Finite time collapse in chemotaxis systems with logistic-type superlinear source

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Abstract.
We consider the following quasilinear Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla(u\nabla v) + g(u), & (x,t) \in \Omega \times [0, T_{max}), \\ \qquad 0 = \Delta v - v + u, & (x,t) \in \Omega \times [0, T_{max}), \end{cases}$$

on a ball $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$, under homogeneous Neumann boundary conditions and non negative initial data. The source term $g(u)$ is superlinear and of logistic type i.e. $g(u) = \lambda u - \mu u^k$, $k > 1$, $\mu > 0$, $\lambda \in \mathbb{R}$ and T_{max} is the blow-up time.
The solution (u, v) may or may not blow up in finite time. Under suitable conditions on data, we prove that the function u, which blows up in $L^\infty(\Omega)$-norm [22], blows up also in $L^p(\Omega)$-norm for some $p > 1$. Moreover a lower bound of the lifespan (or blow-up time when it is finite) T_{max} is derived.
In addition, if $\Omega \subset \mathbb{R}^3$ a lower bound of T_{max} is explicitly computable.

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1 Introduction

In many biological phenomena the chemotaxis, the biased movement of cells (or organisms) in response to chemical gradients, plays an important role in coordinating cell migration (see [7], [1], [4]). The movement is referred to as chemoattractant if the cells move toward the increasing signal concentration ($\chi > 0$), while it is called chemorepulsion whenever the cells move away from

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the increasing signal concentration ($\chi < 0$) with χ in (1.2).
In 1970 Keller and Segel [7] derived a celebrated model to describe this event. The model has been extensively studied since 1970s, and a number of variations have been proposed and examined, and the properties of their solutions investigated, as the existence of global bounded solutions and the question whether the chemotaxis model allows for a chemotactic collapse, that is, if the system possesses solutions that blow up in finite or infinite time ([1], [5], [21]).
The topic of blow up solutions has been addressed by several authors also for more general operators and from different points of view (see for instance [9] for some results concerning the elliptic case, [10], [11] and [12] for parabolic systems under various boundary conditions).
Our aim is to study the parabolic-elliptic problem:

$$
\begin{cases}
u_t = \Delta u - \nabla(u\nabla v) + g(u), & (x,t) \in \Omega \times [0, T_{max}),\\
0 = \Delta v - v + u, & (x,t) \in \Omega \times [0, T_{max}),\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times [0, T_{max}),\\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases} \tag{1.1}
$$

with $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$, $g(u) = \lambda u - \mu u^k$, $k > 1$, $\mu > 0$ and $\lambda \in \mathbb{R}$ and the nonnegative initial datum $u_0 \in C^0(\bar{\Omega})$.

System (1.1) is a particular case of the following initial-boundary value problem

$$
\begin{cases}
u_t = \Delta u - \chi\nabla(u\nabla v) + g(u), & x \in \Omega\ t > 0,\\
\tau v_t = \Delta v - v + u, & x \in \Omega, t > 0,\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega,
\end{cases} \tag{1.2}
$$

with $\Omega \subset \mathbb{R}^n$, $n \geq 1$ a bounded domain with smooth boundary, $\tau > 0$, $\chi \in \mathbb{R}$ and $g(u)$ a source term.

We recall that

► If $\tau = 1$, $\chi > 0$ and $g(u) = 0$, (1.2) is the classical Keller-Segel system introduced by Keller and Segel [7].
► If $\tau = 0$, $\chi > 0$ and $g(u) = 0$, we have a simpler model which reflects that the signal substance diffuses much faster than cells move (Parabolic-Elliptic Keller-Segel system) and the question of blow-up and global existence of solution was studied for instance in [13], [17] and [6].
► If $\tau = 0$, $\chi = 1$ and $g(u) = 0$ and if $\Omega = B_R(0) \subset \mathbb{R}^n$, or $\Omega = \mathbb{R}^n$, $n \geq 3$, $R > 0$ in [18] Ph. Souplet and M. Winkler consider radially symmetric solutions of the following parabolic-elliptic Keller-Segel-Patlak system

$$\begin{cases} u_t = \Delta u - \nabla\big(u \nabla v\big) & x \in \Omega\ t > 0, \\ 0 = \Delta v + u - M, & x \in \Omega, t > 0 \end{cases} \tag{1.3}$$

with Neumann boundary conditions, $u(x, 0) = u_0(x)$ in Ω and

$$M := \begin{cases} \frac{1}{|\Omega|} \int_\Omega u_0(x)dx & if\ \Omega = B_R, \\ 0 & if\ \Omega = \mathbb{R}^n. \end{cases}$$

The authors study the blow-up asymptotics of radially decreasing solutions of (1.3) and show that the final profile satisfies $C_1|x|^{-2} \leq u(x, T) \leq C_2|x|^{-2}$ with convergence in $L^1(\Omega)$ as $t \to T$, the time existence of the solution.

► If $\tau = 0$, $\chi = 1$, and $g(u) = \lambda u - \mu u^k$, if $k = 2, \forall\lambda$ either $n \leq 2$, $\mu > 0$ or $n \geq 3$, $\mu \leq \frac{n-2}{n}$, then no blow-up occurs; if $k > 2$, $\forall\lambda$ the same conclusion holds [19]. If $k > 1$, $\mu > 0$ and $\lambda \in \mathbb{R}$, that is g is a source term of logistic superlinear degradation type, in [22] recently M. Winkler proves that, in low-dimensional spatial settings (compared with higher dimensional case in [22]) under a dimensional dependent range of k, when $\Omega \equiv B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, the solution of (1.2) blows up in finite time in $L^\infty(\Omega)$-norm.

► If $\tau = 1$, $\chi > 0$ and $g(u) = 0$, for the following more general system

$$\begin{cases} u_t = \nabla \cdot [(u + \alpha)^{m_1 - 1}\nabla u - \chi u(u + \alpha)^{m_2 - 2}\nabla v], & \text{in } \Omega \times (0, T). \\ v_t = \Delta v - v + u, & \text{in } \Omega \times (0, T) \end{cases} \tag{1.4}$$

under Neumann boundary conditions and initial conditions, where Ω is a general bounded domain in $\mathbb{R}^n$ with smooth boundary, $\alpha > 0$, $\chi > 0$, $m_1,\ m_2 \in \mathbb{R}$

and $T > 0$, in [15] T. Nishino and T. Yokota derived a lower bound of blow-up time.

