Why We Can Approximate Spheroidal Geopotential Surfaces as Spherical but Can't Approximate True Geopotential Surfaces as Spheroidal in Atmospheric and Oceanic Modeling

Peter Chu[1] and Peter C Chu[2]

[1]Affiliation not available
[2]Department of Oceanography, Naval Postgraduate School

April 15, 2024

Why We Can Approximate Spheroidal Geopotential Surfaces as Spherical but Can't Approximate True Geopotential Surfaces as Spheroidal in Atmospheric and Oceanic Modeling

Peter C. Chu

Department of Oceanography, Naval Postgraduate School, Monterey, CA 93943, USA

Corresponding Author: Peter Chu (pcchu@nps.edu)

Orcid ID: 0000-0002-3101-3555

Key Points:

- Horizontal pressure gradient error is negligible in approximating spheroidal geopotential surfaces as spherical.
- Horizontal pressure gradient error is nonnegligible gravity disturbance vector in approximating true geopotential surfaces as spheroidal.
- Horizontal gravity disturbance vector should be included in any analytical and numerical atmospheric and oceanic models.

Key Words: Nonuniform Earth Mass Density, True Geopotential, Spheroidal Geopotential, Spherical Geopotential, Horizontal Pressure Gradient Error, Horizontal Gravity Disturbance Vector

Abstract

Horizontal pressure gradient error (HPGE) in coordinate transformation is well known in meteorology and oceanography. However, HPGE has been ignored completely in spherical, spheroidal, and true geopotential coordinate transformations. Let gravitational acceleration on a point-mass in atmosphere or oceans be $\mathbf{n}_0$ with uniform Earth mass density and be $\mathbf{n}$ with nonuniform Earth mass density such as 5 structural layers. Combination of $\mathbf{n}_0$ with centrifugal acceleration leads to apparent gravity $\mathbf{g}_a$ and associated spheroidal geopotential Φ_a. Combination of $\mathbf{n}$ with centrifugal acceleration leads to true gravity $\mathbf{g}_t$ and associated true geopotential Φ_t. Subtraction of $\mathbf{n}_0$ from $\mathbf{n}$ is the gravity disturbance vector, $\delta\mathbf{g} \equiv \mathbf{n} - \mathbf{n}_0$. Spherical geopotential approximation (SGA) is to approximate the spheroidal geopotential (Φ_a) as spherical geopotential (Φ_s) corresponding to standard gravity $\mathbf{g}_s$ (i.e., to approximate $\mathbf{g}_a$ as $\mathbf{g}_s$ or to transform spheroidal into spherical geopotential coordinates). Spheroidal (ellipsoidal) geopotential approximation proposed by Chang et al. (2023) (EGA-CWSM) is to approximate the true geopotential surfaces as spheroidal (i.e., to approximate $\mathbf{g}_t$ as $\mathbf{g}_a$ or to transform true into spheroidal geopotential coordinates). EGA-CWSM is totally different from the earlier proposed EGA-SB (Staniforth 2014; Beńard 2014). The horizontal momentum equation does not change from transforming spheroidal to spherical geopotential coordinates due to negligible HPGE but does change evidently from transforming true to spheroidal geopotential coordinates due to nonnegligible HPGE, which equals the horizontal gravity disturbance vector. Thus, EGA-CWSM is invalid. It is urgent to include the horizontal gravity disturbance vector in atmospheric and oceanic models.

Plain Language Summary

The effect of the solid Earth with nonuniform versus uniform mass densities on atmospheric and oceanic dynamics is identified through geopotential coordinate transformation. The true gravity due to the nonuniform Earth mass density is associated with the true geopotential coordinates. The apparent gravity due to the uniform Earth mass density is associated with the spheroidal geopotential coordinates. The spherical geopotential approximation is to approximate the spheroidal geopotential surfaces as spherical. Transformation among the true, spheroidal, and spherical geopotential coordinates leads to the horizontal pressure gradient error, which is negligible in using the spherical geopotential approximation but nonnegligible with equalling the horizontal gravity disturbance vector in approximating the true geopotential surfaces as spheroidal. Thus, we should include the horizontal gravity disturbance vector in atmospheric and oceanic models.

1 Introduction

Spherical, spheroidal, and true geopotential surfaces and associated geopotential coordinates exist in meteorology and oceanography. Among them, the spherical geopotential (Φ_s) coordinates are used most often to represent the global atmosphere with the outward unit vector $\mathbf{k}_s$ in the radial direction. The spheroidal geopotential (Φ_a) coordinates are established more recently for numerical modelling (e.g., Gates 2004; White et el. 2008; Beńard 2014; Staniforth 2014; Staniforth and White 2015) with the outward unit vector $\mathbf{k}_a$ perpendicular to the spheroidal (ellipsoidal) geopotential surfaces. The true geopotential (Φ_t) coordinates are only used recently for theoretical studies (Chu 2021a; Chang et al. 2023; Chu 2021a, 2023, 2024) with the outward unit vector $\mathbf{k}_t$ perpendicular to the true geopotential surfaces. The corresponding standard gravity ($\mathbf{g}_s$), apparent gravity ($\mathbf{g}_a$), and true gravity ($\mathbf{g}_t$) are given by.

$$\mathbf{g}_s = -|\mathbf{g}_s|\mathbf{k}_s, \quad \mathbf{g}_a = -|\mathbf{g}_a|\mathbf{k}_a, \quad \mathbf{g}_t = -|\mathbf{g}_t|\mathbf{k}_t \tag{1}$$

where $|\mathbf{g}_s| = g_0 = 9.81$ m s^{-2} (constant) due to negligible radial variation of $\mathbf{g}_s$ in the oceans and combined troposphere and stratosphere due to their thin thicknesses in comparison to the Earth radius. The angles between $\mathbf{k}_t$ and $\mathbf{k}_a$ and between $\mathbf{k}_a$ and $\mathbf{k}_s$ are small (10^{-5} – 10^{-4} radian) (Gill 1982, Chang, and Wolfe 2022). Deviation of $|\mathbf{g}_a|$ and $|\mathbf{g}_t|$ from g_0 is less than two orders of magnitude than g_0 (e.g., Gill 1982; Staniforth 2014),

$$\frac{\left||\mathbf{g}_a| - g_0\right|}{g_0} < 10^{-2}, \quad \frac{\left||\mathbf{g}_t| - g_0\right|}{g_0} < 10^{-2} \tag{2}$$

If neglecting such small differences, Eq (1) becomes,

$$\mathbf{g}_s = -g_0\mathbf{k}_s, \quad \mathbf{g}_a \cong -g_0\mathbf{k}_a, \quad \mathbf{g}_t \cong -g_0\mathbf{k}_t \tag{3}$$

Let ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}_a$) be the unit vectors of confocal hyperboloids spheroidal geopotential coordinates (λ, φ, ξ) with λ the longitude, φ the geodetic latitude, and ξ the dimensionless parameter for spheroidal geopotential surface as depicted in Gates (2004). The other spheroidal geopotential coordinates using approximated spheroidal geopotentials (Beńard 2014; Staniforth 2014) will be discussed in subsection 5.2. Let ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}_t$) be the unit vectors of the true geopotential coordinates (λ, φ, z_t) with z_t denoting vertical coordinate. Let ($\mathbf{i}$, $\mathbf{j}_s$, $\mathbf{k}_s$) be the unit vectors of the spherical geopotential coordinates (λ, φ_s, r) with φ_s the geocentric latitude and r the radial coordinate. Let three-dimensional velocity vector be $\mathbf{V}_s = (u_s, v_s, w_s)$, using the standard gravity $\mathbf{g}_s$ in the spherical geopotential coordinates; be $\mathbf{V}_a = (u_a, v_a, w_a)$, using the apparent gravity $\mathbf{g}_a$ in the spheroidal geopotential coordinates; and be $\mathbf{V}_t = (u_t, v_t, w_t)$, using the true gravity $\mathbf{g}_t$ in the true geopotential coordinates. The horizontal velocity vectors are represented by $\mathbf{U}_s = (u_s, v_s)$, $\mathbf{U}_a = (u_a, v_a)$, and $\mathbf{U}_t = (u_t, v_t)$ in corresponding geopotential coordinates.

All the three geopotential coordinates are curved and quasi-orthogonal. The unit vectors in the spherical and spheroidal geopotential coordinates vary in space. However, the unit vectors in the true geopotential coordinates vary in space and time. The temporal variation of the true geopotential is on very long time scale in meteorological sense because the physical processes to change the mass density $\sigma(\mathbf{r}, t)$ inside the solid Earth is slow and excluded in this study. For a non-globally Cartesian system such as the true geopotential coordinates, the acceleration vector $D\mathbf{V}/Dt$ (Holton and Hakim 2013),

$$\frac{D\mathbf{V}}{Dt} = \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} + u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} \tag{4a}$$

is separated into two parts,

$$\frac{d\mathbf{V}}{dt} \equiv \mathbf{i}\frac{Du}{Dt} + \mathbf{j}\frac{Dv}{Dt} + \mathbf{k}\frac{Dw}{Dt} \tag{4b}$$

$$\mathbf{m} \equiv u\frac{D\mathbf{i}}{Dt} + v\frac{D\mathbf{j}}{Dt} + w\frac{D\mathbf{k}}{Dt} \tag{4c}$$

where ($\mathbf{V}$, $D\mathbf{V}/Dt$, $d\mathbf{V}/dt$, $\mathbf{m}$, $\mathbf{j}$, $\mathbf{k}$) represents one of [($\mathbf{V}_s$, $D\mathbf{V}_s/Dt$, $d\mathbf{V}_s/dt$, $\mathbf{m}_s$, $\mathbf{j}_s$, $\mathbf{k}_s$), ($\mathbf{V}_a$, $D\mathbf{V}_a/Dt$, $d\mathbf{V}_a/dt$, $\mathbf{m}_a$, $\mathbf{j}$, $\mathbf{k}_a$), ($\mathbf{V}_t$, $D\mathbf{V}_t/Dt$, $d\mathbf{V}_t/dt$, $\mathbf{m}_t$, $\mathbf{j}$, $\mathbf{k}_t$)]; $d\mathbf{V}/dt$ is the acceleration vector as if it is in the global Cartesian system; and $\mathbf{m}$ denotes the metric terms [or called the curvature terms in Holton and Hakim (2013)].

