Gravitational potential energy is conserved along neutral surfaces in ocean

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Abstract

It is recapitulated that the gravitational potential energy that is conserved along the neutral surfaces needs two terms, one from buoyancy and the other from gravity. I also show a mathematical identity for the time change of this gravitational potential energy which can be interpreted as exchange of energy amongst kinetic, internal, and gravitational potential forms. Movements along the neutral surface conserve the gravitational potential energy and it is shown that not only conversions into and out of the gravitational potential energy balance, but that each of the conversion terms is zero.

Introduction

A water parcel in ocean is affected by gravity via two kinds of dynamic forces. One is buoyancy which is the sum of pressure (local force in contact) from surrounding seawater. The other is gravity from the Earth (remote force with distance). The distinction is not always clear because ocean is mostly in the hydrostatic balance where these two forces exactly balance.

In ocean, there locally exists a plane along which the fluid parcel can be moved "without experiencing a buoyant restoring force" (McDougall., 1987). This is the definition of neutral surfaces. Movements along this surface consume "no gravitational potential energy" (McDougall., 1987).

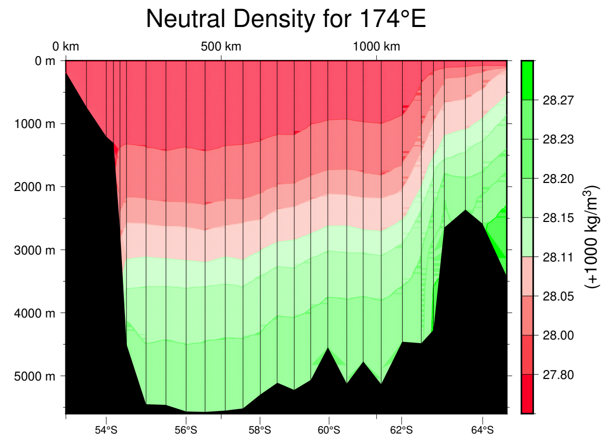

Figure 1: Approximate neutral density γ^n (Jackett & McDougall, 1987) along the GO-SHIP P14S section occupied in 2012. Vertical black lines show the positions of measurements. Black patch is bottom topography.

Figure 1 shows a meridional section of approximate neutral surfaces γ_n (Jackett & McDougall, 1987) along 174°E (GO-SHIP P14S section[1]). A water parcel on γ_n = 1028.11 (kg m^{-3}) around 58°S at 3200 m can be lifted up to a depth shallower than 1000 m further south (64°S) along the approximate neutral surface *without change of gravitational potential energy.*

Such sloping neutral surfaces as seen in Fig.1 are exlained as a response to the westerly wind over the ocean which transports near-surface water northward – the Ekman pumping. If no wind force is applied at the sea surface, the neutral surfaces are horizontal (along the geoid, strictly speaking). From this situation, the sloping neutral surfaces cannot be produced without vertical movement of water parcels, which is across the initial neutral surfaces ("diapycnal") thus accompanied by change of gravitational potential energy. The tilting of the neutral surfaces in Fig.1 is therefore energetically maintained by the surface wind force, but water movements *along* the neutral surfaces ("isopycnal") do not require energy unless accelerated or decelerated.

Definition of gravitational potential energy

The total energy of oceanic water parcel (a compressible binary solution) is the sum of kinetic, internal, and potential energy. This total energy is the Bernoulli function. We define the gravitational potential energy by subtracting internal and kinetic energies from the exact total energy conservation.

We follow the notation and derivation by (Young, 2010) (Y10 thereafter);

$$\frac{D}{Dt}\left(\frac{1}{2}|\boldsymbol{U}|^2 + g_0 Z + e + \frac{P}{\rho}\right) = \frac{1}{\rho}\frac{\partial P}{\partial t}, \tag{1}$$

where $\boldsymbol{U}$ is velocity, g_0 is acceleration due to gravity, Z is geopotential height, e is internal energy, P is pressure, ρ is density. External forcings are omitted. As Y10 points out,

[1] https://cchdo.ucsd.edu/cruise/49NZ20121128

the term on the right means that the Bernoulli function does not satisfy an exact conservation equation. Nonconservation of the Bernoulli density resulting from the unsteady pressure term is a well-known issue with the full equations of motion.

Obviously the kinetic energy is $|\boldsymbol{U}|^2/2$ and the internal energy is e, which leaves

$$\mathrm{P}\frac{}{\rho+g_0 Z} \quad (2)$$

as the gravitational potential energy.