► If $\tau = 0$, $\chi > 0$, $g(u) = 0$ and $M := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx$, in [8], M. Marras, T. Nishino and G. Viglialoro investigate the blow-up solutions of the following

$$
(1.5)\qquad \begin{cases} u_t = \nabla\cdot[(u+\alpha)^{m_1-1}\nabla u - \chi u(u+\alpha)^{m_2-2}\nabla v], \\ 0 = \Delta v - M + u, \\ u_\nu = v_\nu = 0, \\ u(x,0) = u_0(x), \\ \displaystyle\int_\Omega v(x,t)dx = 0, \end{cases}
$$

with $(x,t) \in \Omega \times (0, T_{max})$, Ω a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 1$, T_{max} the blow-up time, $\alpha > 0$ and m_1, m_2 real numbers. Under some links between the above parameters m_1, m_2 and the extra condition $\int_\Omega v(x,t)dx = 0$, they prove that if $p_0 > \frac{n}{2}(m_2 - m_1)$ any blowing up classical solution in $L^\infty(\Omega)$–norm blows up also in $L^{p_0}(\Omega)$–norm and a lower bound of the blow–up time T_{max} is derived.

► If $\tau = 0$ and $\chi > 0$ and $g(u) \leq au - \mu u^2$ a source term of logistic type $(a \geq 0, \mu > 0)$, another interesting model was archivied by X. Cao and S. Zheng in [2]; there the following quasilinear parabolic-elliptic Keller-Segel system is considered

$$
(1.6)\qquad \begin{cases} u_t = \nabla\big(\phi(u)\nabla u\big) - \chi\nabla\big(u\nabla u\big) + g(u), \quad x \in \Omega\ t > 0, \\ 0 = \Delta v - v + u, \quad x \in \Omega, \quad t > 0 \end{cases}
$$

with Neumann boundary conditions and $u(x,0) = u_0(x)$ in $\Omega \subset \mathbb{R}^n$, $n \geq 1$, a bounded convex domain with smooth boundary, $\phi(s) > 0$ for $s > 0$, $\phi(s) \geq ks^p$, $k > 0$, $p \in \mathbb{R}$. There are three nonlinear mechanisms included in this model: the nonlinear diffusion $\nabla\big(\phi(u)\nabla u\big)$, the aggregation $\chi\nabla\big(u\nabla v\big)$ and the logistic absorption $g(u)$; they observe that the nonlinear diffusion with the logistic absorption dominate the aggregation, so that the unique classical solution is global in time and bounded, regardless of the initial data, if $\mu > \chi(1 - \frac{2}{n(1-p)_+})$, which enlarge the parameter range $\mu > \chi\frac{n-2}{n}$ present

when in the system $g(u) = 0$.
For other results see the references in the papers cited above.
Our purpose is to find a lower bound T of the blow up time T_{max}, so that there exists a safe interval of existence of the solution (u, v) to system (1.1), $[0, T]$ with $T < T_{max}$. First we prove that $u(x, t)$, which blows up in $L^\infty(\Omega)$-norm (see [22]), blows up also in $L^p(\Omega)$-norm, $p > \frac{n}{2}$, by improving a result of Freitag ([3]).
In [22], M. Winkler proves that, assuming some restrictions on k and u_0, the solution of (1.1) blows up in finite time, in $L^\infty(\Omega)$-norm, with $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$, $R > 0$.
This result is contained in the following Theorem.

Theorem 1.1 ([22]). *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$ and $k > 1$ be such that*

$$k < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases} \tag{1.7}$$

Then for all $L > 0$, $m > 0$ and $m_0 \in (0, m)$ one can find $r_0 = r_0(R, \lambda, \mu, k, L, m, m_0) \in (0, R)$ with the property that whenever $u_0 \in C^0(\bar{\Omega})$ such that

$$u_0(x) \leq L|x|^{-n(n-1)} \quad \textit{for all } x \in \Omega \tag{1.8}$$

as well as

$$\int_\Omega u_0(x)dx \leq m \quad \textit{but} \quad \int_{B_{r_0}} u_0 \geq m_0, \tag{1.9}$$

there exists $T_{max} \in (0, \infty)$ and a classical solution (u, v) of (1.1) with

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) & \textit{and} \\ v \in C^{2,0}(\bar{\Omega} \times (0, T_{max})), \end{cases} \tag{1.10}$$

which blows up at $t = T_{max}$ in the sense that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.11}$$

Now we can state our first main result which provides that the classical solution of (1.1), blows up in L^p-norm at finite time.

Theorem 1.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, the classical solution (u, v) to system* (1.1) *for $t \in (0, T_{\max})$, provided by Theorem 1.1, is such that for all $p > \frac{n}{2}$*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty. \tag{1.12}$$

Define $\forall p > 1$ the energy function

$$\Psi(t) = \frac{1}{p}||u||_{L^p(\Omega)} \quad \text{with} \quad \Psi_0 = \Psi(0) = \frac{1}{p}||u_0||_{L^p(\Omega)}. \tag{1.13}$$

Theorem 1.3. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, for all $p > \frac{n}{2}$ and positive constants B_1, B_2, B_3 depending on λ, p, n such that the blow up time T_{max} of the classical solution (u, v) to system* (1.1)*, provided by Theorem 1.2, satisfies the following estimate*

$$T_{\max} \geq \int_{\Psi_0}^{\infty} \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2}}, \tag{1.14}$$

with $\gamma_1 = \frac{p+1}{p}$, $\gamma_2 = \frac{2(p+1)-n}{2p-n}$.

In the next Theorem, assuming $\Omega \subset \mathbb{R}^3$, a safe interval of existence of the solution $[0, T]$, $T < T_{max}$ is obtained since we can derive an explicit lower bound for T_{max}.
To this end, we introduce the function

$$\Phi(t) = \|u\|^2_{L^2(\Omega)} \quad \text{with} \quad \Phi_0 = \Phi(0) := \|u_0\|^2_{L^2(\Omega)}. \tag{1.15}$$

We observe that, under the hypotheses of Theorem 1.1, if the solution (u, v) of (1.1) blows up in $L^\infty(\Omega)$-norm, from Theorem 1.2 (with $p = 2$), it blows up also in $L^2(\Omega)$-norm at $t = T_{max}$.