On the base of small metric term difference between the spheroidal ($\mathbf{m}_a$) and spherical ($\mathbf{m}_s$) geopotential coordinates (Gill 1982),

$$\frac{O\left(|\mathbf{m}_a - \mathbf{m}_s|\right)}{O\left(|\mathbf{m}_s|\right)} \simeq 0.17\%, \tag{5}$$

the spherical geopotential approximation (SGA) was proposed to approximate the spheroidal geopotential surfaces for the apparent gravity $\mathbf{g}_a$ as spherical. Such an approximation was confirmed by numerical modelling studies such as Gates (2004), Staniforth and White (2014), however, small systematic differences may accumulate in long-term simulations (Gates 2004). With the SGA, almost all analytical and numerical atmospheric models use the spherical geopotential coordinates (i.e., corresponding to the standard gravity $\mathbf{g}_s$) and related local coordinates.

Two types of spheroidal geopotential coordinates have been established with the one based on the confocal hyperboloids (e.g., Gates 2004) and the other based on the simplified spheroidal geopotential $\hat{\Phi}_a$ to represent Φ_a (e.g., Staniforth and White 2014; Beńard 2014). Such a simplification (i.e., to approximate Φ_a as $\hat{\Phi}_a$) is called the spheroidal geopotential approximation by Staniforth (2004) and Beńard (2014), hereafter referred to EGA-SB. Note that the EGA-SB is only for the spheroidal geopotential coordinates and does not involve the spherical and true geopotential coordinates.

On the base of small metric errors between the true and spheroidal geopotential coordinates (Chang and Wolfe 2022), Chang et al. (2023) proposed a different spheroidal geopotential approximation "*to approximate the true geopotential surfaces for the true gravity* $\mathbf{g}_t$ *as spheroidal*," which is referred here as the EGA-CWSM to distinguish from the EGA-SB. A question arises: Can we confirm the SGA and EGA-CWSM only on the base of small metric errors? The answer is obviously negative because these two approximations involve the curved quasi-orthogonal coordinate transformation from spheroidal to spherical geopotential coordinates (SGA) and from true to spheroidal geopotential coordinates (EGA-CWSM), where the horizontal pressure gradient error (HPGE) needs to be investigated.

Chu (2021a, b, c) introduced the gravity disturbance vector $\delta\mathbf{g}$ (then called horizontal gravity $\mathbf{g}_h$) into the horizontal equations of motion in atmosphere and oceans in the spherical geopotential coordinates, and used the publicly available meteorological, oceanographic, and geodetic datasets to confirm $\delta\mathbf{g}$ nonnegligible. Comments on Chu (2021 a, b, c) by Chang and Wolfe (2022) and Stewart and McWilliams (2022), claimed $\delta\mathbf{g}$ negligible in atmospheric and oceanic dynamics, were based on small metric errors, wrong comparison, and wrong derivations, and ignorance of the HPGE in the true to spherical geopotential coordinate transformation.

Comments and critics on research papers are common in scientific journals. Replies versus comments largely advance scientific knowledge. It is quite unusual in this case that Chang and Wolfe (2022) and Stewart and McWilliams (2022) with severe mistakes (see Appendix A) were published in the Scientific Reports (SR). However, Chu's replies submitted to SR (also sent to Chang, McWilliams, Stewart, and Wolfe on 20 April 2022) (see website https://ars.els-cdn.com/content/image/1-s2.0-S0377026523000209-mmc1.pdf) was rejected for publication in SR and the paper (Chu, 2021a) was mistakenly retracted by the Chief Editor of SR. Later, the then Editor-in-Chief of the Journal of Geophysical Research – Atmospheres (Minghua Zhang) disregarded responses from Chu and mistakenly retract the paper (Chu, 2021b) on 30 September 2022. Chu (2023) demonstrated the importance of the horizontal gravity disturbance vector in atmospheric dynamics. Chang et al. (2023) commented on Chu (2023) with the same mistakes such as neglecting the HPGE in the geopotential coordinate transformation and others.

As prominent atmospheric and oceanic fluid dynamitists, Chang, McWilliams, Stewart, and Wolfe have misled and continue to mislead the meteorological and oceanographic communities. To counter their negative influences, Chu (2024) replied to the comments by Chang et al. (2023) and showed the **EGA-CWSM invalid due to the nonnegligible HPGE** (equaling the horizontal gravity disturbance vector) in transforming the true to spheroidal geopotential coordinates. However, a question remains: **Why is the SGA valid but not the EGA-CWSM?** To answer this question, two types of spheroidal geopotential coordinates are used to identify the HPGE in transforming the spheroidal to spherical geopotential coordinates.

The rest of the paper is organized as follows. Section 2 shows the gravitational acceleration with nonuniform and uniform Earth mass density. Section 3 presents the horizontal momentum equations in the spherical, spheroidal, and true geopotential coordinates with corresponding gravities. Section 4 lists the mathematical expressions of the SGA and EGA-CWSM. Section 5 uses the horizontal atmospheric equations of motion with two types of spheroidal geopotential coordinates to confirm negligible HPGE in transforming spheroidal to spherical geopotential coordinates and to confirm the SGA. Section 6 uses the relationship between the orthometric, and ellipsoidal (spheroidal) heights commonly employed in the geodetic community to confirm the nonnegligible HPGE in transforming true to spheroidal geopotential coordinates (same as in Chu 2024). Section 7 presents the conclusions. Appendix A lists the mistakes in Chang and Wolfe (2022), Stewart and McWilliams (2022), and Chang et al. (2023).

2 Gravitational Acceleration with Nonuniform and Uniform Earth Mass Density

Newton's law of universal gravitation states that every point mass attracts every other point mass by a force acting along the line intersecting the two points. The force is proportional to the product of the two-point masses, and inversely proportional to the square of the distance between them. The Newton's gravitational force ($\mathbf{F}_N$) of solid Earth on an atmospheric point mass (m_A) at location $\mathbf{r}_A$ is the volume integration over all the point masses located at $\mathbf{r}$ inside the solid Earth (Figure 1) with the formula [Equation (6.4) in Vaniček and Krakiwsky 1986]

$$\mathbf{F}_N(\mathbf{r}_A) = m_A\mathbf{n}, \;\; \mathbf{n} = G\iiint_{\Pi} \frac{\sigma(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_A|^3}(\mathbf{r}-\mathbf{r}_A)d\Pi \tag{6}$$

where $G = 6.67408\times10^{-11}\text{Nm}^2\text{kg}^{-2}$, is the Newtonian gravitational constant; [$\sigma(\mathbf{r})$, Π] are the mass density and volume of the solid Earth; $\mathbf{n}$ is the true gravitational acceleration, and the Earth center is the origin of the position vectors $\mathbf{r}$ and $\mathbf{r}_A$. Combination of $\mathbf{n}$ and centrifugal acceleration leads to the true gravity $\mathbf{g}_t$.

Let σ_0 be the average mass density. With σ_0, Eq (6) becomes,

$$\mathbf{F}_N(\mathbf{r}_A) = -m_A\frac{GM}{|\mathbf{r}_A|^3}\mathbf{r}_A + Gm_A\iiint_{\Pi} \frac{[\sigma(\mathbf{r})-\sigma_0]}{|\mathbf{r}-\mathbf{r}_A|^3}(\mathbf{r}-\mathbf{r}_A)d\Pi \tag{7}$$

where $M = \sigma_0\Pi = 5.98\times10^{24}$ kg is the total mass of the solid Earth.

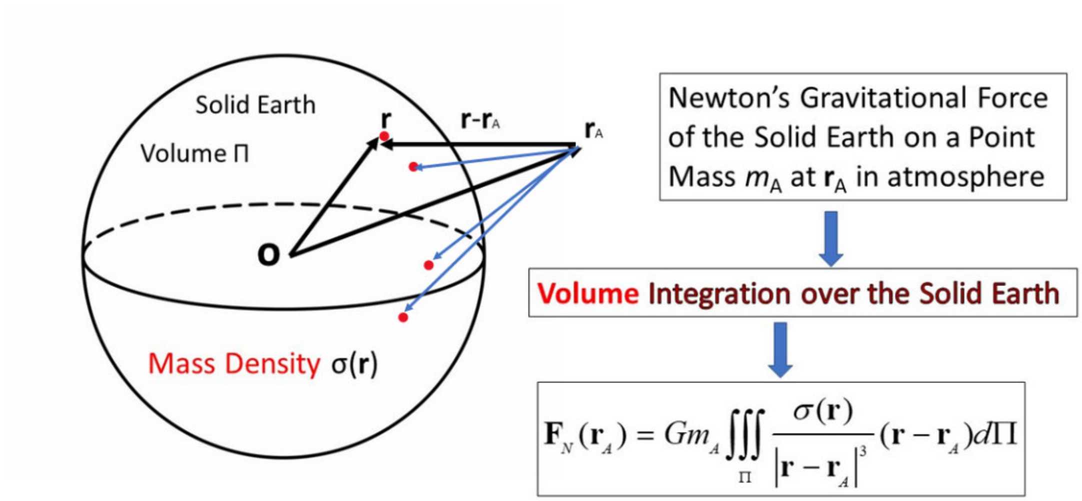

Figure 1. Newtonian gravitation of a point mass located at **r** inside the solid Earth on a point mass located at $\mathbf{r}_A$ in atmosphere. The gravitation of the solid Earth on a point mass m_A at $\mathbf{r}_A$ is the volume integration, and non-radial [i.e., $\mathbf{F}_N(\mathbf{r}_A)$ is not pointing to the center O], with associated true gravitational acceleration, $\mathbf{n} = \mathbf{F}_N(\mathbf{r}_A)/m_A$. Combination of **n** and the centrifugal acceleration leads to the true gravity $\mathbf{g}_t$.

