac{1}{2}}$ and i_{c_*} has no minimizer (see Theorem 1.1 in [19]); However, if $f(t) = l(2 + \frac{4}{N})|t|^{1+\frac{4}{N}}t + |t|^{q-2}t$, $2 + \frac{4}{N} < q < 2 + \frac{8}{N}$, then we can prove that $0 < c_* < \left(\frac{a}{l(2+\frac{4}{N})}\right)^{\frac{N}{4}} (\int_{\mathbb{R}^N} |Q_{2+\frac{4}{N}}|^2)^{\frac{1}{2}}$ and i_{c_*} has a minimizer (see details in Remark 3.1).

The conditions $(f_1)-(f_3)$ are elementary for a L^2-subcritical general nonlinearity problem. Compared with the pure power nonlinearity case, the main difficulties to prove Theorems 1.1 and 1.2 are the L^2-preserving scaling may fail to prove $i_c < 0$ and the strict subadditivity condition as the nonlinearity term is general. We need to look for other scaling techniques.

Our methods to prove Theorems 1.1 and 1.2 can be similarly used to prove the existence of L^2-constrained minimizers for Schrödinger equation with L^2-subcritical general nonlinearity, which was proved by Shibata in [14].

Throughout this paper, we use standard notations. For simplicity, we write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^N$. $L^p := L^p(\mathbb{R}^N)$ $(1 \leq p < +\infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function space respectively. C will denote a positive constant unless specified. We use " $:=$ " to denote definitions. We denote a subsequence of a sequence $\{u_n\}$ as $\{u_n\}$ to simplify the notation unless specified.

The paper is organized as follows. In § 2, we present some preliminary results for Theorems 1.1 and 1.2. In § 3, we prove Theorems 1.1 and 1.2.

2 Preliminary Results for Theorems 1.1 and 1.2

In this section, we give some preliminary results.

Lemma 2.1. *Assume that $(f_1) - (f_3)$ hold and $N \leq 3$. Then for any $c > 0$, $I(u)$ is bounded from below and coercive on S_c.*

Proof. By $(f_1) - (f_3)$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{1+\frac{8}{N}} + C_\varepsilon |t|, \qquad \forall\, t \in \mathbb{R}.$$

Then for any $u \in H^1(\mathbb{R}^N)$, there exists $C_\varepsilon > 0$ such that

$$\left| \int_{\mathbb{R}^N} F(u) \right| \leq \varepsilon \int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}} + C_\varepsilon \int_{\mathbb{R}^N} |u|^2.$$

So by the Gagliardo-Nirenberg inequality (1.4), we see that for any $c > 0$ and any $u \in S_c$, there exists $C_\varepsilon > 0$ such that

$$I(u) \geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \varepsilon \frac{N+4}{N |Q_{2+\frac{8}{N}}|_2^{\frac{8}{N}}} c^{\frac{8}{N}-2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - C_\varepsilon c^2$$

By taking $0 < \varepsilon \leq \frac{bN |Q_{2+\frac{8}{N}}|_2^{\frac{8}{N}}}{4(N+4) c^{\frac{8}{N}-2}}$ small enough, then there exists $C > 0$ such that

$$I(u) \geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - Cc^2 \geq -Cc^2,$$

which implies that $I(u)$ is bounded from below and coercive on S_c for any $c > 0$. $\square$

For any $c > 0$, set

$$i_c := \inf_{u \in S_c} I(u),$$

then by Lemma 2.1, i_c is well defined.

Lemma 2.2. *Assume that* $(f_1) - (f_3)$ *hold and* $N \leq 3$. *Then for any* $c > 0$, $i_c \leq 0$.

Proof. By (f_1) and (f_2), we see that $\frac{F(t)}{t^2} \to 0$ as $t \to 0$. For any $c > 0$ and $u \in S_c$, set $u_t(x) := t^{\frac{N}{2}} u(tx)$ for any $t > 0$, then $u_t \in S_c$ and

$$i_c \leq I(u_t) = t^2 \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + t^4 \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^N} \frac{F(t^{\frac{N}{2}} u)}{|t^{\frac{N}{2}} u|^2} |u|^2 \to 0$$

as $t \to 0^+$, hence $i_c \leq 0$ for all $c > 0$.