We remark that the choice of the domain $\Omega \subset \mathbb{R}^3$ is due to the use of a Sobolev type inequality valid only in $\mathbb{R}^3$.

Theorem 1.4. *Let $\Omega = B_R(0) \subset \mathbb{R}^3$, $R > 0$ and (u, v) be a classical solution of* (1.1) *for $t \in (0, T_{\max})$, provided by Theorem 1.1. Then Φ, defined in* (1.15)*, satisfies a first order differential inequality:*

$$\Phi'(t) \leq A\Phi^3(t), \tag{1.16}$$

with A a positive constant dependent on $||u_0||_{L^2(\Omega)}$, k, λ, μ, $|\Omega|$.

From Theorem 1.4, as a consequence, we have

Corollary 1.1. *Under the assumptions of Theorem (1.4), let* (u, v) *be a solution of* (1.1) *and* $\Phi(t)$ *and* Φ_0 *defined in* (1.15). *Then there exists a safe interval of existence of* (u, v) *say* $[0, T]$ *with*

$$T = \frac{1}{2A\Phi_0^2} \leq T_{max}. \tag{1.17}$$

We remark that $\frac{1}{2A\Phi_0^2}$ is explicitly computable.

This paper is organized as follows.
In Section 2 we collect some results to be used in the proofs of the main theorems. In Section 3, we prove that $u(x, t)$, which blows up in $L^\infty(\Omega)$-norm, blows up also in $L^p(\Omega)$- norm with $p > \frac{n}{2}$ (Theorem 1.2). Moreover, by using a Gagliardo-Niremberg inequality, we prove Theorem 1.3. The Section 4 is dedicated to the case $\Omega \subset \mathbb{R}^3$ and contains the proofs of Theorem 1.4 and a corollary where a safe interval of existence of (u, v) say $[0, T]$ is derived with T an explicit lower bound of the blow up T_{max}.

2 Preliminaries

In this section we state some known results to be used in the proofs of the main theorems.
Throughout the paper we will assume the conditions contained in the Theorem1.1.
We need the following Gagliardo-Niremberg inequality.

Lemma 2.1. *Let* Ω *be a bounded and smooth domain of* $\mathbb{R}^n$, $n \geq 1$. *Let* $\mathsf{r} \geq 1$, $0 < \mathsf{q} \leq \mathsf{p} \leq \infty$, $\mathsf{s} > 0$, *then there exists a constant* $c_{GN} > 0$ *such that*

$$\|w\|_{L^{\mathsf{p}}(\Omega)} \leq c_{GN} \left(\|\nabla w\|^a_{L^{\mathsf{r}}(\Omega)} \|w\|^{1-a}_{L^{\mathsf{q}}(\Omega)} + \|w\|_{L^{\mathsf{s}}(\Omega)} \right) \tag{2.1}$$

for all $w \in L^q(\Omega)$ *with* $\nabla w \in L^r(\Omega)$, *and* $a := \frac{\frac{1}{\mathsf{q}} - \frac{1}{\mathsf{p}}}{\frac{1}{\mathsf{q}} + \frac{1}{n} - \frac{1}{\mathsf{r}}} \in [0, 1)$.

Proof. See [14] pag. 125. □

Lemma 2.2. *Let* $n \in \mathbb{N}$, $n \geq 3$. *Assume*

$$p_0 > \frac{n}{2}, \tag{2.2}$$

then for all $p > p_0$ it holds that

$$0 < \theta_1 < 1, \quad \theta_1 = \frac{\frac{p}{2p_0} - \frac{p}{2(p+1)}}{\frac{p}{2p_0} - \frac{1}{2} + \frac{1}{n}} \tag{2.3}$$

$$0 < \beta_1 < 1, \quad \beta_1 = \frac{p+1}{p}\theta_1, \tag{2.4}$$

$$0 < \theta_2 < 1, \quad \theta_2 = \frac{n}{2(p+1)}, \tag{2.5}$$

$$0 < \beta_2 < 1, \quad \beta_2 = \frac{p+1}{p}\theta_2 = \frac{n}{2p}. \tag{2.6}$$

Proof. From $p > p_0 > \frac{n}{2}$ we have $p > 1 - \frac{2}{n}$ and $\frac{p}{p+1} > 1 - \frac{2}{n}$ and (2.3) follows. The result (2.4) follows from hypothesis (2.2), infact we have $\frac{1}{2p_0} < \frac{1}{n}$ from which we obtain $\frac{p+1}{2p_0} - \frac{1}{2} < \frac{p}{2p_0} - \frac{1}{2} + \frac{1}{n}$ and (2.4) follows. Easily we obtain also (2.5) and (2.6). □

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$ be a bounded and smooth domain, $u \in C^0(\Omega)$ a positive function, and p, k, two positive real number such that $p + k - 1 > p > 0$. Then we have*

$$\int_\Omega u^{p+k-1} dx \geq |\Omega|^{\frac{1-k}{p}} \Big(\int_\Omega u^p dx \Big)^{\frac{p+k-1}{p}}. \tag{2.7}$$

Proof. The inequality follows from the Hölder's inequality. □

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$ be a bounded and smooth domain, and $\lambda \in \mathbb{R}$, $\mu > 0$, $k > 1$. Then for the solution (u, v) of (1.1) we have*

$$\int_\Omega u dx \leq \bar{m}, \quad \text{for all } t \in (0, T_{max}), \tag{2.8}$$

with

$$\bar{m} = \max\Big\{ \int_\Omega u_0 dx, \ \big(\frac{\lambda}{\mu}\ |\Omega|^{k-1}\big)^{\frac{1}{k-1}} \Big\}. \tag{2.9}$$

Proof. From the first equation in (1.1) we obtain

$$\frac{d}{dt}\int_\Omega udx = \lambda\int_\Omega udx - \mu\int_\Omega u^k dx \leq \lambda\int_\Omega udx - \mu|\Omega|^{1-k}\Big(\int_\Omega udx\Big)^k \tag{2.10}$$

where, in the last term we used the inverse of Hölder's inequality: $\int_\Omega u \leq |\Omega|^{\frac{k-1}{k}}\left(\int_\Omega u^k\right)^{\frac{1}{k}}$.
From (2.10) we infer that $y = \int_\Omega udx$ satisfies

$$\begin{cases} y'(t) \leq \lambda y(t) - \bar{\mu}y^k(t), \quad \bar{\mu} = \mu|\Omega|^{1-k}, \quad \textit{for all } t \in ([0, T_{max}) \\ y(0) = y_0. \end{cases} \tag{2.11}$$