For uniform mass density, $\sigma(\mathbf{r}) = \sigma_0 = \text{const}$, the Earth gravitation (7) becomes,

$$\mathbf{F}_0(\mathbf{r}_A) = -m_A\mathbf{n}_0, \quad \mathbf{n}_0 \equiv -\frac{GM}{|\mathbf{r}_A|^3}\mathbf{r}_\mathbf{A} \tag{8}$$

which is radial and equivalent to treating the solid Earth as a point mass located at the Earth center **O** with the total Earth mass to attract the atmospheric point mass (m_A) at location $\mathbf{r}_A$ by $\mathbf{F}_0(\mathbf{r}_A)$ (Figure 2). Here, $\mathbf{n}_0$ is the gravitational acceleration by the solid Earth with the uniform mass density. The combination of $\mathbf{n}_0$ and centrifugal acceleration leads to the apparent gravity $\mathbf{g}_a$. Subtraction of (8) from (6) and use of (7) lead to

$$\delta\mathbf{g} \equiv \mathbf{n} - \mathbf{n}_0 = G\iiint_\Pi \frac{[\sigma(\mathbf{r}) - \sigma_0]}{|\mathbf{r}-\mathbf{r}_A|^3}(\mathbf{r}-\mathbf{r}_A)d\Pi = \mathbf{g}_t - \mathbf{g}_a \tag{9}$$

which is the gravity disturbance vector. δ**g** is neglected completely in atmospheric modelling although it is a major variable in geodesy. The gravity disturbance vector at the Earth spheroidal surface (z = 0) is given by (Sandwell and Smith 1997),

$$\delta\mathbf{g}|_{z=0} = g_0\nabla N \tag{10}$$

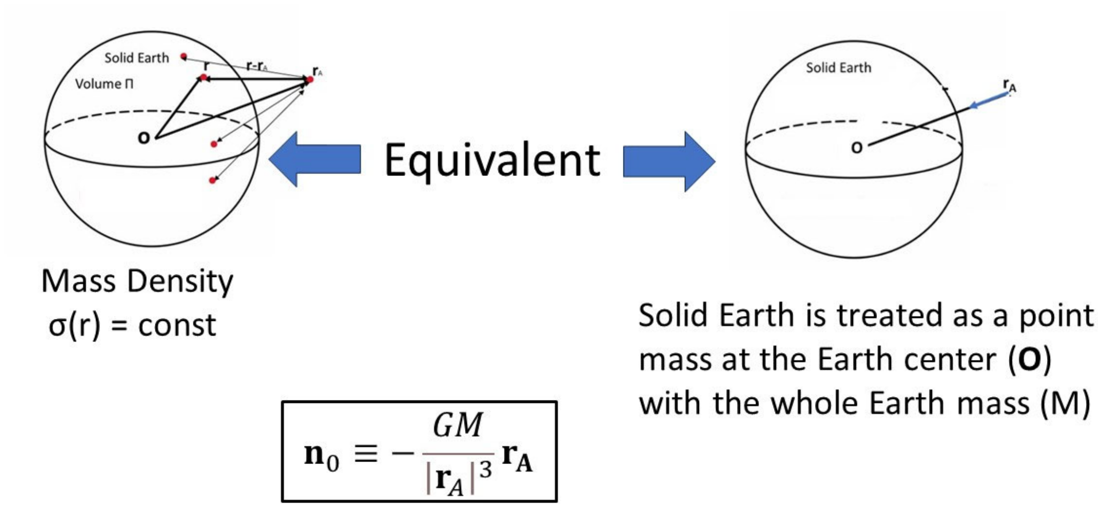

Figure 2. Corresponding to the spherical and spheroidal geopotentials, the solid Earth is treated either as uniform mass density or as a point mass located at the Earth centre (**O**) with the whole Earth mass to attract a point mass m_A in atmosphere. The associated hypothetical gravitational acceleration ($\mathbf{n}_0$) is radial (pointing to the Earth centre **O**). Combination of $\mathbf{n}_0$ and the centrifugal acceleration leads to the apparent gravity $\mathbf{g}_a$, which is non-radial.

3 Horizontal Momentum Equations

The horizontal momentum equation without friction is given by.

$$\left(\frac{d\mathbf{U}_s}{dt}\right)_s + \mathbf{m}_s + 2\mathbf{\Omega}\times(\mathbf{U}_s)_s = -\left(\frac{1}{\rho}\nabla p\right)_s \tag{11a}$$

for the standard gravity $\mathbf{g}_s$ in the spherical geopotential coordinates, by

$$\left(\frac{d\mathbf{U}_a}{dt}\right)_a + \mathbf{m}_a + 2\mathbf{\Omega}\times(\mathbf{U}_a)_a = -\left(\frac{1}{\rho}\nabla p\right)_a \tag{11b}$$

for the apparent gravity $\mathbf{g}_a$ in the spheroidal geopotential coordinates, and by

$$\left(\frac{d\mathbf{U}_t}{dt}\right)_t + \mathbf{m}_t + 2\mathbf{\Omega}\times(\mathbf{U}_t)_t = -\left(\frac{1}{\rho}\nabla p\right)_t \tag{11c}$$

for the true gravity $\mathbf{g}_t$ in the true geopotential coordinates. Here, the symbols $(\ )_s$, $(\ )_a$, $(\ )_t$ represent the corresponding geopotential coordinates; $\mathbf{\Omega}$ is the angular velocity vector of the Earth self-spinning; ρ is the density; and p is pressure. Note that gravity vanishes in the horizontal momentum equations (11a), (11b), (11c). The meteorological and oceanographic communities have reached the consensus of negligible metric terms $\mathbf{m}_s$, $\mathbf{m}_a$, $\mathbf{m}_t$ (e.g., Gill 1982; Gates 2004; Holton and Hakim 2013; Staniforth 2014; Chang and Wolfe 2022; Chang et al. 2023). We may delete the metric terms in (11a), (11b), (11c) for this study,

$$\left(\frac{d\mathbf{U}_s}{dt}\right)_s + 2\mathbf{\Omega}\times(\mathbf{U}_s)_s = -\left(\frac{1}{\rho}\nabla p\right)_s \tag{12a}$$

$$\left(\frac{d\mathbf{U}_a}{dt}\right)_a + 2\boldsymbol{\Omega}\times\left(\mathbf{U}_a\right)_a = -\left(\frac{1}{\rho}\nabla p\right)_a \tag{12b}$$

$$\left(\frac{d\mathbf{U}_t}{dt}\right)_t + 2\boldsymbol{\Omega}\times\left(\mathbf{U}_t\right)_t = -\left(\frac{1}{\rho}\nabla p\right)_t \tag{12c}$$

4 Mathematical Expressions of the SGA and EGA-CWSM

Mathematically, the SGA is to extend (12a) into

$$\left(\frac{d\mathbf{U}_a}{dt}\right)_s + 2\boldsymbol{\Omega}\times\left(\mathbf{U}_a\right)_s = -\left(\frac{1}{\rho}\nabla p\right)_s + \varepsilon_1 \tag{13a}$$

for the apparent gravity $\mathbf{g}_a$ in the spherical geopotential coordinates to get the same velocity and acceleration vectors as in the spheroidal geopotential coordinates,

$$\left(\mathbf{U}_a\right)_a = \left(\mathbf{U}_a\right)_s,\ \left(\frac{d\mathbf{U}_a}{dt}\right)_a = \left(\frac{d\mathbf{U}_a}{dt}\right)_s \tag{13b}$$

Here, ε_1 is the SGA error, which can be identified through subtracting (13a) from (12b) and using (13b),

$$\varepsilon_1 = \left(\frac{1}{\rho}\nabla p\right)_s - \left(\frac{1}{\rho}\nabla p\right)_a \tag{13c}$$

which is the HPGE in transforming the spheroidal to spherical geopotential coordinates. Similarly, the EGA-CWSM is to extend (12b) into

$$\left(\frac{d\mathbf{U}_t}{dt}\right)_a + 2\boldsymbol{\Omega}\times\left(\mathbf{U}_t\right)_a = -\left(\frac{1}{\rho}\nabla p\right)_a + \varepsilon_2 \tag{14a}$$

for the true gravity $\mathbf{g}_t$ in the spheroidal coordinates to get the same velocity and acceleration vectors in the true geopotential coordinates,

$$\left(\mathbf{U}_t\right)_t = \left(\mathbf{U}_t\right)_a,\ \left(\frac{d\mathbf{U}_t}{dt}\right)_t = \left(\frac{d\mathbf{U}_t}{dt}\right)_a \tag{14b}$$

Here, ε_2 is the EGA-CWSM error, which can be identified through subtracting (14a) from (12c) and using (14b),

$$\varepsilon_2 = \left(\frac{1}{\rho}\nabla p\right)_a - \left(\frac{1}{\rho}\nabla p\right)_t \tag{14c}$$

which shows that ε_2 is the HPGE in transforming the true to spheroidal geopotential coordinates. Validity of SGA and EGA-CWSM should be justified on the magnitudes of (ε_1, ε_2), i.e., only small $|\varepsilon_1|$ verifies the SGA, and only small $|\varepsilon_2|$ verifies the EGA-CWSM.