□

Lemma 2.3. *Assume that* $(f_1) - (f_3)$ *hold and* $N \leq 3$. *Then the function* $c \mapsto i_c$ *is continuous on* $(0, +\infty)$.

Proof. For any $c > 0$, it is enough to prove that $i_{c_n} \to i_c$ for any sequence $\{c_n\} \subset (0, +\infty)$ satisfying that $c_n \to c$.

For any $u \in S_c$, let $v_n := \frac{c_n}{c} u$. Then $v_n \in S_{c_n}$. By $(f_1) - (f_3)$ and the Dominated Convergence Theorem, we see that $\int_{\mathbb{R}^N} F(v_n) \to \int_{\mathbb{R}^N} F(u)$. Thus

$$\begin{aligned} i_{c_n} \leq I(v_n) &= \left(\frac{c_n}{c} \right)^2 \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \left(\frac{c_n}{c} \right)^4 \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^4 - \int_{\mathbb{R}^N} F(v_n) \\ &= I(u) + o_n(1), \end{aligned}$$

where $o_n(1) \to 0$ as $c_n \to c$. By the arbitrary of $u \in S_c$, we see that $\overline{\lim_{c_n \to c}} i_{c_n} \leq i_c$.

On the other hand, we suppose that $\{u_n\} \subset S_{c_n}$ is a sequence satisfying that $I(u_n) \leq i_{c_n} + \frac{1}{n}$. By Lemmas 2.1 and 2.2, we see that $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Let $w_n := \frac{c}{c_n} u_n \in S_c$, then similarly we have $\int_{\mathbb{R}^N} F(w_n) = \int_{\mathbb{R}^N} F(u_n) + o_n(1)$. Hence

$$\begin{aligned} i_c \leq I(w_n) &= \left(\frac{c}{c_n} \right)^2 \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \left(\frac{c}{c_n} \right)^4 \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^4 - \int_{\mathbb{R}^N} F(w_n) \\ &= I(u_n) + o_n(1) \\ &\leq i_{c_n} + \frac{1}{n} + o_n(1), \end{aligned}$$

which implies that $i_c \leq \underline{\lim}_{c_n \to c} i_{c_n}$. So $\lim_{c_n \to c} i_{c_n} = i_c$. The lemma is proved.

□

Lemma 2.4. *Assume that $(f_1)-(f_3)$ hold and $N \leq 3$. Then there exists $c_* \geq 0$ such that $i_c < 0$ if $c > c_*$. Moreover, if $c_* > 0$, then $i_c = 0$ for all $0 < c \leq c_*$.*

Proof. For any $u \in S_1$, set $u^c(x) = u(c^{-\frac{2}{N}}x), \forall\, c > 0$. Then $u^c \in S_c$. Moreover,

$$i_c \leq I(u^c) = c^{2-\frac{4}{N}}\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + c^{4-\frac{8}{N}}\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^2 - c^2\int_{\mathbb{R}^N}F(u).$$

Since $2-\frac{4}{N}, 4-\frac{8}{N} < 2$, $i_c \leq I(u^c) \to -\infty$ as $c \to +\infty$. Hence $i_c < 0$ for all $c > 0$ large enough. So

$$\{c \in (0,+\infty)|\ i_c < 0\} \neq \varnothing.$$

Define

$$c_* := \inf\{c > 0|\ i_c < 0\}, \tag{2.1}$$

then $c_* \in [0,+\infty)$ is well defined and $i_c < 0$ if $c > c_*$.