Upon an ODE comparison argument this entails that

$$y(t) \leq \bar{m}, \quad \textit{for all } t \in (0, T_{max}).$$

This clearly proves the lemma. $\square$

Lemma 2.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ assumed to be star-shaped and convex in two orthogonal directions. For any nonnegative $w \in C^1(\Omega)$, the following inequality holds*

$$\begin{aligned} \int_\Omega w^3 dx \leq & \sqrt{2}\Big[a_1^{\frac{3}{2}}\Big(\int_\Omega w^2 dx\Big)^{\frac{3}{2}} + \frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3}\Big(\int_\Omega w^2\Big)^3 \\ & + \frac{3a_2^{\frac{3}{2}}\epsilon_1}{4}\int_\Omega |\nabla w|^2 dx\Big], \end{aligned} \tag{2.12}$$

with $\epsilon_1 > 0$ a suitable constant, and

$$a_1 = \frac{3}{2\rho_0}, \quad a_2 = \frac{d}{\rho_0} + 1, \quad \rho_0 = \min_{\partial\Omega} x_i\nu_i > 0, \;\; d^2 = \max_{\bar{\Omega}} x_i x_i.$$

Proof. The proof easily follows from the inequality (see Lemma A2 in [16])

$$\int_\Omega w^3 dx \leq \Big\{\frac{3}{2\rho_0}\int_\Omega w^2 dx + \Big(\frac{d}{\rho_0}+1\Big)\Big(\int_\Omega w^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{1}{2}}\Big\}^{\frac{3}{2}}. \tag{2.13}$$

In fact, in (2.13), firstly we apply the following arithmetic inequality

$$(2.14) \qquad (a+b)^{\frac{3}{2}} \le \sqrt{2}(a^{\frac{3}{2}}+b^{\frac{3}{2}}), \quad a,b>0,$$

to have

$$\int_\Omega w^3 dx \le \sqrt{2}\Big\{\Big(\frac{3}{2\rho_0}\int_\Omega w^2 dx\Big)^{\frac{3}{2}} + \Big(\frac{d}{\rho_0}+1\Big)^{\frac{3}{2}}\Big(\int_\Omega w^2 dx\Big)^{\frac{3}{4}}\Big(\int_\Omega |\nabla w|^2 dx\Big)^{\frac{3}{4}}\Big\},$$

and then, from an application of Young's inequality we get to (2.12). □

3 Blow up in L^p norm.

Throughout this section we are under the hypotheses of Theorem 1.1.

The goal of this section is to extend the result of Freitag (Theorem 2.2 in [3]) to solution (u, v) of problem (1.1).
In order to prove Theorem 1.3, first we state the following Lemmas.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, a bounded and smooth domain and (u, v) be a solution of* (1.1)*. If for some $p_0 > \frac{n}{2}$ exists a constant C such that*

$$(3.1) \qquad \|u\|_{L^{p_0}(\Omega)} \le C, \quad for\ all\ t \in (0, T_{max}),$$

then, for some $\hat{C} > 0$ and $p > p_0$

$$(3.2) \qquad \|u\|_{L^{p}(\Omega)} \le \hat{C}, \quad for\ all\ t \in (0, T_{max}),$$

Proof. Let $\Psi(t)$ be defined in (1.13) with $p > p_0$. Differentiating $\Psi(t)$, we have

$$(3.3) \qquad \begin{aligned} \Psi'(t) &= \int_\Omega u^{p-1}u_t dx = \int_\Omega u^{p-1}\Delta u dx - \int_\Omega u^{p-1}\nabla\cdot\Big(u\nabla v\Big)dx \\ &+ \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx = J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Now we estimate separately the four terms of (3.3)

$$J_1 = \int_\Omega \nabla\Big(u^{p-1}\nabla u\Big)dx - (p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx = \tag{3.4}$$
$$- \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx$$

$$J_2 = -\int_\Omega u^{p-1}\nabla u \nabla v dx - \int_\Omega u^p \Delta v dx. \tag{3.5}$$

We can estimate the first term in (3.5) as follow

$$\int_\Omega u^{p-1}\nabla u\nabla v dx = \int_\Omega \nabla\Big(u^{p-1}u\nabla v\Big)dx - \int_\Omega u\nabla\Big(u^{p-1}\nabla v\Big)dx$$
$$= -(p-1)\int_\Omega u^{p-1}\nabla u\nabla v dx - \int_\Omega u^p\Delta v dx$$

from which we obtain

$$\int_\Omega u^{p-1}\nabla u\nabla v dx = -\frac{1}{p}\int_\Omega u^p \Delta v dx. \tag{3.6}$$

Replacing (3.6) into (3.5) we arrive at

$$J_2 = -\Big(1-\frac{1}{p}\Big)\int_\Omega u^p\Delta v dx = -\Big(1-\frac{1}{p}\Big)\int_\Omega u^p v dx \tag{3.7}$$
$$+\Big(1-\frac{1}{p}\Big)\int_\Omega u^{p+1}dx \le \Big(1-\frac{1}{p}\Big)\int_\Omega u^{p+1}dx$$

In the last term of (3.7) we now use the Gagliardo-Niremberg inequality (2.1) with $w = u^{\frac{p}{2}}$, $\mathsf{r} = 2$, $\mathsf{p} = 2\frac{p+1}{p}$, $\mathsf{q} = \frac{2p_0}{p}$, $\mathsf{s} = \frac{2}{p}$. We have

$$\int_\Omega u^{p+1}dx = \|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^{2\frac{p+1}{p}}} \le c_{GN}\Big(\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+1}{p}\theta_1}_{L^2}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}(1-\theta_1)}_{L^{\frac{2p_0}{p}}} + \|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^{\frac{2}{p}}}\Big),$$

with $\theta_1 \in (0,1)$ and $\frac{p+1}{p}\theta_1 \in (0,1)$ defined in Lemma 2.2, having also made use of

$$(a+b)^\alpha \le 2^\alpha(a^\alpha + b^\alpha), \quad \text{for any } \ a,b\ge 0, \ \alpha > 0. \tag{3.8}$$