5 HPGE in Transforming Spheroidal to Spherical Geopotential Coordinates

5.1 Use of confocal hyperboloids for spheroidal geopotential coordinates

Let location of an atmospheric (or oceanic) point-mass be represented by (λ, φ, ξ) in the confocal hyperboloid type of spheroidal geopotential coordinates (Gates 2004) and by (λ, φ_s, r) in the spherical geopotential coordinates. Let (a, b) be the Earth semimajor and semi minor axes; $c = (a^2 - b^2)^{1/2}$ be the Earth focal distance; $\alpha = (\sin^2\varphi + \sinh^2\xi)^{1/2}$ be the separation parameter [same as the symbol '*R*' used in Gates (2004)]. The horizontal pressure gradient in the spheroidal geopotential coordinates is given by [see equation (35) in Gates (2004)],

$$\left(\nabla p\right)_a = \mathbf{i}\frac{1}{c\cos\varphi\cosh\xi}\frac{\partial p}{\partial\lambda} + \mathbf{j}\frac{1}{c\alpha}\frac{\partial p}{\partial\varphi} \tag{15}$$

The hydrostatic balance is represented by [see equation (54) in Gates (2004)],

$$-\frac{1}{c\alpha}\frac{\partial p}{\partial\xi} = -\frac{\partial p}{\partial z} = \rho g_0 \tag{16}$$

where z is the spheroidal (ellipsoidal) height along the vertical ξ coordinate from the Earth reference spheroid. The horizontal pressure gradient in the spherical geopotential coordinates is given by,

$$\left(\nabla p\right)_s = \mathbf{i}\frac{1}{r\cos\varphi_s}\frac{\partial p}{\partial\lambda} + \mathbf{j}_s\frac{1}{r}\frac{\partial p}{\partial\varphi_s} \tag{17}$$

Substitution of (15) and (17) into (13c) leads to,

$$\varepsilon_1 = \frac{1}{\rho}\left[\mathbf{i}\left(\frac{1}{r\cos\varphi_s} - \frac{1}{c\cos\varphi\cosh\xi}\right)\frac{\partial p}{\partial\lambda} + \mathbf{j}_s\frac{1}{r}\frac{\partial p}{\partial\varphi_s} - \mathbf{j}\frac{1}{c\alpha}\frac{\partial p}{\partial\varphi}\right]_s . \tag{18}$$

The direction of the standard gravity, $\mathbf{g}_s = -g_0\mathbf{k}_s$, is towards the Earth's center. The direction of the apparent gravity, $\mathbf{g}_a = -g_0\mathbf{k}_a$, is perpendicular to the Earth spheroidal surface. Let δ be the geodetic latitude minus the geocentric latitude,

$$\delta = \varphi - \varphi_s \tag{19}$$

which is represented in the spheroidal geopotential coordinates by [see equation (39) in Gates (2004)],

$$\delta = \tan^{-1}\left(\frac{\sin\varphi\cos\varphi}{\sinh\xi\cosh\xi}\right) \tag{20}$$

The unit vector $\mathbf{j}_a$ can be represented in the spherical geopotential coordinates by

$$\mathbf{j} = \mathbf{j}_s\cos\delta - \mathbf{k}_s\sin\delta \tag{21}$$

where the component ($-\mathbf{k}_s \sin\delta$) is in the radial direction of the spherical geopotential coordinates and does not affect the horizontal pressure gradient in the spherical geopotential coordinates. Thus (18) becomes,

$$\varepsilon_1 = \frac{1}{\rho}\left[\mathbf{i}\left(\frac{1}{r\cos\varphi_s} - \frac{1}{c\cos\varphi\cosh\xi}\right)\frac{\partial p}{\partial\lambda} + \mathbf{j}_s\left(\frac{1}{r}\frac{\partial p}{\partial\varphi_s} - \frac{\cos\delta}{c\alpha}\frac{\partial p}{\partial\varphi}\right)\right] \tag{22}$$

Since φ_s depends on (φ, δ) as shown in Eq (19), latitudinal pressure gradients are connected by

$$\frac{\partial p}{\partial\varphi} = \frac{\partial p}{\partial\varphi_s}\frac{\partial\varphi_s}{\partial\varphi} = \frac{\partial p}{\partial\varphi_s}\left(1 - \frac{\partial\delta}{\partial\varphi}\right) \tag{23}$$

between the spheroidal and spherical geopotential coordinates. Substitution of (23) into (22) gives,

$$\varepsilon_1 = \frac{1}{\rho}\left[\mathbf{i}\left(\frac{1}{r\cos\varphi_s} - \frac{1}{c\cos\varphi\cosh\xi}\right)\frac{\partial p}{\partial\lambda} + \mathbf{j}_s\left[\frac{1}{r} - \frac{\cos\delta}{c\alpha}\left(1 - \frac{\partial\delta}{\partial\varphi}\right)\right]\frac{\partial p}{\partial\varphi_s}\right] \tag{24}$$

The relative longitudinal horizontal pressure error is given by,

$$\gamma_\lambda = 1 - \frac{r\cos\varphi_s}{c\cos\varphi\cosh\xi} . \tag{25}$$

The relative latitudinal horizontal pressure error is given by,

$$\gamma_{\varphi_s} = 1 - \frac{r\cos\delta}{c\alpha}\left(1 - \frac{\partial\delta}{\partial\varphi}\right) \tag{26}$$

Derivative of (20) respect to φ gives,

$$\frac{\partial \delta}{\partial \varphi}=\left(\frac{\cos 2\varphi}{\sinh\xi\cosh\xi}\right)\left[1+\left(\frac{\sin 2\varphi}{2\sinh\xi\cosh\xi}\right)^2\right]^{-1} \tag{27}$$

Substitution of (27) into (26) leads to

$$\gamma_{\varphi_s}=1-\frac{r\cos\delta}{c\alpha}\left\{1-\left(\frac{\cos 2\varphi}{\sinh\xi\cosh\xi}\right)\left[1+\left(\frac{\sin 2\varphi}{2\sinh\xi\cosh\xi}\right)^2\right]^{-1}\right\}. \tag{28}$$

As pointed out by Gates (2004), the dimensionless parameter ξ for the Earth spheroidal surface ξ_E = 3.193; the values of ξ are only slightly larger than ξ_E; it is reasonably to approximate sinh ξ and cosh ξ at ξ_E, i.e., sinh ξ ≈ 12.09, cosh ξ ≈ 12.22. Besides, the other parameters in (25) and (28) are also taken from Gates (2004) such as a = 6378.4 km, b = 6356.9 km, $c\alpha \approx$ 6323 km, δ_{max} = max(δ) = 11'35"; the radius of Earth spherical surface r is taken as a. All these parameters are listed in Table 1.

Table 1. Values of parameters used to identify the HPGEs in the spherical geopotential approximation obtained from Gates (2004).

Parameter	Mathematic Formula	Value
Earth Semimajor Axis a		6378.4 km
Earth Semiminor Axis b		6356.9 km
Earth Focal Distance c	$c=\sqrt{(a^2-b^2}$	523.0 km
Earth Spherical Radius r	$r=a$	6378.4 km
Separation Parameter α	$\alpha=\sqrt{\left(\sin^2\varphi+\sinh^2\xi\right)}$	$c\alpha \cong$ 6323 km
Dimensionless Parameter ξ for Spheroidal Geopotential Surfaces		
Earth Spheroidal Surface ξ_E	$\xi_E=\tanh^{-1}(b/a)$	3.192
sinh ξ		≅ 12.15
cosh ξ		≅ 12.22
Geodetic Latitude Minus Geocentric Latitude δ	$\delta=\tan^{-1}\left(\frac{\sin\varphi\cos\varphi}{\sinh\xi\cosh\xi}\right)$	$0\le\delta\le\delta_{\max}=11'35"$ $0.99999432\le\cos\delta\le 1$

Thus Eqs (25) and (28) becomes,

$$\gamma_\lambda=1-\frac{0.9980191\cos\varphi_s}{\cos\varphi}==1-0.9980191\left[1-\frac{\cos\delta\sin^2\varphi}{\sinh\xi\cosh\xi}\right] \tag{29}$$

$$\cong 1-0.9980191\left[1-0.006769\cos\delta\sin^2\varphi\right]=\mathrm{O}\ (10^{-3})$$

$$\gamma_{\varphi_s}=1-0.9980191\cos\delta\left\{1-\left(\frac{\cos 2\varphi}{147.7398}\right)\left[1+\left(\frac{\sin 2\varphi}{295.4796}\right)^2\right]^{-1}\right\}=\mathrm{O}(10^{-3}) \tag{30}$$

Here, the definition for δ [i.e., Eq (20)] is used in (29). The relative longitudinal and latitudinal HPGEs in the SGA is on the order of 10^{-3},

$$\boxed{O\left(\left|\gamma_\lambda\right|\right)=O\left(\left|\gamma_\varphi\right|\right)=10^{-3}} \tag{31}$$

which confirms the validity of SGA [i.e., Eq (13a)] (the approximation of spheroidal geopotential surfaces as spherical) on both negligible metric error and HPGE.