Movements along neutral surface

Suppose a small seawater parcel at (Θ_1, S_1, p_1) is carried along a neutral surface to (Θ_2, S_2, p_2). Here Θ_i and S_i are conservative temperature and Absolute Salinity, respectively. This movement is without heating or mixing such that Θ_1 and S_1 are conserved. The water parcel at the new location $\boldsymbol{r}_2$ at pressure p_2 with small area δA and height δz satisfies the equation of motion

$$\rho(\Theta_1, S_1, p_2)\delta A \cdot \delta z \frac{d^2\boldsymbol{r}_2}{dt^2} = (p_2(z)\delta A - p_2(z+\delta z)\delta A)\hat{\boldsymbol{k}} - \rho(\Theta_1, S_1, p_2) g \delta z \cdot \delta A \hat{\boldsymbol{k}}, \quad (3)$$

where $\hat{\boldsymbol{k}}$ is vertical unit vector. The first two terms on the right-hand side are buoyancy force (local force in contact) and the last term is gravity (remote force from the Earth). Since the movement is along the neutral surface, the system is in equilibrium at $\boldsymbol{r}_2$ and the left-hand side is zero. This equation with hydrostacy for the surrounding water

$$\frac{\partial p_2}{\partial z} = -\rho(\Theta_2, S_2, p_2) g \quad (4)$$

gives

$$\rho(\Theta_2, S_2, p_2) = \rho(\Theta_1, S_1, p_2). \quad (5)$$

This is equivalent to the definition of neutral surface

$$\left(\frac{\partial\rho}{\partial\Theta}\right)_{S,p}\nabla\Theta+\left(\frac{\partial\rho}{\partial S}\right)_{\Theta,p}\nabla S=0$$
(6)

introduced by (McDougall., 1987).

A small change in the potential energy (2) with this movement satisfies

$$\delta\left(\frac{P}{\rho}+g_0Z\right)=\frac{\delta P}{\rho}+g_0\delta Z-\frac{P}{\rho^2}\delta\rho=0$$
(7)

because the first two terms balance in the hydrostatic pressure field and the last term is zero thanks to (5).

Energy conversion

Noting that Z does not depend on time,

$$\frac{DZ}{Dt}=U\cdot\nabla(0,0,Z)=w$$
(8)

we rewrite time change of the gravitational potential energy as

$$\mathrm{D}\frac{}{Dt\left(\frac{P}{\rho}+g_0Z\right)=M-C+\frac{1}{\rho}\frac{\partial P}{\partial t}}$$
(9)

where

$$M=P\frac{D}{Dt}\frac{1}{\rho}=\frac{P}{\rho}\nabla\cdot\boldsymbol{U} \tag{10}$$

and

$$C=-\left(\frac{\boldsymbol{U}}{\rho}\cdot\nabla P+g_0\boldsymbol{U}\cdot\nabla(0,0,Z)\right). \tag{11}$$

Mathematically, this is just an identity. In terms of physics, the kinetic energy equation (vector inner product of $\boldsymbol{U}$ and (18) of Y10),

$$\frac{D}{Dt}\frac{|U|^2}{2}=C \tag{12}$$

and the internal energy conservation (Eq.(20) of Y10)

$$\frac{De}{Dt} = -M \tag{13}$$

demonstrate that M is conversion from the internal energy to the gravitational potential energy and C is conversion from the gravitational potential energy to the kinetic energy. We note movements along the neutral surface do not change density ρ (see (5)) such that $M = 0$. It was shown previously that the left hand side of (9) is zero for such movements. We therefore conclude that $C = 0$ for such movements.

Boussinesq approximation

The arguments above can be paraphrased under the Boussinesq approximation by following the rigorous formulation by Y10. By separating out the small terms in density and pressure

$$\begin{aligned} \rho &= \rho_0 - \frac{\rho_0}{g_0} b \\ P &= P_0 - g_0 \rho_0 Z + \rho_0 p \end{aligned}$$

where $b = g_0(1 - \rho/\rho_0)$ is buoyancy. and the last terms on the right-hand sides are much smaller than the other terms, the pressure term in the potential energy (2) becomes

$$\frac{P}{\rho} \approx \frac{P_0}{\rho_0} - (g_0 + b)Z + \frac{P_0}{\rho_0 g_0} b + p, \tag{14}$$

which is Eq.(A1) of (Young, 2010). Then the potential energy (2) under the Boussinesq approximation is

$$\frac{P}{\rho} + g_0 Z \approx \frac{P_0}{\rho_0} - bZ + \frac{P_0}{\rho_0 g_0} b + p. \tag{15}$$

A small change of this quantity along a neutral surface satisfy

$$\left(\frac{P_0}{\rho_0 g_0} - Z \right) \delta b + \delta p - b\delta Z = 0. \tag{16}$$

The first term vanishes because $\delta b = 0$ (see Eq.(5)). We assume that the perturbation is hydrostatic such that the last two terms on the left-hand side balance.