If $c_* > 0$, then we conclude from Lemma 2.2 that $i_c = 0$ for all $c < c_*$. Moreover, by the continuity of the function $c \mapsto i_c$, we see that $i_{c_*} = 0$. □

Lemma 2.5. *Assume that $(f_1)-(f_3)$ hold and $N \leq 3$. For each $c > c_*$, it holds $i_c < i_\alpha + i_{\sqrt{c^2-\alpha^2}}$ for any $0 < \alpha < c$.*

Proof. Since $c > c_*$, by Lemma 2.4 we have $i_c < 0$. Let $\{u_n\} \subset S_c$ be a minimizing sequence of i_c, i.e. $\lim\limits_{n\to\infty} I(u_n) = i_c$, then $I(u_n) = i_c + o_n(1) \leq 1$ for n large enough, where $o_n(1) \to 0$ as $n \to \infty$. By Lemma 2.1, we see that $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

Hence there exist two positive constants $C_2 > C_1 > 0$ independent of n such that

$$C_1 \leq \int_{\mathbb{R}^N}|\nabla u_n|^2 \leq C_2. \tag{2.2}$$

Indeed by contradiction, if $\int_{\mathbb{R}^N}|\nabla u_n|^2 \to 0$ as $n \to \infty$, then by $(f_1)-(f_3)$ and the Gagliardo-Nirenberg inequality (1.4), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\int_{R^N}F(u_n) \leq \varepsilon\int_{\mathbb{R}^N}|u_n|^2 + C_\varepsilon\int_{\mathbb{R}^N}|u_n|^{2+\frac{8}{N}} \leq \varepsilon c^2 + c^{\frac{8}{N}-2}C_\varepsilon\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^2,$$

which implies that $\int_{\mathbb{R}^N}F(u_n) \leq \varepsilon c^2$ as $n \to \infty$. Then by the arbitrary of ε we have $\lim\limits_{n\to\infty}\int_{\mathbb{R}^N}F(u_n) = 0$. So $I(u_n) \to 0$ as $n \to \infty$, which contradicts to the definition of $\{u_n\}$.

Set $u_n^\theta := u_n(\theta^{-\frac{2}{N}}x)$ with $\theta > 1$. Then $u_n^\theta \in S_{\theta c}$ and by (2.2) we see that

$$\begin{aligned} I(u_n^\theta) &= \theta^2 I(u_n) + \theta^2\left[(\theta^{-\frac{4}{N}}-1)\frac{a}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + (\theta^{2-\frac{8}{N}}-1)\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^2\right] \\ &\leq \theta^2 I(u_n) - \theta^2\left[(1-\theta^{-\frac{4}{N}})\tfrac{aC_1}{2} + (1-\theta^{2-\frac{8}{N}})\tfrac{bC_1^2}{4}\right]. \end{aligned} \tag{2.3}$$

Letting $n \to \infty$ and notice that the second term of r.h.s. above is strictly negative and independent of n, it follows that

$$i_{\theta c} < \theta^2 i_c, \quad \forall \theta > 1. \tag{2.4}$$

For any $0 < \alpha < c$, without loss of generality, we may assume that $\alpha \geq \sqrt{c^2 - \alpha^2}$. We prove this lemma by discussing the following three cases.

If $c_* \geq \alpha \geq \sqrt{c^2 - \alpha^2}$, then $i_\alpha = i_{\sqrt{c^2-\alpha^2}} = 0$, hence $i_c < 0 = i_\alpha + i_{\sqrt{c^2-\alpha^2}}$.

If $\alpha > c_* \geq \sqrt{c^2 - \alpha^2}$, then $i_\alpha < 0$ and $i_{\sqrt{c^2-\alpha^2}} = 0$. Hence by (2.4) we see that $i_c < \frac{c^2}{\alpha^2} i_\alpha < i_\alpha = i_\alpha + i_{\sqrt{c^2-\alpha^2}}$.