5.2 Use of approximated spheroidal geopotential for spheroidal geopotential coordinates

The spheroidal geopotential (Φ_a) is given by [see Eq (18) in Staniforth (2014)]

$$\Phi_a(\varphi_s, r) = \frac{GM}{r}\left\{1 - \left[\left(\frac{2\varepsilon - \mu}{2}\right)\frac{a^2}{R^2}\left(\frac{r}{R}\right)^{-2} + \frac{\mu}{2}\frac{R^3}{a^3}\left(\frac{r}{R}\right)^3\right]\sin^2\varphi_s\right\} + \frac{GM}{r}\left[\left(\frac{2\varepsilon - \mu}{6}\right)\frac{a^3}{R^3}\left(\frac{r}{R}\right)^{-3} + \frac{\mu}{2}\frac{R^2}{a^2}\left(\frac{r}{R}\right)^2\right] \tag{32}$$

with

$$\varepsilon \equiv \frac{a-b}{a}, \quad \mu \equiv \frac{\Omega^2 a^3}{GM} \tag{33}$$

where φ_s is the geocentric latitude; a and b are the equatorial and polar semi-axes of Earth's assumed spheroidal surface with ε the measure of the ellipticity; and μ is ratio of centrifugal and gravitational forces. Note that the spheroidal geopotential (Φ_a) is independent on the longitude (λ). Both ε and μ are small parameters (Staniforth (2014),

$$O(\varepsilon^2, \varepsilon\mu, \mu^2) = O(10^{-5}) \tag{34}$$

Since the thickness of combined troposphere and stratosphere in much thinner in comparison to the Earth radius [see Eq (19) in Staniforth (2014)],

$$\frac{r}{R} = 1 + \frac{r-R}{R} = 1 + O(\varepsilon, \mu) \tag{35}$$

where R = 6378 km, is the equatorial Earth radius. Eq (32) is approximated by [see Eq (20) in Staniforth (2014)],

$$\hat{\Phi}_a(\varphi_s, r) = \frac{GM}{r}\left\{1 - \left[\left(\frac{2\varepsilon - \mu}{2}\right)\frac{a^2}{R^2} + \frac{\mu}{2}\frac{R^3}{a^3}\right]\sin^2\varphi_s\right\} + \frac{GM}{R}\left[\left(\frac{2\varepsilon - \mu}{6}\right)\frac{a^2}{R^2} + \frac{\mu}{2}\frac{R^3}{a^3}\right] \tag{36}$$

Thus, the EGA-SB (i.e., to approximate $\hat{\Phi}_a$ for Φ_a) is totally different from EGA-CWSM (i.e., to approximate the true geopotential surfaces as spheroidal) (Chang et al. 2023).

The zonal and latitudinal coefficients (h_λ, $h_{\varphi s}$) in the metric are given by [see Eqs (47) and (48) in Staniforth (2014)],

$$h_\lambda = R\left\{1 - \left[\left(\frac{2\varepsilon - \mu}{2}\right)\frac{a^2}{R^2} + \frac{\mu}{2}\frac{R^3}{a^3}\right]\sin^2\varphi_s\right\}\cos\varphi_s \tag{37}$$

$$h_{\varphi_s}^2 = R^2 \left\{ 1 + \left[\left(\frac{2\varepsilon - \mu}{2} \right) \frac{a^2}{R^2} + \frac{\mu}{2} \frac{R^3}{a^3} \right] \left(2 - 3\sin^2\varphi_s \right) \sin^2\varphi_s \right\}^2 \sin^2\varphi_s$$
$$+ R^2 \left\{ 1 + \left[\left(\frac{2\varepsilon - \mu}{2} \right) \frac{a^2}{R^2} + \frac{\mu}{2} \frac{R^3}{a^3} \right] \left(-3\sin^2\varphi_s \right) \sin^2\varphi_s \right\}^2 \cos^2\varphi_s \qquad (38)$$

For limiting case, $\varepsilon \to 0$, $\mu \to 0$, Eqs (37) and (38) reduce to the coefficients for the spherical geopotential coordinates ($R \cos \varphi_s$, R). The relative longitudinal horizontal pressure gradient error is given by,

$$\gamma_\lambda = -\left[\left(\frac{2\varepsilon - \mu}{2} \right) \frac{a^2}{R^2} + \frac{\mu}{2} \frac{R^3}{a^3} \right] \sin^2\varphi_s \qquad (39)$$

The relative latitudinal horizontal pressure error is given by,

$$\gamma_{\varphi_s}^2 = \left\{ \left[\left(\frac{2\varepsilon - \mu}{2} \right) \frac{a^2}{R^2} + \frac{\mu}{2} \frac{R^3}{a^3} \right] \left(2 - 3\sin^2\varphi_s \right) \sin^2\varphi_s \right\}^2 \sin^2\varphi_s$$
$$+ \left\{ \left[\left(\frac{2\varepsilon - \mu}{2} \right) \frac{a^2}{R^2} + \frac{\mu}{2} \frac{R^3}{a^3} \right] \left(-3\sin^2\varphi_s \right) \sin^2\varphi_s \right\}^2 \cos^2\varphi_s \qquad (40)$$

Here,

$$|\sin^2\varphi_s| \le 1, \;\; |\cos^2\varphi_s| \le 1, \;\; \frac{a}{R} \sim 1, \qquad (41)$$

From (34) we have

$$\varepsilon \sim 10^{-5/2}, \;\; \mu \sim 10^{-5/2} \qquad (42)$$

Use of (41) and (42) for (39) and (40) leads to

$$\boxed{O\left(\left|\gamma_\lambda\right|\right) = O\left(\left|\gamma_\varphi\right|\right) = O\left(10^{-5/2}\right) \simeq 0.3\%} \qquad (43)$$

which also confirms the validity of SGA (13a) (the approximation of spheroidal geopotential surfaces as spherical) on both negligible metric error and HPGE.

6 HPGE in Transforming True to Spheroidal Geopotential Coordinates

Chu (2024) shows that the HPGE is non-negligible and equals the horizontal gravity disturbance vector in transforming true to spheroidal geopotential coordinates, which is presented in this section for comparison to the HPGE in transforming spheroidal to spherical geopotential coordinates.

6.1 Orthometric and spheroidal (ellipsoidal) heights

The spheroidal coordinates (λ, φ, ξ) can be changed into (λ, φ, z) according to Eq (16). Let location of an atmospheric point-mass A be determined by (λ, φ, z) in the spheroidal geopotential coordinates and by (λ, φ, z_t) in the true geopotential coordinates (with irregular geometry) (Figure 3). Here, z is the spheroidal (ellipsoidal) height; z_t is the orthometric height. The spheroidal geopotential surfaces are represented by,

$$z = \text{const} \qquad (44)$$

The true geopotential surfaces are represented by,

$$z_t = \text{const} \qquad (45)$$

with

$$\boxed{z_t = z - N(\lambda, \varphi)} \tag{46}$$

which is commonly used in the geodetic community with N the geoidal height.

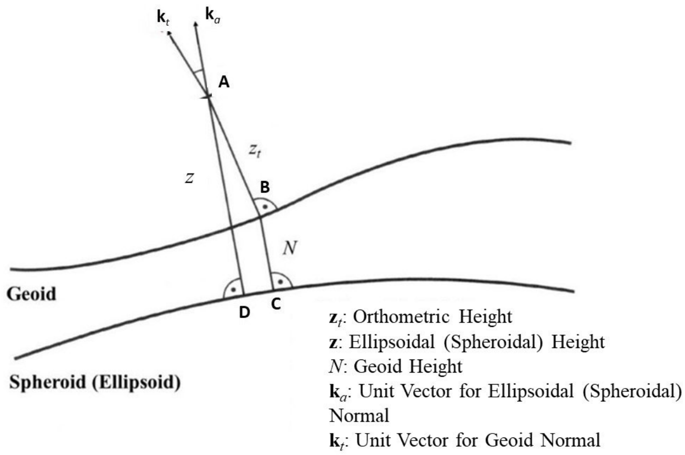

Figure 3. Orthometric height (z_t), spheroidal height (z), geoid height (N), and unit vertical vectors ($\mathbf{k}_a$, $\mathbf{k}_t$) for the spheroidal and true geopotential surfaces most fitted to the global mean sea level. Here, z = AD, z_t = AB, N = BC. The angle between ($\mathbf{k}_a$, $\mathbf{k}_t$) is over exaggerated since it is only around 2×10^{-5} radian. The formula $z_t = z - N$ is quite accurate and commonly used in the geodetic community.

6.2 HPGE equaling horizontal gravity disturbance vector.

In the true geopotential coordinate (λ, φ, z_t), the true gravity $\mathbf{g}_t$ does not have component on the true geopotential surfaces (i.e., the true horizontal surfaces). The hydrostatic balance equation with the true gravity $\mathbf{g}_t$ is given by,

$$\frac{\partial p}{\partial z_t} = -\rho g_0 \tag{47}$$

A derivative with respect to λ between the z and z_t as the vertical coordinates is given by,

$$\left(\frac{\partial}{\partial \lambda}\right)_z = \left(\frac{\partial}{\partial \lambda}\right)_{z_t} + \left(\frac{\partial z_t}{\partial \lambda}\right)_z \frac{\partial}{\partial z_t} \tag{48}$$

Using (48) to the derivative of p gives

$$\left(\frac{\partial p}{\partial \lambda}\right)_z = \left(\frac{\partial p}{\partial \lambda}\right)_{z_t} + \left(\frac{\partial z_t}{\partial \lambda}\right)_z \frac{\partial p}{\partial z_t} \tag{49}$$

Substitution of (46) and (47) into (49) leads to

$$\left(\frac{\partial p}{\partial \lambda}\right)_z = \left(\frac{\partial p}{\partial \lambda}\right)_{z_t} + \rho g_0 \frac{\partial N}{\partial \lambda} \tag{50}$$

We obtain the following relationship after conducting similar operation for φ,

$$(\nabla p)_a = (\nabla p)_t + \rho g_0 (\nabla N)_a \tag{51}$$

Substitution of (51) into (14c) leads to

$$\varepsilon_2 = \left[\left(\frac{1}{\rho}\nabla p\right)_a - \left(\frac{1}{\rho}\nabla p\right)_t\right] = g_0 (\nabla N)_a \tag{52}$$

which shows that the error of the horizontal pressure gradient force equals the horizontal gravity disturbance vector $g_0(\nabla N)_a$ in transforming the true geopotential to spheroidal geopotential coordinates. The horizontal gravity disturbance vector $g_0\nabla N$ has comparable order of magnitudes as the Coriolis force with the ratio changing from 0.6168 (max) at 1,000 hPa to 0.1573 (min) at 200 hPa, and mean of 0.3052 in the troposphere (Chu 2023) using the two publicly available and independent datasets with the geoid height (N) from the static gravity field model EIGEN-6C4 (http://icgem.gfz-potsdam.de/home) and long-term mean atmospheric data such as wind velocity **U**, and temperature (T) at 12 pressure levels (1000 to 100 hPa) in troposphere from the NCEP/NCAR reanalyzed climatology https://psl.noaa.gov/data/gridded/data.ncep.reanalysis.derived.pressue.html.