Since Lemma 2.2 and (3.1) hold, we apply Young's inequality in the previous inequality arriving to

$$
\begin{aligned}
\int_\Omega u^{p+1}dx \leq c_{GN}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx\Big)^{\frac{p+1}{p}\theta_1}\Big(\int_\Omega u^{p_0}dx\Big)^{(p+1)(1-\theta_1)}+ \\
c_{GN}\Big(\int_\Omega u dx\Big)^{p+1} \leq c_{GN}\epsilon_1\beta_1\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \\
c_{GN}\epsilon_1^{\frac{\beta_1}{1-\beta_1}}(1-\beta_1)C^{\frac{(p+1)(1-\theta_1)}{1-\beta_1}} + c_{GN}\bar{m}^{p+1} = \\
c_1\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_2,
\end{aligned} \tag{3.9}
$$

valid for any $\epsilon_1 > 0$, $c_1 = c_1(\epsilon_1) = c_{GN}\epsilon_1\beta_1$, $\beta_1 = \frac{p+1}{p}\theta_1 \in (0,1)$, $c_2 = c_2(\epsilon_1) = c_{GN}\epsilon_1^{\frac{\beta_1}{1-\beta_1}}(1-\beta_1)C^{\frac{(p+1)(1-\theta_1)}{1-\beta_1}} + c_{GN}\bar{m}^{p+1}$, with $\bar{m}$ and C defined respectively in (2.9) and (3.1) .
Replacing (3.9) into (3.7) leads to

$$
J_2 \leq c_1\Big(1-\frac{1}{p}\Big)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_2\Big(1-\frac{1}{p}\Big) \tag{3.10}
$$

In the third term of (3.3) we use, in order, Hölder's and Young's inequalities to obtain

$$
\begin{aligned}
J_3 \leq \lambda\Big(\int_\Omega u^{p+k-1}dx\Big)^{\frac{p}{p+k-1}}\Big|\Omega\Big|^{\frac{k-1}{p+k-1}} \leq \\
\lambda\epsilon_2\frac{p}{p+k-1}\int_\Omega u^{p+k-1}dx + \lambda\frac{k-1}{p+k-1}\epsilon_2^{-\frac{p}{k-1}}|\Omega| = \\
c_3\int_\Omega u^{p+k-1}dx + c_4,
\end{aligned} \tag{3.11}
$$

with $\epsilon_2 > 0$, $c_3 = c_3(\epsilon_2) = \lambda\epsilon_2\frac{p}{p+k-1}$, $c_4 = c_4(\epsilon_2) = \lambda\frac{k-1}{p+k-1}\epsilon_2^{-\frac{p}{k-1}}|\Omega|$.

$$
J_4 = -\mu\int_\Omega u^{p+k-1}dx. \tag{3.12}
$$

We now substitute (3.4), (3.10)-(3.12) in (3.3)

$$
\begin{aligned}
\Psi' \leq -\frac{(p-1)}{p}\big(\frac{4}{p}-c_1\big)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx - \\
(\mu - c_3)\int_\Omega u^{p+k-1}dx + c_5.
\end{aligned} \tag{3.13}
$$

with $c_5 = c_2\left(1 - \frac{1}{p}\right) + c_4$.
In (3.13) we choose ϵ_1, such that $\frac{4}{p} - c_1 \geq 0$ and ϵ_2, such that $\mu - c_3 \geq 0$. Neglecting the negative term $-\frac{(p-1)}{p}\left(\frac{4}{p} - c_1\right)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx$ and using (2.7) in the second term of (3.13) we obtain

$$\Psi' \leq -c_6 \Psi^\gamma + c_5,$$

with $c_6 = (\mu - c_3)|\Omega|^{\frac{1-k}{p}} p^{\frac{p+k-1}{p}}$ and $\gamma = \frac{p+k-1}{p}$.

Thanks to this result, we arrive at this initial problem

$$\begin{cases} \Psi'(t) \leq c_5 - c_6 \Psi^\gamma(t) & t \in (0, T_{max}), \\ \Psi(0) = \frac{1}{p}\int_\Omega u_0^p, \end{cases}$$

so, an application of a comparison principle leads to

$$\Psi(t) \leq \max\left\{\Psi(0), \left(\frac{c_5}{c_6}\right)^{\frac{1}{\gamma}}\right\} =: \hat{C} \quad \text{for all} \quad t \in (0, T_{max}). \tag{3.14}$$

Moreover, from this bound, elliptic regularity results applied to the second equation of system (1.1), i.e. $-\Delta v + v = u$, imply $v \in L^\infty((0, T_{max}); W^{2,p}(\Omega))$ and, hence, $\nabla v \in L^\infty((0, T_{max}); W^{1,p}(\Omega))$ and from Sobolev embedding theorems we have $v \in L^\infty((0, T_{max}); C^{[2-n/p]}(\bar{\Omega}))$ and $\nabla v \in L^\infty((0, T_{max}); L^q(\Omega))$ for all $n < q < p^* =: \frac{np}{n-p}$.

□

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, a bounded and smooth domain and (u, v) be the classical solution to system* (1.1). *If for some $\frac{n}{2} < p < n$, there exists $\hat{C} > 0$ such that:*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq \hat{C} \quad \textit{for all } t \in (0, T_{max}), \tag{3.15}$$

then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \textit{for all } t \in (0, T_{max}). \tag{3.16}$$

Proof. For any $x \in \Omega$ and $t \in (0, T_{max})$, we set $t_0 = \max\{t_0, t-1\}$ and we consider the representation formula for u:

$$
\begin{aligned}
u(\cdot,t) \leq & e^{(t-t_0)\Delta}u(\cdot,t_0) - \int_{t_0}^{t} e^{(t-s)\Delta}\nabla\cdot(u(\cdot,t)\nabla v)(\cdot,t)ds \\
& + \int_{t_0}^{t} e^{(t-s)\Delta}\big[\lambda u(\cdot,t) - \mu u^k(\cdot,t)\big]ds \\
& =: u_1(\cdot,t) + u_2(\cdot,t) + u_3(\cdot,t),
\end{aligned}
\tag{3.17}
$$

and

$$
\|u(\cdot,t)\|_{L^\infty} \leq \|u_1(\cdot,t)\|_{L^\infty} + \|u_2(\cdot,t)\|_{L^\infty} + \|u_3(\cdot,t)\|_{L^\infty}, \tag{3.18}
$$

