On variational approach to fourth order problems with unbounded weight

Marek Galewski[1] and Dumitru Motreanu[2]

[1]Politechnika Lodzka Instytut Matematyki
[2]Universite de Perpignan Via Domitia

March 7, 2023

Abstract

We investigate fourth order equations with Dirichlet type boundary conditions with perturbation unbounded from above making the problem non-potential. We apply variational method to some auxiliary problem and conclude about the existence and uniqueness to the original one. Multiple solutions are also considered. We conclude our note with the result pertaining to the continuous dependence on parameters.

Received day month year; | Revised day month year; | Accepted day month year

DOI: xxx/xxxx

ARTICLE TYPE

On variational approach to fourth order problems with unbounded weight

Marek Galewski*[1] | Dumitru Motreanu[2]

[1]Institute of Mathematics, Lodz University of Technology, al. Politechniki 8, 93-590 Lodz, Poland
[2]Department of Mathematics, University of Perpignan, 66860 Perpignan, France

Correspondence
*Marek Galewski
Email: marek.galewski@p.lodz.pl.com

Summary

We investigate fourth order equations with Dirichlet type boundary conditions with perturbation unbounded from above making the problem non-potential. We apply variational method to some auxiliary problem and conclude about the existence and uniqueness to the original one. Multiple solutions are also considered. We conclude our note with the result pertaining to the continuous dependence on parameters.

KEYWORDS:
variational method, dependence on parameters, Dirichlet problem, existence and uniqueness, unbounded weight

1 | INTRODUCTION

We are interested in the following variant of the elastic beam equation, i.e. the fourth order problem with unbounded from above perturbation g and with a numerical parameter $\lambda > 0$

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\right) = \lambda f(t, u(t)), \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{1}$$

where $f : [0,1]\times\mathbb{R}\to\mathbb{R}$, $g : [0,1]\times\mathbb{R}_+\to\mathbb{R}$ are functions subject to some conditions provided below. We underline following[8] that in such a case the problem cannot be directly tackled by monotonicity methods due to the fact the relevant operator is in this case unbounded. The variational approach cannot be used due to the lack of potentiality which results from the fact that g is unbounded from above. Thus in order to overcome these difficulties we will apply some truncation technique from[8] and[9] which was introduced for second order partial differential equations in connection with the usage of the theory of monotone operators, namely the (generalized) Browder-Minty Theorem. With this truncation method the auxiliary problem which we obtain is now variational, i.e. solutions correspond in a $1-1$ manner to critical points of the relevant action functional. We underline,that in the sources mentioned the authors obtained the existence of at least one solution without any information about the multiplicity. The feature of the present note is to apply a recent multiplicity theorem from[4] in order to get the existence of multiple solutions for coercive functionals as well. We consider the coercive case which means that we have an upper bounded for any solution, provided it exists. At the same time, the coercivity means that we have restricted number of multiplicity results at our disposal.

We seek weak solutions in the space

$$H_0^2(0,1) = \left\{u \in H_0^1(0,1) : \ddot{u} \in L^2(0,1), \dot{u}(0) = \dot{u}(1) = 0\right\}$$

normed by

$$\|u\|_{H_0^2} = \sqrt{\int_0^1 \left|\frac{d^2}{dt^2}u(t)\right|^2 dt}.$$

Put

$$\|u\|_C := \max_{t\in[0,1]} |u(t)|.$$

As is the case of the well known space $H_0^1(0,1)$ the Sobolev and Poincaré inequalities read as follows: for any $u \in H_0^2(0,1)$ it holds

$$\|u\|_C \le \|u\|_{H_0^1} \le \frac{1}{\pi}\|u\|_{H_0^2}$$

and

$$\|u\|_{L^2} \le \frac{1}{\pi}\|u\|_{H_0^1} \le \frac{1}{\pi^2}\|u\|_{H_0^2}.$$

Moreover, we have

$$\|\dot u\|_C \le \|u\|_{H_0^2}.$$

As we seek for solutions in $H_0^2(0,1)$, we mean the so called weak solutions, namely $u \in H_0^2(0,1)$ solves (1) when

$$\int_0^1 \tfrac{d^2}{dt^2}u(t)\tfrac{d^2}{dt^2}v(t)\,dt + \int_0^1 g_R\left(\tfrac{d}{dt}u_R(t)\right)\tfrac{d}{dt}u(t)\tfrac{d}{dt}v(t)\,dt =$$

$$\lambda \int_0^1 f(t,u(t))v(t)\,dt$$

for all $v \in H_0^2(0,1)$. Further on we follow also the issue of higher regularity which is due to the variant of the celebrated du Bois-Reymond Lemma, see for example[7].

Here are the conditions which we will use. Let us recall for $p \ge 1$ that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is an L^p-Carathéodory function if the following conditions are satisfied (the first two meaning it is Carathéodory):

(i). $t \mapsto f(t,x)$ is measurable on $[0,1]$ for each fixed $x \in \mathbb{R}$,

(ii). $x \mapsto f(t,x)$ is continuous on $\mathbb{R}$ for a.e. $t \in [0,1]$,

(iii). for each $d \in \mathbb{R}^+$ function $t \mapsto \max_{|x|\le d}|f(t,x)|$ belongs to $L^p(0,1)$.