If $\alpha \geq \sqrt{c^2 - \alpha^2} > c_*$, then by (2.4) we see that $i_c < \frac{c^2}{\alpha^2} i_\alpha = i_\alpha + \frac{c^2-\alpha^2}{\alpha^2} i_\alpha \leq i_\alpha + i_{\sqrt{c^2-\alpha^2}}$. □

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1

Proof. (i) For any $c > c_*$, $i_c < 0$. Let $\{u_n\} \subset S_c$ be a minimizing sequence of i_c, then by Lemma 2.1, $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

Let $\delta := \lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2$. Then $\delta \geq 0$. If $\delta = 0$, then by the vanishing lemma (see e.g. [18]), $u_n \to 0$ in $L^p(\mathbb{R}^N)$, $\forall 2 < p \leq 2 + \frac{8}{N}$. By $(f_1) - (f_3)$, for any $\varepsilon > 0$, there exists C_ε such that

$$\int_{\mathbb{R}^N} F(u_n) \leq \varepsilon c^2 + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{2+\frac{8}{N}}.$$

Since ε is arbitrary, $\int_{\mathbb{R}^N} F(u_n) \to 0$ as $n \to \infty$. Thus

$$i_c = \lim_{n\to\infty} I(u_n) \geq -\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) = 0,$$

which is a contradiction. So $\delta > 0$.

There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^2 \geq \frac{\delta}{2}$. Set $\tilde{u}_n(x) = u_n(x + y_n)$, then

$$\int_{\mathbb{R}^N} |\tilde{u}_n|^2 \geq \frac{\delta}{2} > 0. \tag{3.1}$$

Moreover, by the translation invariance of $\mathbb{R}^N$, we see that $I(\tilde{u}_n) \to i_c$ and $\{\tilde{u}_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Then there exists $\tilde{u} \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}, & \text{in } H^1(\mathbb{R}^N), \\ \tilde{u}_n \to \tilde{u}, & \text{in } L^q_{loc}(\mathbb{R}^N), \quad q \in [1, 2 + \frac{8}{N}], \\ \tilde{u}_n(x) \to \tilde{u}(x), & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{3.2}$$

as $n \to \infty$, which and (3.1) implies that $\tilde{u} \neq 0$.

Set $\alpha := |\tilde{u}|_2$, then $\alpha \in (0, c]$. Next we try to prove that $\alpha = c$. By contradiction, we assume that $\alpha < c$. By (3.2), we have

$$|\tilde{u}_n|_2^2 = |\tilde{u}_n - \tilde{u}|_2^2 + |\tilde{u}|_2^2 + o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$, i.e. $\lim\limits_{n\to\infty} |\tilde{u}_n - \tilde{u}|_2 = \sqrt{c^2 - \alpha^2} > 0$. By (3.2), the Brezis-Lieb Lemma and Lemma 2.3, we see that

$$i_c = \lim_{n\to\infty} I(\tilde{u}_n) \geq I(\tilde{u}) + \lim_{n\to\infty} I(\tilde{u}_n - \tilde{u}) \geq i_\alpha + \lim_{n\to\infty} i_{|\tilde{u}_n - \tilde{u}|_2} = i_\alpha + i_{\sqrt{c^2-\alpha^2}},$$

which contradicts Lemma 2.5. So $|\tilde{u}|_2 = c$. Thenwe conclude from (3.2) and the Fatou's Lemma that $i_c \leq I(\tilde{u}) \leq \lim\limits_{n\to\infty} I(\tilde{u}_n) = i_c$. So $\tilde{u} \in S_c$ is a minimizer of i_c and then $\tilde{u}$ is a critical point of $I(u)$ constrained on S_c.

(ii) By contradiction, for some $c_0 \in (0, c_*)$, we suppose that there exists a $u_{c_0} \in S_{c_0}$ such that $I(u_{c_0}) = i_{c_0} = 0$. Then

$$\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2\right)^2 = \int_{\mathbb{R}^N} F(u_{c_0}).$$

Set $u_{c_*} := u_{c_0}((\frac{c_*}{c_0})^{-\frac{2}{N}}x)$. Then $u_{c_*} \in S_{c_*}$ and

$$\begin{aligned} i_{c_*} &\leq I(u_{c_*}) \\ &= (\frac{c_*}{c_0})^{2-\frac{4}{N}}\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2 + (\frac{c_*}{c_0})^{4-\frac{8}{N}}\frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2\right)^2 - (\frac{c_*}{c_0})^2\int_{\mathbb{R}^N} F(u_{c_0}) \\ &= (\frac{c_*}{c_0})^2\left[\left[(\frac{c_*}{c_0})^{-\frac{4}{N}} - 1\right]\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2 + \left[(\frac{c_*}{c_0})^{2-\frac{8}{N}} - 1\right]\frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_{c_0}|^2\right)^2\right] < 0, \end{aligned}$$

which is a contradiction with $i_{c_*} = 0$. So i_c has no minimizer for all $c \in (0, c_*)$.