Following the steps of Lemma 4.1 in [20], we obtain

$$
\|u_1(\cdot,t)\|_{L^\infty} \leq \max\Big\{\|u_0\|_{L^\infty},\ 2\bar{m}C_S\Big\} := c_7, \tag{3.19}
$$

C_S a positive constant and $\bar{m}$ defined in (2.9), and

$$
\|u_2(\cdot,t)\|_{L^\infty} \leq c_8, \tag{3.20}
$$

with c_8 a positive constant which plays the analogous role of the constant $\hat{C}_S$ of Lemma 4.1 in [20].
Now we prove that there exists a constant $c_9 > 0$ such that $\|u_3\|_{L^\infty} \leq c_9$. To this end, we firstly observe that

$$
h(u) = \lambda u - \mu u^k \leq h(u_*) := c_9,
$$

with $u_* = \left(\frac{\lambda}{\mu k}\right)^{\frac{1}{k-1}}$.

We have

$$
\begin{aligned}
\|u_3(\cdot,t)\|_{L^\infty} & \leq \int_{t_0}^{t} \|e^{(t-s)\Delta}\big[\lambda u(\cdot,t) - \mu u^k(\cdot,t)\big]\|_{L^\infty(\Omega)}ds \\
& \leq \int_{t_0}^{t} \|c_9 e^{(t-s)\Delta}\|_{L^\infty}ds = c_9(t-t_0) \leq c_9
\end{aligned}
\tag{3.21}
$$

From (3.19), (3.20), (3.21) we arrive at (3.16) with $\hat{C} = c_7 + c_8 + c_9$.

□

Proof of Theorem 1.2

Proof. From Theorem 1.1, the unique local classical solution of (1.1) blows up at $t = T_{max}$ in the sense $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ ((1.11)). By contradiction we prove that it blows up also in L^p-norm. In fact, if exist $p_0 > \frac{n}{2}$ and $C > 0$ such that,

$$\|u\|_{L_{p_0}(\Omega)} \leq C$$

then, from Lemma 3.1 exists a constant $\hat{C} > 0$ such that

$$\|u\|_{L^p(\Omega)} < \hat{C} \text{ for all } t \in (0, T_{max}),$$

and

$$\begin{cases} u \in L^\infty((0,T_{max}); L^p(\Omega)) & (\text{for } p > \frac{n}{2}), \\ u\nabla v \in L^\infty((0,T_{max}); L^{q_1}(\Omega)) & \text{for all } q_1 > n+2. \end{cases} \tag{3.22}$$

From Lemma 3.2 there exists $\hat{C} > 0$ such that,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \text{for all} \quad t \in (0, T_{max}),$$

which is in contradiction to the hypothesis (1.11), so that, if u blows up in L_∞-norm and $p > p_0 > \frac{n}{2}$ then u blows up also in L_p-norm. □

Proof of Theorem 1.3

Proof. We start from (3.3) and we use (3.4), (3.7) to write

$$\begin{aligned} \Psi'(t) \leq & -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \Big(1 - \frac{1}{p}\Big)\int_\Omega u^{p+1}dx + \lambda \int_\Omega u^p dx \\ & - \mu \int_\Omega u^{p+k-1} dx. \end{aligned} \tag{3.23}$$

In the second term of (3.23) we apply the Gagliardo-Niremberg inequality (2.1) with $\mathsf{p} = 2\frac{p+1}{p}$, $\mathsf{r} = \ \mathsf{q} = \ \mathsf{s} = 2$,

$$\begin{aligned} \int_\Omega u^{p+1} = & \big\|u^{\frac{p}{2}}\big\|^{2\frac{p+1}{p}}_{L^{2\frac{p+1}{p}}} \leq \bar{c}_{GN}\big\|\nabla u^{\frac{p}{2}}\big\|^{2\frac{p+1}{p}\theta_2}_{L^2}\big\|u^{\frac{p}{2}}\big\|^{2\frac{p+1}{p}(1-\theta_2)}_{L^2} \\ & + \bar{c}_{GN}\big\|u^{\frac{p}{2}}\big\|^{2\frac{p+1}{p}}_{L^2}, \end{aligned} \tag{3.24}$$

where θ_2 is defined in (2.5), having also made use (3.8).
Using the expression of θ_2, we rewrite (3.24) and then applying Young's inequality, we have

$$
\begin{aligned}
\int_\Omega u^{p+1} \leq & \bar{c}_{GN}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx\Big)^{\frac{n}{2p}}\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-n}{2p}} \\
& + \bar{c}_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \leq a_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx \\
& + a_2\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-n}{2p-n}} + \bar{c}_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}},
\end{aligned}
\tag{3.25}
$$

with $a_1 = a_1(\varepsilon_1) = \frac{n}{2p}\varepsilon_1 \bar{c}_{GN},\ a_2 = a_2(\varepsilon_1) = \frac{2p-n}{2p}\varepsilon_1^{-\frac{n}{(2-n)p}}\bar{c}_{GN},\ \varepsilon_1 > 0.$
By replacing (3.25) and (2.7) into (3.23) we arrive at

$$
\begin{aligned}
& \Psi'(t) \leq -\Big(\frac{p-1}{p}\Big)\Big(\frac{4}{p} - a_1\Big)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \\
& \Big(1-\frac{1}{p}\Big)a_2\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-n}{2p-n}} + \Big(1-\frac{1}{p}\Big)\bar{c}_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} + \\
& \lambda\int_\Omega u^p dx - \mu|\Omega|^{\frac{1-k}{p}}\Big(\int_\Omega u^p dx\Big)^{\frac{p+k-1}{p}}.
\end{aligned}
\tag{3.26}
$$

Choosing ε_1 in (3.26) such that $\Big(\frac{p-1}{p}\Big)\Big(\frac{4}{p} - a_1\Big) \geq 0$ we can neglect the first term and the fifth (negative) term in (3.26). Using the definition of $\Psi(t) = \frac{1}{p}\int_\Omega u^p dx$ we obtain the following first order differential inequality on Ψ

$$
\Psi'(t) \leq B_1\Psi + B_2\Psi^{\frac{p+1}{p}} + B_3\Psi^{\frac{2(p+1)-n}{2p-n}},
\tag{3.27}
$$

with $B_1 = \lambda p,\quad B_2 = \Big(1-\frac{1}{p}\Big)\bar{c}_{GN}p^{\frac{p+1}{p}},\quad B_3 = \Big(1-\frac{1}{p}\Big)a_2 p^{\frac{2(p+1)-n}{2p-n}}.$
Integrating (3.27) from 0 to T_{max} we obtain (1.14).