A time change of the Boussinesq gravitational potential energy (15) in arbitrary direction is

$$\mathrm{D}\frac{}{Dt\left(\frac{P_0}{\rho_0}-bZ+\frac{P_0}{\rho_0 g_0}b+p\right)=\left(\frac{P_0}{\rho_g g_0}-Z\right)\frac{Db}{Dt}-b\frac{DZ}{Dt}+\frac{Dp}{Dt}.}$$
(17)

The rightmost two terms can be identified as the conversion between the kinetic energy and the gravitational potential energy;

$$C' = -\left(-b\frac{Dz}{Dt} + \nabla(\boldsymbol{u}p)\right), \tag{18}$$

which is Eq.(15) of Y10. Under the Boussinesq approximation, the kinetic energy equation (12) becomes

$$\frac{D}{Dt}\left(\frac{|\boldsymbol{u}|^2}{2}\right) = C',$$
(19)

where $\boldsymbol{u}$ is incompressible part of the velocity (i.e. $\boldsymbol{U}$ under the Boussinesq approximation).

Under the Boussinesq approximation, a key thermodynamic quantity is the Boussinesq dynamic enthalpy

$$h^{\ddagger}(\Theta, Z, S) = \int_Z^0 b(\Theta, Z', S)dZ'. \tag{20}$$

From $h^{\ddagger}$, the Boussinesq internal energy is defined as

$$\mathrm{e}^{\ddagger} = h^{\ddagger} + Zb.$$
(21)

Time change of the internal energy is

$$\frac{De^{\ddagger}}{Dt} = -M', \tag{22}$$

where

$$M' = -Z\frac{Db}{Dt} \tag{23}$$

which is Eq.(A4) of Y10. As noted in Eq.(40) of Y10, the quantity M' can be neglected under the Boussinesq approximation but serves as a diagnostics for the divergent component of the velocity field $\boldsymbol{U}$. This explains why the first term on the right-hand side in (17) can be neglected. Alternatively, one can argue that $P_0/(\rho_0 g_0) \approx 10$ m such that $|Z| \gg P_0/(\rho_0 g_0)$ for deep water parcels.

In summary, under the Boussinesq approximation the gravitational potential energy satisfies

$$\frac{D}{Dt}\left(\frac{P_0}{\rho_0} - bZ + \frac{P_0}{\rho_0 g_0} b + p\right) = M' - C', \tag{24}$$

where $M' > 0$ is the conversion from the internal energy to the gravitational potential energy and $C' > 0$ is the conversion from the gravitational potential energy to the kinetic energy.

Discussion

When discussing ocean energetics (Hughes et al., 2009; Zemskova et al., 2015), gravitational potential energy is defined as $\rho g z$. Thus, movement of water parcel along the neutral surface does not conserve the gravitational potential energy because it is the form (2) that is conserved along the neutral surface. Change of $\rho g z$ along the neutral surface is compensated by the energy input from the westerly wind which drives the diapycnal upwelling, as discussed in the thought experiment in Introduction. The energy in the P/ρ part of (2) is counted as part of the internal energy in (Hughes et al., 2009; Zemskova et al., 2015).

The definition in (21) implies that it is more natural to use enthalpy

$$h = \frac{P}{\rho} + e \tag{25}$$

than internal energy e. Indeed, enthalpy is closely related the well-conserved quantity, conservative temperature Θ (McDougall, 2003) and has physically appealing interpretation (See Eq.(33) of Y10 and following paragraphs). In this formulation with enthalpy, the graviational potental energy is simply $\rho g z$. Again, a water parcel moving along a neutral surface does not conserve this quantity as $\delta(\rho g z) = (\delta\rho) g z + \rho g(\delta z) = \rho g(\delta z)$. The purpose of this short note is to point out this ambiguity of the term "gravitational potential energy".

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