□

Proof of Theorem 1.2

Proof. (1) For any $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, by $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{1+\frac{4}{N}}} = +\infty$ and the Fatou's Lemma we have

$$\lim_{t\to 0}\int_{\mathbb{R}^N} \frac{F(t^{\frac{N}{2}}u)}{|t^{\frac{N}{2}}u|^{2+\frac{4}{N}}}|u|^{2+\frac{4}{N}} = +\infty.$$

For any $c > 0$ and any $u \in S_c$, set $u_t(x) = t^{\frac{N}{2}}u(tx)$ with $t > 0$, then $u_t \in S_c$ and

$$i_c \leq I(u_t) = t^2\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + t^4\frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - t^2\int_{\mathbb{R}^N} \frac{F(t^{\frac{N}{2}}u)}{|t^{\frac{N}{2}}u|^{2+\frac{4}{N}}}|u|^{2+\frac{4}{N}},$$

hence $\frac{i_c}{t^2} \to -\infty$ as $t \to 0^+$. So $i_c < 0$ for all $c > 0$, which implies that $c_* = 0$.

(2) By $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{1+\frac{4}{N}}} < +\infty$ and (f_3), there exists a constant $C > 0$ such that

$$F(t) \le C|t|^{2+\frac{4}{N}} + |t|^{2+\frac{8}{N}}, \qquad \forall\, t \in \mathbb{R}.$$

For any $c > 0$ and any $u \in S_c$, by the Gagliardo-Nirenberg inequality (1.4), there exist a constant C independent of u such that

$$\left|\int_{\mathbb{R}^N} F(u)\right| \le C\left[c^{\frac{4}{N}} \int_{\mathbb{R}^N} |\nabla u|^2 + c^{\frac{8}{N}-2}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2\right].$$

Then

$$I(u) \ge \left(\frac{a}{2} - Cc^{\frac{4}{N}}\right)\int_{\mathbb{R}^N} |\nabla u|^2 + \left(\frac{b}{4} - Cc^{\frac{8}{N}-2}\right)\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2,$$

By taking $c > 0$ sufficient small satisfying that $0 < c \le \min\{(\frac{b}{4C})^{\frac{N}{8}-2}, (\frac{a}{4C})^{\frac{N}{4}}\}$, then

$$I(u) \ge \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u|^2 \ge 0,$$

which imply that $i_c \ge 0$ for $c > 0$ small. By Lemma 2.2 we see that $i_c = 0$ for c small enough. So $c_* > 0$.

(3) Suppose that $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = 0$, for any $\varepsilon > 0$ and any $u \in H^1(\mathbb{R}^N)$, by (f_3) there exists a constant $C_\varepsilon > 0$ such that

$$\left|\int_{\mathbb{R}^N} F(u)\right| \le \varepsilon \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} + C_\varepsilon \int_{\mathbb{R}^N} |u|^{2+\frac{8}{N}}. \tag{3.3}$$

Let $\{u_n\} \subset S_{c_*}$ be a minimizing sequence of i_{c_*}, then by Lemma 2.1, $\{u_n\}$ is uniformly bounded in $H^1(\mathbb{R}^N)$.

Set $\delta := \lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \ge 0$. If $\delta = 0$, by the vanishing lemma and (3.3) we have $\int_{\mathbb{R}^N} F(u_n) \to 0$ as $n \to \infty$. By $I(u_n) \to i_{c_*} = 0$ we see that $\int_{\mathbb{R}^N} |\nabla u_n|^2 \to 0$ as $n \to \infty$. By (3.3) and the Gagliardo-Nirenberg inequality (1.4), there exist two constants $C_1, C_2 > 0$ independent of n such that

$$I(u_n) \ge \left(\frac{a}{2} - \varepsilon C_1 c_*^{\frac{4}{N}}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 + \left(\frac{b}{4} - C_\varepsilon C_2 c_*^{\frac{8}{N}-2}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\right)^2.$$