□

4 An explicit lower bound of T_{max} in $\Omega \subset \mathbb{R}^3$

In this section we consider the L^2-norm of u defined in (1.15) as $\Phi(t) = \|u\|_2^2, t \in [0, T_{max})$ with $\Phi_0 = \Phi(0) := \|u_0\|_2^2$.

Under the hypotheses of Theorem 1.1, we assume the spatial convex domain $\Omega \subset \mathbb{R}^3$. Let $[0,T]$, $T < T_{max}$ be the time interval of existence of the solution of (1.1): we have $\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. From Theorem 1.2, selecting $p = 2$ (which fits with the choice $n = 3$ in the condition $p > \frac{n}{2}$), necessarily the classical solution (u,v) of (1.1) blows up in L^2-norm at $t = T_{max}$. In this situation we prove that $\Phi(t)$ satisfies a differential inequality of the first order stated in Theorem 1.4 and as a consequence we determine a lower bound of the lifespan T_{max} by proving Corollary 1.1.

Proof of Theorem 1.4

Proof. By differentiating (1.15) and using the equation in (1.1), we have

$$\begin{aligned} \Phi'(t) =& 2\int_\Omega u u_t dx = 2\int_\Omega u\Delta u dx - 2\int_\Omega u\nabla\cdot\Big(u\nabla v\Big)dx \\ &+ 2\lambda\int_\Omega u^2 dx - 2\mu\int_\Omega u^{k+1}dx = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.1}$$

We now estimate the terms in (4.1) in order to arrive to a first order differential inequality in terms of powers of Φ.

$$I_1 = 2\int_\Omega u\Delta u dx = 2\int_\Omega \nabla\cdot\Big(u\nabla u\Big)dx - 2\int_\Omega |\nabla u|^2 dx = -2\int_\Omega |\nabla u|^2 dx. \tag{4.2}$$

Using the divergence theorem and the second equation in (1.1) we can write

$$\begin{aligned} I_2 &= -2\int_\Omega u\nabla\cdot\Big(u\nabla v\Big)dx = -\int_\Omega \nabla\cdot\Big(u^2\Delta v\Big)dx - \int_\Omega u^2\Delta v dx \\ &= -\int_\Omega u^2\Delta v dx = -\int_\Omega u^2 v dx + \int_\Omega u^3 dx. \end{aligned} \tag{4.3}$$

To bound the last term in (4.3) in term of Φ and $\int_\Omega |\nabla u|^2 dx$ firstly we make use of (2.12) (with $w = u$) in Lemma 2.5 and neglecting the negative term $-\int_\Omega u^2 v dx$ we obtain

$$I_2 \le \sqrt{2}a_1^{\frac{3}{2}}\Big(\int_\Omega u^2 dx\Big)^{\frac{3}{2}} + \sqrt{2}\frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3}\Big(\int_\Omega u^2\Big)^3 + \sqrt{2}\frac{3a_2^{\frac{3}{2}}\epsilon_1}{4}\int_\Omega |\nabla u|^2 dx. \tag{4.4}$$

Using Holder inequality, we bound the last term in (4.1)

$$I_4 = -2\mu \int_\Omega u^{k+1} dx \leq -2\mu |\Omega|^{\frac{1-k}{2}} \Big(\int_\Omega u^2 dx \Big)^{\frac{k+1}{2}} \tag{4.5}$$

We replace (4.2), (4.4) and (4.5) in (4.1) we arrive at

$$\begin{aligned}\Phi'(t) \leq & 2\lambda \int_\Omega u^2 dx - 2\mu |\Omega|^{\frac{1-k}{2}} \Big(\int_\Omega u^2 dx \Big)^{\frac{k+1}{2}} + \sqrt{2} a_1^{\frac{3}{2}} \Big(\int_\Omega u^2 dx \Big)^{\frac{3}{2}} \\ & + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3} \Big(\int_\Omega u^2 \Big)^3 + \Big(\sqrt{2} \frac{3 a_2^{\frac{3}{2}} \epsilon_1}{4} - 2 \Big) \int_\Omega |\nabla u|^2 dx.\end{aligned}$$

Choosing $\epsilon_1 = \frac{4\sqrt{2}}{3} a_2^{-\frac{3}{2}}$, we have

$$\Phi'(t) \leq 2\lambda \Phi - 2\mu |\Omega|^{\frac{1-k}{2}} \Phi^{\frac{k+1}{2}} + \sqrt{2} a_1^{\frac{3}{2}} \Phi^{\frac{3}{2}} + \sqrt{2} \frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3} \Phi^3. \tag{4.6}$$

Since $1 < k < 7/6$, and $p > \frac{n}{2}$, $u(x,t)$ blows up in L^2-norm at finite time T_{max} then $\Phi(t)$ can be non decreasing, so that $\Phi(t) \geq \Phi_0$ with $t \in [0, T_{max})$, or non increasing (possibly with some kind of oscillations), in which case there exists a time $t_1 \in [0, T_{max})$ where $\Phi(t_1) = \Phi_0$. As a consequence, $\Phi(t) \geq \Phi_0$ $\forall t \in [t_1, T_{max})$. It implies that

$$\frac{\Phi}{\Phi_0} \leq \Big(\frac{\Phi}{\Phi_0} \Big)^{\frac{k+1}{2}}, \quad t \in [t_1, T_{max})$$

from which

$$-\Phi^{\frac{k+1}{2}} \leq -\Phi \Phi_0^{\frac{k-1}{2}}, \quad t \in [t_1, T_{max}). \tag{4.7}$$

Moreover

$$\Phi^{\frac{3}{2}} \leq \Phi^3 \Phi_0^{-\frac{3}{2}}, \quad t \in [t_1, T_{max}). \tag{4.8}$$