Nonnegligible HPGE shows the invalidity of the EGA-CWSM (the approximation of true geopotential surfaces as spheroidal). Substitution of (52) into (14a) gives the horizontal momentum equation in the spheroidal geopotential coordinates for the true gravity $\mathbf{g}_t$

$$\left(\frac{d\mathbf{U}_t}{dt}\right)_a + 2\mathbf{\Omega}\times\left(\mathbf{U}_t\right)_a = -\left(\frac{1}{\rho}\nabla p\right)_a + g_0(\nabla N)_a \tag{53}$$

which shows the occurrence of the horizontal gravity vector in the horizontal momentum equation in the spheroidal geopotential coordinates using the true gravity $\mathbf{g}_t$. Section 5 confirms the validity of SGA (approximating the spheroidal geopotential surfaces as spherical) based on both negligible metric error and HPGE, Eq (53) can be reasonably transformed from the spheroidal geopotential to spherical geopotential coordinates,

$$\left(\frac{d\mathbf{U}_t}{dt}\right)_s + 2\mathbf{\Omega}\times\left(\mathbf{U}_t\right)_s = -\left(\frac{1}{\rho}\nabla p\right)_s + g_0(\nabla N)_s \tag{54}$$

which shows that the horizontal gravity disturbance vector $g_0(\nabla N)_s$ also occurs in the spherical geopotential coordinates using the true gravity $\mathbf{g}_t$.

7 Conclusions

Metric and horizontal pressure gradient errors exist in geopotential coordinate transformation, but only the metric error is recognized in the meteorological and oceanographic communities. Due to negligible metric error, it is to approximate the true geopotential surfaces as spheroidal (i.e., EGA-CWSM) and to approximate the spheroidal geopotential surfaces as spherical (i.e., the SGA). Almost all the analytical and numerical models use spherical geopotential coordinates.

The horizontal pressure gradient error is identified in this study as negligible in transforming the spheroidal to spherical geopotential coordinates, and as nonnegligible with equaling the horizontal gravity disturbance vector ($g_0\nabla N$) in transforming the true to spheroidal geopotential coordinates. Such identification confirms the SGA and rejects the EGA-CWSM proposed by Chang et al. (2023). It is urgent to include the horizontal gravity disturbance vector ($g_0\nabla N$) in any analytical or numerical atmospheric models.

Appendix A. Mistakes Identified in Chang and Wolfe (2022), Stewart and McWilliams (2022), and Chang et al. (2023)

In this appendix, Chang and Wolfe (2022) is referred to CW22; Stewart and McWilliams (2022) is referred to SM22; Chang et al. (2023) is referred to CWSM23. Mistakes have been identified in CW22, CW22 Supplementary, SM22, SM22 Supplementary and CWSM23. The quoted contents with italic font are directly copied from these references.

A1. Wrong comparison leads to wrong statement of "negligible impact of δg"

SM22 used the following equations,

$$\rho_0 \frac{D\mathbf{U}}{Dt} + \rho_0 f\mathbf{k}\times\mathbf{U} + \nabla_h p = \rho\nabla_h V + \rho_0\mathbf{F} \qquad \text{Eq (1) in SM22}$$

$$V \approx g_0(N - z) \qquad \text{Eq (3) in SM22}$$

$$\begin{aligned}&\rho_0 \frac{D\mathbf{U}}{Dt} + \rho_0 f\mathbf{k}\times\mathbf{U} - \rho_0\mathbf{F}\\ &= g_0\int_{z'=S}^{z'=z} \nabla_h\rho dz' - \rho_0 g_0\nabla_h(S-N) + g_0(\rho-\rho_0)\nabla_h N\end{aligned} \qquad \text{Eq (5) in SM22}$$

to claim that

"At the surface z = S the "horizontal gravity anomaly" term is zero by construction because ρ = ρ0. In the subsurface, while the "horizontal gravity anomaly" term in ***(5)*** *is non-zero, it is approximately three orders of magnitude smaller than the "horizontal gravity" term in* ***(1)*** *…… Consequently, "horizonal gravity" would likely have a* ***negligible impact*** *on ocean circulation even in a model formulated in absolute spherical coordinates."*

Anyone with basic scientific knowledge knows that the importance of a forcing term in atmospheric and oceanic dynamics should be compared to other terms **in the same dynamic equation**. SM22 compared $[g_0(\rho-\rho_0)\nabla_h N]$ in [**Eq (5)** SM22] to $[g_0\rho\nabla_h N]$ in [**Eq (1)** SM22]. Such comparison is wrong. The correct comparison should be between the horizontal gravity disturbance vector $[g_0(\rho-\rho_0)\nabla_h N]$ and the baroclinic pressure gradient $g_0\int_{z'=S}^{z'=z}\nabla_h\rho dz'$ **in the same equation** [i.e., Eq (5) SM22].

Besides, ρ_0 is a constant (e.g., 1028 kg/m^3) using the Boussinesq approximation, not the surface density. The statement in SM22

"At the surface z = S the 'horizontal gravity anomaly' term is zero by construction because ρ = ρ0"

is also wrong. The horizontal gravity disturbance vector $[g_0(\rho-\rho_0)\nabla_h N]$ is NOT zero at the ocean surface.

A2. Wrong derivation led to wrong statement on "shift in the reference density in oceanic Ekman layer."

SM22 Supplementary used the following four equations (∇_h is the horizontal vector differential operator),

"$$\int_z^0 \mathbf{U} dz' = \frac{1}{\rho_0 f}\int_z^0 \mathbf{k}\times\nabla_h \hat{p} dz' + \frac{1}{f}\int_z^0 b\mathbf{k}\times\nabla_h N dz' - \frac{\mathbf{k}\times\boldsymbol{\tau}}{\rho_0 f} \qquad \text{(13) in SM22 Supp}$$

$$b^* = -g_0(\rho-\rho_0^*)/\rho_0^* \approx b+\delta b_0,\quad \delta b_0 = g_0\delta\rho_0/\rho_0^* \qquad \text{(14) in SM22 Supp}$$

$$\int_z^0 \mathbf{U} dz' = \frac{1}{\rho_0 f}\int_z^0 \mathbf{k}\times\nabla_h \hat{p}^* dz' + \frac{1}{f}\int_z^0 b\mathbf{k}\times\nabla_h N dz' - z\frac{\delta b_0}{f}\mathbf{k}\times\nabla_h N - \frac{\mathbf{k}\times\boldsymbol{\tau}}{\rho_0 f} \qquad \text{(15) in SM22 Supp}$$

$$\nabla_h \hat{p} = \nabla_h \hat{p}^* + g_0\delta\rho_0\nabla_h N \qquad \text{(16) in SM22 Supp}$$

Here, ρ is the density; $b = -g_0(\rho-\rho_0)/\rho_0$*, is the buoyancy;* $(\rho_0^*,\ b^*, \hat{p}^*)$ *are the shifted (*ρ_0*,* $b,\ \hat{p}$ *) due to* $\delta\rho_0$*.*"

to claim that:

"An arbitrary change in the reference density leads to a vertically-uniform addition to the "horizontal gravity"-driven component of the flow, and thus a vertically-integrated transport that increases linearly with depth. This implies that the "horizontal gravity"-driven component of the flow is ill-defined, and thus that analyzing this flow in isolation, or as part of the 'Ekman' transport (as done by Chu[1]) is misleading."

[Eq(15) SM22 Supplementary] has two severe mistakes: (a) the sign for the term $z(\delta b_0/f)\mathbf{k}\times\nabla_h N$ should be '+' not '-'; (b) the buoyancy b in [Eq.(15) SM22 Supplementary] is based on the **unshifted reference density** ρ_0, but the dynamic pressure $\hat{p}^*$ is based on **the shifted reference density** ρ_0^*. If the shifted reference density ρ_0^* is used for both buoyancy b and dynamic pressure $\hat{p}$, and the sign for the term $z(\delta b_0/f)\mathbf{k}\times\nabla_h N$ is corrected from '-' to '+', [Eq.(15) SM22 Supplementary] becomes [substitution of Eq.(14) into Eq.(15) in SM22 Supplementary]

$$\underbrace{\int_z^0 \mathbf{U} dz'}_{\text{Total}} = \underbrace{\frac{1}{\rho_0 f}\int_z^0 \mathbf{k}\times\nabla_h \hat{p}^* dz'}_{\text{Pressure Gradient}} + \underbrace{\frac{1}{f}\int_z^0 b^*\mathbf{k}\times\nabla_h N dz'}_{\text{Gravity Disturbance}} \underbrace{- \frac{\mathbf{k}\times\boldsymbol{\tau}}{\rho_0 f}}_{\text{Wind Stress (Ekman)}} \qquad \text{(A1)}$$

which shows that the Ekman transport driven by the horizontal gravity disturbance vector is **well-defined**, and there is no vertically integrated transport that increases linearly with depth.

A3. Mistakenly used neutral atmosphere to get wrong statement on "shift to absolute spherical coordinates in atmospheric Ekman layer."

SM22 Supplementary used the following equations,

$$\rho = \rho_0 + \tilde{\rho} \qquad \text{(5) in SM22 Supplementary}$$

$$\mathbf{F} = \frac{\partial}{\partial z}(K \frac{\partial \mathbf{U}}{\partial z}) \qquad \text{(10) in SM22 Supplementary}$$

$$\rho_0 f\mathbf{k} \times \mathbf{U} + \nabla_3 p \approx \rho(z)\mathbf{g} + \rho_0 \mathbf{F} \qquad \text{(20) in SM22 Supplementary}$$

$$\rho_0 f\mathbf{k} \times \mathbf{U}_g = -\nabla_h p + \rho(z)\mathbf{g}_h, \ \mathbf{g}_h \equiv \delta\mathbf{g} \qquad \text{(24) in SM22 Supplementary}$$

$$\rho_0 f\mathbf{k} \times (\mathbf{U} - \mathbf{U}_g) = \rho_0 \mathbf{F} \qquad \text{(25) in SM22 Supplementary}$$

to claim that

"*Thus the 'Ekman' flow and pumping are unchanged by the shift to absolute spherical coordinates*."