By taking $\varepsilon = \frac{a}{4C_1 c_*^{\frac{4}{N}}}$ there exists a positive constant C independent of n such that

$$I(u_n) \ge \frac{a}{4}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \left(\frac{b}{4} - Cc_*^{\frac{8}{N}-2}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\right)^2 > 0 \tag{3.4}$$

for n large enough since $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 = 0$. This contradicts to the choice of $\{u_n\}$. So $\delta > 0$. There exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n|^2 \ge \frac{\delta}{2}$.

Set $v_n(x) := u_n(x+y_n)$, then $\int_{\mathbb{R}^N} |v_n|^2 \geq \frac{\delta}{2} > 0$ and $\{v_n\} \subset S_{c_n}$ is a uniformly bounded minimizing sequence of i_{c_*}. There exists $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^N), \quad v_n \to v \text{ in } L^q_{loc}(\mathbb{R}^N),\ q \in [1, 2+\frac{8}{N}], \quad v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N.$$

So $0 < |v|_2 \leq c_*$ and

$$\lim_{n\to\infty} |v_n - v|_2 = \lim_{n\to\infty} \sqrt{c_n^2 - |v|_2^2} = \sqrt{(c_*)^2 - |v|_2^2} < c_*.$$

If $|v|_2 < c_*$, then we conclude from lemma 2.3 that $\lim\limits_{n\to\infty} I(v_n - v) \geq \lim\limits_{n\to\infty} i_{|v_n - v|_2} = i_{\sqrt{(c^*)^2 - |v|_2^2}} = 0$. By the Brezis-Lieb Lemma and Lemma 2.2, we see that

$$0 = \lim_{n\to\infty} I(v_n) \geq I(v) + \lim_{n\to\infty} I(v_n - v) \geq I(v) \geq i_{|v|_2} = 0$$

which implies that $I(v) = i_{|v|_2}$ with $|v|_2 < c^*$. It contradicts to Theorem 1.1 (2). Thus $|v|_2 = c_*$. So $v \in S_{c_*}$ is a minimizer of i_{c_*} and then v is a critical point of $I|_{S_{c_*}}$.

Remark 3.1. *When $\lim\limits_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = l > 0$, then $c_* > 0$. For any $u \in S_{c_*}$, $I(u^t) \geq 0$, where $u_t(x) = t^{\frac{N}{2}} u(tx)$ with $t > 0$. By the Dominated Covergence Theorem, we see that*

$$\begin{aligned}
\lim_{t\to 0^+} \frac{I(u^t)}{t^2} &= \lim_{t\to 0^+} \left[\frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + t^2 \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^N} \frac{F(t^{\frac{N}{2}} u)}{|t^{\frac{N}{2}} u|^{2+\frac{4}{N}}} |u|^{2+\frac{4}{N}} \right] \\
&= \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - l \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}},
\end{aligned}$$

which implies that $\Phi(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - l \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} \geq 0$ for all $u \in S_{c_*}$. Then

$$c_* \leq \left(\frac{a}{l(2+\frac{4}{N})} \right)^{\frac{N}{4}} |Q_{2+\frac{4}{N}}|_2$$

(For each $c > (\frac{a}{l(2+\frac{4}{N})})^{\frac{N}{4}} |Q_{2+\frac{4}{N}}|_2$, $\inf\limits_{u\in S_c} \Phi(u) < 0$ since we conclude from (1.4) that $\Phi(\frac{c}{|Q_{2+\frac{4}{N}}|_2} Q_{2+\frac{4}{N}}) < 0$).