We substitute (4.7) and (4.8) into (4.6) to have

$$
\begin{aligned}
\Phi'(t) \leq & 2\Big(\lambda - \mu|\Omega|^{\frac{1-k}{2}}\Phi_0^{\frac{k-1}{2}}\Big)\Phi + \Big(\sqrt{2}a_1^{\frac{3}{2}}\Phi_0^{-\frac{3}{2}} + \sqrt{2}\frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3}\Big)\Phi^3 \\
= & A_1\Phi + A_2\Phi^3, \quad t \in [t_1, T_{max}),
\end{aligned}
$$

with $A_1 = 2\Big(\lambda - \mu|\Omega|^{\frac{1-k}{2}}\Phi_0^{\frac{k-1}{2}}\Big)$, $A_2 = \sqrt{2}a_1^{\frac{3}{2}}\Phi_0^{-\frac{3}{2}} + \sqrt{2}\frac{a_2^{\frac{3}{2}}}{4\epsilon_1^3}$.
At last we can write

$$
\Phi'(t) \leq A\Phi^3, \quad t \in [t_1, T_{max}) \tag{4.9}
$$

where the positive constant A depends on $\|u_0\|_2$, k, ; μ, $|\Omega|$, so defined

$$
A = \begin{cases} A_1\Phi_0^{-2} + A_2, \quad \text{if } \lambda > \mu|\Omega|^{\frac{1-\mathrm{k}}{2}}\Phi_0^{\frac{\mathrm{k}-1}{2}}, \\ A_2, \quad \text{if } \ \lambda \leq \mu|\Omega|^{\frac{1-\mathrm{k}}{2}}\Phi_0^{\frac{\mathrm{k}-1}{2}}, \end{cases} \tag{4.10}
$$

and (1.16) is proved. □

Proof of Corollary 1.1.

Proof. Integrating (4.9) from t_1 to T_{max} we lead to

$$
\frac{1}{2\Phi_0^2} = \int_{\Phi_0}^{\infty} \frac{d\eta}{\eta^3} \leq \int_{t_1}^{T_{max}} A\, d\tau \leq \int_0^{T_{max}} A\, d\tau = A\, T_{max} \tag{4.11}
$$

from which we obtain (1.17): it means that the solution of (1.1) exists bounded in the interval $[0, T]$, with $T = \frac{1}{2A\Phi_0^2}$, the lower bound of the lifespan T_{max}. □

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References

[1] N. Bellomo, A. Belloquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel model of pattern formation in biological tissues, *Math. Mod. Meth. Appl. Sci.,* **25** (9) (2015), 1663–1763.

[2] X. Cao and S. Zheng, Boundedness of solutions to a quasilinear parabolic elliptic Keller-Segel system with logistic source, *Mathematical Methods in the Applied Sciences,* **37** (15), (2014) 2185-2340.

[3] M.Freitag, Blow–up profiles and refined extensibility criteria in quasilinear Keller–Segel systems, *J. Math. Anal. Appl.*, **463**(2) (2018), 964–988.

[4] T. Hillen and K.J. Painter, A User's Guide to PDE Models for Chemotaxis, *J. Math. Biol.*, **58** (1), (2009), 183–217.

[5] D. Horstmann, From1970 until present: The Keller-Segel model in chemotaxis and its consequences, *Der Jahresbericht der Deutschen Mathematiker- Vereinigung* (2003); **105**(3):103–165.

[6] W. Jager and S. Luckhaus, On explosion of solutions to a system of partial differential equations modelling chemotaxis, *Trans. Amer. Math. Soc*, **329**, n.2, (1992) 819–824.

[7] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.* **26** (1970), 399-–415.

[8] M. Marras, T. Nishino and G. Viglialoro, A refined criterion and lower bounds for the blow-up time in a parabolic–elliptic chemotaxis system with nonlinear diffusion, *Nonlinear Analysis* (2019) 111725, https://doi.org/10.1016/j.na.2019.111725.

[9] M. Marras and G. Porru, Estimates and uniqueness for boundary blow-up solutions of P-Laplace equations, *Electronic Journal of Differential Equations* **119** (2011), 1–10.

[10] M. Marras and S. Vernier-Piro, Blow-up time estimates in nonlocal reaction-diffusion systems under various boundary conditions, *Boundary value problems* **2017**, 2, (2017), https://doi.org/10.1186/s13661-016-0732-2.

[11] M. Marras, S. Vernier-Piro and G. Viglialoro, Lower bounds for blow-up time in a parabolic problem with a gradient term under various boundary conditions, *Kodai Mathematical Journal* **37**, 3, (2014), 532–543.

[12] M. Marras and G. Viglialoro, Blow-up time of a general Keller-Segel system with source and damping terms, *Comptes Rendus de l'Academie Bulgare des Sciences* **69** (6), (2016), 687–696 .

[13] T. Nagai, Behavior of solution to a parabolic-elliptic system modelling chemotaxis, *J. Korean Math. Soc.* **37** (2000), 721–733.

[14] L. Nirenberg, Onelliptic partial differential equations, *Ann. Scuola Norm. Sup.Pisa* **3** (1959), 13:115–162.

[15] T. Nishino and T. Yokota, Effect of nonlinear diffusion on a lower bound for the blow-up time in a fully parabolic chemotaxis system, *J. Math. Anal. Appl.*, **479** (2019), no.1, 1078–1098.

[16] L.E. Payne, G.A. Philippin and S. Vernier-Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition II, *Nonlinear Analysis*, **73** (2010), 971-978

ck *Appl. Anal.*, **85** (2006), 1301–1311.

[17] L.E. Payne and J.C. Song, Lower bound for blow-up time in a model of chemotaxis, *J. Math. Anal. Appl.*, **385**(2012), 672–676.

[18] Ph. Souplet and M. Winkler, Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimension $n \geq 3$, *Communications in Mathematical Physics*, **367**, (2019), 665-681.

[19] Tello and J.I., Winkler, M. A chemotaxis system with logistic source, *Comm. Part. Differential Eq.* **32** (6), (2007), 849–877.

[20] G. Viglialoro and T. E. Woolley. Solvability of a Keller–Segel system with signal-dependent sensitivity and essentially sublinear production, *Applicable Analysis*, (2019) DOI: 10.1080/00036811.2019.1569227

[21] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *Journal de Mathematiques Pures et Appl.*, **100** (5),(2013), 748–767.

[22] M. Winkler, Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation, *Z. Angew. Math. Phys.*, **69**(40), (2018), 1–25.