SM22 supplementary mistakenly or intentionally used neutral atmosphere, i.e., (ρ, $\nabla_h p$) independent on z. In fact, the atmospheric density used in Chu (2021c) varies with z [see Eq (23) in Chu 2021c]:

$$\frac{\rho}{\rho_0} = s(z), \ s(z) \equiv \exp\left(-\frac{z}{H}\right), \quad H = 10.4 \text{ km} \qquad \text{(A2)}$$

Anyone with basic knowledge on college ordinary differential equations knows that solution of a linear ordinary differential equation is invariant with the shift of the independent variable **only if all the coefficients in the equation are constants**; but is variant even if even only one coefficient is not constant (i.e., a function of the independent variable). [Eq (25) in SM Supplementary] is a second order ordinary differential equation with **U** the dependent variable, and z the independent variable, and (K, $\mathbf{U}_g$) the coefficients.

Invariant solution of [Eq (25) SM Supplementary] with the shift to the absolute spherical coordinates (i.e., moving z-surfaces up and down) is valid only for very special conditions: neutral atmosphere and constant K, which leads to the constant coefficients (K, $\mathbf{U}_g$) (i.e., independent on z) in [Eq (25) SM Supplementary].

However, with gravity disturbance vector $\delta\mathbf{g} \neq 0$ and stratified atmosphere $\rho(z)$, the term $\rho(z)\delta\mathbf{g}$ depends on z, and so the coefficient $\mathbf{U}_g$ [see Eq (24) SM Supplementary]. Thus, Eq (25) in SM Supplementary is a second order ordinary differential equation with z-varying coefficient $\mathbf{U}_g$. The solution of [Eq (25) in SM Supplementary] varies with the shift to the absolute spherical coordinates. The Ekman flow and Ekman pumping change with the shift to absolute spherical coordinates as shown in Chu (2021c). The gravity disturbance vector $\delta\mathbf{g}$ **does affect** the atmospheric Ekman flow and Ekman pumping. The Ekman pumping velocity is the same by Eq (41) in Chu (2021b, retracted by JGR – Atmospheres) as by Eq (46) in Chu (2023).

A4. Mistakenly treated the metric terms as the only errors in the geopotential coordinate transformations.

The metric terms are treated as the only errors among the spheroidal, spherical, and true geopotential coordinate transformations in CW22 and CWMS23. The Second Paragraph on Page 2 in CW22:

"*Let us estimate how large this error might be. Mathematically, the exact form of the metric terms is*:

$$\boxed{\frac{D\mathbf{U}}{Dt}=\mathbf{i}\frac{Du}{Dt}+\mathbf{j}\frac{Dv}{Dt}+\mathbf{k}\frac{Dw}{Dt}+u\frac{D\mathbf{i}}{Dt}+v\frac{D\mathbf{j}}{Dt}+w\frac{D\mathbf{k}}{Dt}} \qquad \text{Eq (4) in CW22}$$

*where u, v, w are the three velocity components, and **i**, **j**, and **k** are the three are the three unit vectors of the coordinate system. The last 3 terms on the RHS of (4) are the metric terms, which arise due to the local unit vectors changing direction following the fluid motion. …… This estimate confirms that the errors made by approximating the near oblate spheroidal coordinate in which the true gravity is exact vertical with a truly oblate spheroidal coordinate system is negligible, as suggested in ocean dynamics texts*[3,4]"

Line 13-17 in the Second Paragraph in CWSM23

"*As shown by CW22, the **metric errors** introduced in the calculus of the spheroidal geopotential approximation are small, reaffirming the long-standing practice of using this coordinate system for atmospheric and oceanic modeling (Gill 1982, Staniforth 2022). Based on these and similar analyses, CW22 and SM22 concluded that the horizontal components of the true gravity are not relevant to ocean (and atmospheric) dynamics because these horizontal components vanish when the coordinate system is interpreted correctly*."

It is incorrect because the HPGE in transforming the true to spheroidal/spherical geopotential coordinates is non-negligible and equals the horizontal gravity disturbance vector in addition to the metric terms (see Section 5). Such an important error (HPGE) is totally neglected in CW22 and CWSM23.

A5. Irrelevant scale analysis on the metric terms

To ignore the HPGE in the geopotential coordinate transformations completely, detailed scale analysis on the metric terms depicted in CW22 and CWSM23 is irrelevant because the metric errors are negligible in comparison to HPGE in the geopotential coordinate transformations.

A6. Invalid EGA-CWSM

Any approximation needs to be verified. However, the EGA-CWSM proposed in CWSM23 has never been verified. Section 5 shows that the HPGE is non-negligible in transforming the true to spheroidal geopotential coordinates. Thus, the comments below by CWSM23 are incorrect.

First Paragraph in CWSM23

"*…… Chang and Wolfe (2022; hereafter CW22) and Stewart and McWilliams (2022; hereafter SM22) pointed out that atmospheric and oceanic scientists express the equations of motion in coordinate form by defining the "vertical" direction in the coordinate system to be opposite to **g**, effectively using a geopotential coordinate (see, e.g., Gill 1982).*
Importantly, in this coordinate system, the true gravity, $\mathbf{g} = \mathbf{g}_{eff} + \delta\mathbf{g}$, is exactly vertical – with no horizontal components. Furthermore, in this coordinate system "horizontal" geopotential surfaces are not exactly spheroidal but nearly spheroids with some bumps due to the inhomogeneities of the

*Earth's mass distribution. For mathematical simplicity, atmospheric and oceanic scientists approximate these geopotential coordinate surfaces geometrically as exact spheroids; that is, they use a coordinate system in which **true gravity is exactly aligned with the vertical coordinate r and approximate the shapes of the iso-surfaces of r as spheroids**. For clarity we will henceforth refer to this approximation as the spheroidal geopotential approximation*."

Lines 9-13 in the Second Paragraph in CWSM23

"*However, as noted by CW22 and SM22, this analysis only quantifies the error introduced by making the absolute spheroidal approximation; that is, neglecting the horizontal component of gravity in an absolute spheroidal coordinate system. **It does not quantify the error in the community-standard spheroidal geopotential approximation** described in the preceding paragraph; that is, in adopting geopotential coordinates and then approximating the shapes of the geopotentials as spheroids*."

The community-standard spheroidal geopotential approximation is for the use of approximated spheroidal geopotential for spheroidal geopotential coordinates (i.e., EGA-SB) as depicted in Subsection 5.2. It is totally different from EGA-CWSM.

A7. Mistakenly treated the fluid dynamics in rotating frame as in non-rotating frame.

CW22, SM22, atrue3 confuse the fluid dynamics in rotating with non-rotating frame and mistakenly claim the static horizontal pressure gradient force largely cancels the horizontal component of the true gravity. Last paragraph in CWSM23:

"*Physically, as pointed by CW22 and SM22, the reason why the horizontal components of gravity in a spheroidal (or spherical) coordinate system are not dynamically relevant is that in a fluid, **static forces are largely balanced by a static pressure gradient force**. The presence of horizonal gravity in the equation of motion will drive a static horizontal pressure gradient force that largely cancels this component of gravity.*

*…… Failure to account for this cancelation is also the fundamental flaw of Chu (2021), in which the author assumed that **the horizontal components of gravity** will drive Ekman transport instead of **being largely balanced by a static horizontal pressure gradient force** in spheroidal coordinates (see equations 17-20 of Chu 2021)*."

Anyone with basic knowledge of fluid dynamics and geophysical fluid dynamics knows that static forces are largely balanced by a static pressure gradient force only in nonrotating frame, not in rotating frame. Due to the Earth rotation, the steady-state dynamics under low Rossby number is the balance among the gravity, the pressure gradient force, and the Coriolis force. Since the climatological datasets (or called static datasets) for the horizontal component of the true gravity, horizontal pressure gradient force, and the Coriolis force (from horizontal velocity vector) are all available online, the best way is to use these data rather than to use vague statement "*static* *forces are largely balance by a static pressure gradient force*" to identify if the static horizontal pressure gradient force largely cancels the horizontal component of the true gravity or not. Chu (2021a, b, c; 2023, 2024) clearly shows that **the static horizontal pressure gradient force does not cancel the horizontal component of the true gravity, i.e., the horizontal gravity disturbance vector ($g_0\nabla N$).**

A8. Mistakenly decomposed the gravity into gravitational and centrifugal accelerations.

The ultimate cause to use gravity in atmospheric and oceanic dynamics is to make the centrifugal acceleration vanish in the equation of motion. Thus, two basic rules are always followed by meteorologists and oceanographers consciously or unconsciously:

Rule-1 The centrifugal acceleration should never occur in the atmospheric and oceanic dynamics such as in the equation of motion.
Rule-2 The gravity should never be split into gravitational acceleration and centrifugal acceleration.

Breaking these two rules would be equivalent to not conforming to the foundational atmospheric and oceanic dynamics. However, the centrifugal force was stated explicitly in CW22 Supplementary, and implicitly in CWSM23 as the "neglected horizontal" component of $\mathbf{g}_e$. The "neglected horizontal" component of $\mathbf{g}_e$ in an exact spherical coordinate system is the centrifugal acceleration. In CW22 Supplementary:

"*Note that while the horizontal component of the **centrifugal force** is stronger than the "horizontal" component of gravity associated with the wiggles in the true geopotential surfaces, the scale over which the **centrifugal force** varies is larger, hence the error associated with ignoring its variation can be smaller*."