If $f(u) = l(2+\frac{4}{N})|u|^{1+\frac{4}{N}} u + |u|^{q-2} u$ with $2 + \frac{4}{N} < q < 2 + \frac{8}{N}$, then by the definition of $Q_{2+\frac{4}{N}}$,

$$\begin{aligned}
& I\Big(\big(\frac{a}{l(2+\frac{4}{N})}\big)^{\frac{N}{4}} (Q_{2+\frac{4}{N}})_t\Big) \\
&= t^4 \frac{b}{4} \Big(\frac{a}{l(2+\frac{4}{N})} \Big)^N \left(\int_{\mathbb{R}^N} |\nabla Q_{2+\frac{4}{N}}|^2 \right)^2 - t^{\frac{N(q-2)}{2}} \frac{1}{q} \Big(\frac{a}{l(2+\frac{4}{N})} \Big)^{\frac{Nq}{4}} \int_{\mathbb{R}^N} |Q_{2+\frac{4}{N}}|^q < 0
\end{aligned}$$

for $t>0$ small enough. Hence $i_{\left(\frac{a}{l(2+\frac{4}{N})}\right)^{\frac{N}{4}}|Q_{2+\frac{4}{N}}|_2}<0$, *so* $c_*<\left(\frac{a}{l(2+\frac{4}{N})}\right)^{\frac{N}{4}}|Q_{2+\frac{4}{N}}|_2$.

For each minimizing sequence $\{u_n\}$ *of* i_{c_*}, *by the Gagliardo-Nirenberg inequality* (1.4) *there exists a constant* $C>0$ *such that*

$$I(u_n)\geq\left(\frac{a}{2}-\frac{l(2+\frac{4}{N})}{2|Q_{2+\frac{4}{N}}|_2^{\frac{4}{N}}}c^{\frac{N}{4}}\right)\int_{\mathbb{R}^N}|\nabla u_n|^2+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^2-C\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^{\frac{N(q-2)}{4}},$$

where $\frac{a}{2}-\frac{l(2+\frac{4}{N})}{2|Q_{2+\frac{4}{N}}|_2^{\frac{4}{N}}}c^{\frac{N}{4}}>0$. *This may bring about the same contradiction as* (3.4). *So similarly to the proof of Theorem 1.2 (3), we can proved* i_{c_*} *has a minimizer.*

□

References

[1] C. O. Alves, F. J. S. A. Crrea, On existence of solutions for a class of problem involving a nonlinear operator, Comm. Appl. Nonlinear Anal. 8 (2001) 43-56.

[2] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc. 348 (1996) 305-330.

[3] S. Bernstain, Sur une classe d'*é*quations fonctionelles aux d*é*riv*é*es partielles, Bull. Acad. Sci. URSS. S*é*r. 4 (1940) 17-26.

[4] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation, Adv. Differential Equations 6 (2001) 701-730.

[5] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108 (1992) 247-262.

[6] X. M. He, W. M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. J. Differential Equations 252 (2012) 1813-1834.

[7] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[8] G. B. Li, H. Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differential Equations, 257 (2014) 566-600.

[9] G. B. Li, H. Y. Ye, Existence of positive solutions for nonlinear Kirchhoff type problems in $\mathbb{R}^3$ with critical Sobolev exponent, Math. Methods Appl. Sci. 37 (2014) 2570-2584.

[10] G. B. Li, H. Y. Ye, On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials, J. Differential Equations 266 (2019) 7101-7123.

[11] X. Luo, Q. F. Wang, Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in $\mathbb{R}^3$, 33 (2017) 19-32.

[12] Y. Li, et al., Existence of a positive solution to Kirchhoff type problems without compactness conditions, J. Differential Equations 253 (2012) 2285-2294.

[13] J. L. Lions, On some questions in boundary value problems of mathmatical physics, in: Contemporary Development in Continuum Mechanics and Partial Differential Equations, in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, New York, 1978, pp. 284-346.

[14] M. Shibata, Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term, Manuscripta Math. 143 (2014) 221-237.

[15] S. I. Pohozaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (NS) 96 (1975) 152-166 (in Russian).

[16] J. Wang et al., Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, J. Differential Equations, 253 (2012) 2314-2351.

[17] M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87 (1983) 567-576.

[18] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

[19] H. Y. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, Math. Methods Appl. Sci.38 (2015) 2663-2679.

[20] H. Y. Ye, The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations, Z. Angew. Math. Phys. 66 (2015) 1483-1497.