Lines 10-12 in the Third Paragraph in CWSM23:

"*If we proceeded with Chu23's analysis and compared the magnitude of the **'neglected horizontal' component of** $\mathbf{g}_{eff}$ in an exact spherical coordinate system to the Coriolis force (equivalent to the C number of Chu23), we would find C > 10*."

Lines 13-16 in the Third Paragraph in CWMS23:

"*On the contrary, this apparent paradox is resolved in the community-standard treatment of the spherical geopotential approximation (see Staniforth 2022) by redefining the vertical direction to be opposite* $\mathbf{g}_{eff}$*, such that horizontal component of* $\mathbf{g}_{eff}$ *becomes exact zero. The approximation then becomes an approximation of the geometry (i.e., approximating spheroids as spheres) rather than the neglect of **the horizontal component of*** $\mathbf{g}_{eff}$*, resulting in errors that are small (e.g., Benard)*."

CW22 and CWSM23 split $\mathbf{g}_e$ into gravitational acceleration and centrifugal acceleration. Such an intention is equivalent to destroying the foundation of the atmospheric and oceanic dynamics.

A9. Mistakenly treated the Earth mass density as the Earth surface mass distribution.

The mass density σ(**r**) represents mass distribution inside the Earth and related to the internal structure of the Earth such as crust, mantle, inner core, and outer core. **It is not the Earth surface mass distribution from spherical to near spheroid**. The Earth gravitational acceleration is the volume integration over the whole solid Earth with σ(**r**) as part of the integrand [see Eq (6) in Section 2].

The following statement in the First Paragraph CWSM23 is wrong:

"*The rotation of the Earth produces a centrifugal force which distorts the **Earth's mass distribution from spherical to nearly spheroidal** with small spatial inhomogeneities. ……. If Earth's mass distribution were exactly spheroidal, the geopotential would also be exactly spheroidal, and net gravity due to this hypothetical geopotential would be perpendicular to spheroidal surfaces – this is* $\mathbf{g}_{eff}$ *defined by Chu (2023; hereafter Chu23). However, **the Earth's mass distribution is not exactly spheroidal**, and the (slightly) uneven mass distribution gives rise to a perturbation field* δ**g**. *The true (or) total gravity* **g** *is the sum of* $\mathbf{g}_{eff}$ *and* δ**g**."

A10. Mistakenly extended the SGA into the EGA-CWSM

The authors of CW22, SM22, and CWSM23 are not aware of the difference between the SGA (negligible HPGE, see Section 5) and the EGA-CWSM (non-negligible HPGE, see Section 6) and used the SGA (Lines 3-6 in the Third Paragraph):

"*This is analogous to the spheroidal (spherical?) geopotential approximation described above: the vertical coordinate is aligned with geopotentials, and then those geopotentials are approximated as spheres instead of spheroids. This approximation is also adopted by Chu23, stating that the errors of such an approximation are small (last paragraph in section 2.2 of Chu23)*."

to extend to the EGA-CWSM (Lines 6-9 in Third Paragraph):

"*It is inconsistent of Chu23 to apply this spherical geopotential approximation while insisting that spheroidal geopotential approximation cannot be applied to the smaller variations in the geopotential field due to the Earth's uneven mass distribution*."

These statements are incorrect.

A11. The claim of "geometric approximation" in CWSM23 is wrong.

This paper clearly shows that the spheroidal geopotential surface approximation mentioned by CW22, SM22, CWSM23 is invalid. The horizontal pressure gradient error is the same as the horizontal component of the true gravity ($g_0 \nabla N$). The statement in the Abstract of CWSM23 especially the **geometrical approximation** is completely wrong:

"*In recent papers by the authors [Chang and Wolfe (2022; CW22) and Stewart and McWilliams (2022; CW22)], we explained that the actual interpretation of the approximation made in atmospheric and oceanic modeling is not neglecting the horizontal component of the true gravity, but is **a geometrical approximation, approximating nearly spheroidal geopotential surfaces with bumps on which the true gravity is vertical by exactly spheroidal surfaces**.*"

A12. Wrong comments by CW22 and SM22 led to wrong retractions.

The Chief Editor of the Scientific Reports mistakenly retracted Chu (2021a) on the base of wrong comments by CW22 and SW22 (see https://www.nature.com/articles/s41598-022-10846-0). The Statement-1, "*In practice, this component can be taken to be zero, because the errors associated with this neglect are smaller than the error of assuming the resting ocean surface appears locally level, as shown by Chang and Wolfe*", is wrong because the horizontal pressure gradient error in transforming the true to spheroidal geopotential coordinates is non-negligible and equals the horizontal gravity disturbance vector $g_0\nabla N$ (see Section 11). The Statement-2, "*This is further expanded upon in Stewart and McWilliams, who also show that in a model formulated in absolute spherical coordinates, the horizontal component of gravity has a negligible impact on ocean circulation,*" is also wrong since the comments by Stewart and McWilliams are based on wrong comparisons (Subsection A1), wrong derivation (Subsection A2), and wrong treatment of z-varying coefficient as constant in a second order differential equation (Subsection A3).

The then Editor-in-Chief, Minghua Zhang, of the Journal of Geophysical Research – Atmospheres mistakenly retracted Chu (2021b) (see https://agupubs.onlinelibrary.wiley.com/doi/10.1002/jgrd.58211) on the base of wrong comment by SM22 "Thus the 'Ekman' flow and pumping are unchanged by the shift to absolute spherical coordinates" (see Subsection A3). The retract statement by Minghua Zhang "*The retraction has been agreed because the conclusions of the paper were found to be incorrect, since they depend on the choice of the coordinate system that does not apply to practical application of the theory of atmospheric Ekman boundary layer*" is wrong since the comment on the atmospheric Ekman layer dynamics by Stewart and McWilliams is based on wrong comparison (Subsection A1), wrong derivation (Subsection A2) and wrong treatment of z-varying coefficient as constant in a second order ordinary differential equation (Subsection A3).

Acknowledgments

The author thanks the Department of Defense Strategic Environmental Research and Development Program (SERDP) for financial support (N6227C21WA00AWC) and the Research Office of the Naval Postgraduate School for paying the publication cost.

Open Research

The datasets used in this study are publicly available with [$N(\lambda, \varphi)$, $T_z(\lambda, \varphi, 0)$] data at http://icgem.gfz-potsdam.de/home, and long-term annual mean (Z, u, v, T_a) at 12 pressure levels 1,000, 925, 850, 700, 600, 500, 400, 300, 250, 200, 150, and 100 hPa at https://psl.noaa.gov/data/gridded/data.ncep.reanalysis.derived.pressue.html.

References

Beńard, P. (2014). An oblate-spheroidal geopotential approximation for global meteorology. *Quarterly Journal of Royal Meteorological Society*, **140**, 170-184. DOI:10.1002/qj.2141.

Chang, E.K.M. and Wolfe, C.L.P. (2022). The "horizontal" components of the real gravity are not relevant to ocean dynamics. *Scientific Reports, Matters Arising* https://www.nature.com/articles/s41598-022-09967-3.

Chang, E.K.M., Wolfe, C.L.P., Stewart, A.L., & McWilliams, J.C., 2023. Comments on "Horizontal gravity disturbance vector in atmospheric dynamics (https://www.sciencedirect.com/science/article/pii/S0377026523000209)"; *Dynamics of Atmospheres and Oceans,* **103**, https://doi.org/10.1016/j.dynatmoce.2023.101382.

Chu, P. C. (2021a). Ocean dynamic equations with real gravity. https://doi.org/10.1038/s41598-021-82882-1, mistakenly retracted by the Chief Editor, *Scientific Reports*.

Chu, P. C. (2021b). True gravity in atmospheric Ekman layer dynamics. https://doi.org/10.1029/2021JD035293, mistakenly retracted by the then Editor in Chief (Minghua Zhang), *Journal of Geophysical Research – Atmospheres*.

Chu, P. C. (2021c). True gravity in ocean dynamics Part-1 Ekman transport. *Dynamics of Atmospheres and Oceans*, **96**, 101268, https://doi.org/10.1016/j.dynatmoce.2021.101268.

Chu, P. C. (2023). Horizontal gravity disturbance vector in Atmospheric dynamics. *Dynamics of Atmospheres and Oceans,* **102**, 101369, https://doi.org/10.1016/j.dynatmoce.2023.101369.

Chu, P.C. (2024). Invalid spheroidal geopotential approximation and non-decomposable centrifugal acceleration from gravity - Reply to: Comments on "Horizontal gravity disturbance vector in atmospheric dynamics" by Chang, Wolfe, Stewart, McWilliams. *Dynamics of Atmospheres and Oceans*. **106**, 101450, https://doi.org/10.1016/j.dynatmoce.2024.101450.

Gates, W.L. (2004). Derivation of the equations of atmospheric motion in oblate spheroidal coordinates. *Journal of Atmospheric Sci*ences, **61**, 2478-2487.

Gill, A. E. (1982). *Atmosphere-Ocean Dynamics*. Academic Press (see Page 46. Equation 3.5.2).

Holton, J. R., & Hakim, G. J. (2013). An Introduction to Dynamic Meteorology, Academic Press, P34.

Stewart, A.L., & McWilliams, J.C. (2022). Gravity is vertical in geophysical fluid dynamics. *Scientific Reports, Matters Arising*, https://www.nature.com/articles/s41598-022-10023-3.

Staniforth, A., & White, A. (2015). Geophysically realistic, ellipsoidal, analytically tractable (GREAT) coordinates for atmospheric and oceanic modeling. *Quarterly Journal of Royal Meteorological Society*, **141**, 1646-1657.

Vaniček, P. & Krakiwsky, E. (1986) Geodesy: the Concepts. North-Holland, Amsterdam [see Equation (6.4) on Page 72].

White A.A., Staniforth A., & Wood, N. (2008). Spheroidal coordinate systems for modelling global atmospheres. *Quarterly Journal of Royal Meteorological Society*, **134**: 261–270. DOI: 10.1002/qj.208.