Observe that $F : [0,1]\times\mathbb{R} \to \mathbb{R}$ given by

$$F(t,x) = \int_0^x f(t,s)\,ds \text{ for a.e. } t \in [0,1] \text{ and all } x \in \mathbb{R}$$

is a Carathéodory function as well in case f is Carathéodory. We see that $\frac{d}{dx}F(t,x) = f(t,x)$ for a.e. $t \in [0,1]$ and all $x \in \mathbb{R}$. The assumptions are required for various results as follows,

A) about the existence:

A1 $g : [0,1]\times\mathbb{R}_+ \to \mathbb{R}$ *is a continuous function for which there are a constant* $g_0 \in (0,\pi^2)$ *and a function* $g_1 : \mathbb{R}_+ \to \mathbb{R}$ *such that* $g(t,x) \ge g_1(t) \ge g_0$ *for all* $t \in [0,1]$ *and* $x \in \mathbb{R}$ *and* $\lim_{t\to\infty} g_1(t) = +\infty$.

A2 $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is an* L^2-*Carathéodory function.*

A3 *There exist* $a \in L^\infty(0,1;\mathbb{R}_+)$, $b \in L^1(0,1)$ *such that for a.e.* $t \in [0,1]$ *and all* $x \in \mathbb{R}$ *it holds*

$$f(t,x)x \le a(t)|x|^2 + b(t)$$

B) in addition to the above in connection with uniqueness:

A4 *For a.e.* $t \in [0,1]$ *function* $x \mapsto f(t,x)$ *is nonincreasing.*

A5 *It holds that*

$$g(t,x)x - g(t,y)y \ge 0$$

for all $t \in [0,1]$ *and* $x \ge y \ge 0$.

C) in connection with the existence of multiple solutions we must replace condition **A3** with some other condition and additionally impose:

A6 *There exist a constant* $\theta \in (1,2)$ and functions $a \in L^{\nu}\left(0,1;\mathbb{R}_+\right)$, $b \in L^1(0,1)$ *such that for a.e.* $t \in [0,1]$ *and all* $x \in \mathbb{R}$ *it holds*

$$f(t,x)x \leq a(t)\,|x|^{\theta} + b(t)$$

and where $\nu = \frac{2}{2-\theta}$.

A7 There is a function $\overline{u} \in H_0^2(0,1)$ such that

$$\int_0^1 F\left(t,\overline{u}(t)\right) > 0$$

A8 $\lim_{x\to 0} \frac{f(t,x)}{x} = 0$ uniformly a.e. for $t \in [0,1]$.

Note that for **A1** we can take any positive lower bound, since we can always decrease it, so the assumption that $g_0 \in \left(0,\pi^2\right)$ is not restrictive, but is important from the techniques applied in the proofs. From assumption **A4** we have that $x \mapsto -f(t,x)$ is nondecreasing for a.e. $t \in [0,1]$ which is what we further need in order to make the auxiliary action functional (strictly) convex. Assumption **A6** is some version of **A3** and both lead towards the coercivity for various values of numerical parameter λ. We introduce this parameter because of the methodology applied in connection with the existence of at least three solutions.

The equation which we investigate in this note pertains to the theory of elastic deflection which was considered from various point of view for example in[1],[2],[13],[14] in which the three critical point theorem due to Ricceri see[10], the Sturm comparison theorem combined with the shooting method and also the Guo-Krasnosel'skij fixed point theorem of cone-expansion compression type are employed. The authors mainly considered, as we do here, rigidly fastened beams, i.e. fourth order equation

$$\frac{d^4}{dt^4}x = f(t,x) \tag{2}$$

pertaining to boundary conditions

$$x(0) = x(1) = \dot{x}(0) = \dot{x}(1) = 0$$

or simply supported beams, i.e. the equation (2) with conditions

$$x(0) = x(1) = \ddot{x}(0) = \ddot{x}(1) = 0$$

are considered. Equation (2) with either boundary conditions is a simplified version of the following one

$$\frac{d^2}{dt^2}\left(E(t)\,I(t)\,\frac{d^2}{dt^2}x(t)\right) + w(t)\,x(t) = f(t,x(t))$$

with suitable assumptions placed on f and where $E : [0,1] \to R$ is Young's modulus of elasticity for the beam, $I : [0,1] \to R$ is the moment of inertia of cross section of the beam and w is the load density (force per unit length of a beam). It is usually assumed that that $w(t) > 0$, $E(t) \geq E_0 > 0$, $I(t) \geq I_0 > 0$ for $t \in [0,1]$ and that $E, I, w \in L^{\infty}(0,1)$ or else that functions E, I are constant (and therefore equal 1) and function w is incorporated into the nonlinear term. The assumptions which we impose allow for having the load density as it is and some minor technical changes would allow us for having functions E, I included into the main setting. Now we provide some examples of nonlinear terms which satisfy our assumptions.

Example 1. Concerning the nonlinear perturbation $g : [0,1]\times\mathbb{R}_+ \to \mathbb{R}$ we may consider the following unbounded from above function

$$g(t,x) = e^x + 1.5 + \sin(\pi t)$$

which is bounded from below and satisfies the monotonicity condition **A5**.

Example 2. Concerning the nonlinear term $f : [0,1]\times\mathbb{R}_+ \to \mathbb{R}$ satisfying **A2**, **A3**, **A4** we may consider the following function (where we drop the dependence on t for clarity)

$$f(x) = \ln\left(x^2+1\right) - 2x$$

which satisfies the growth condition with $a = 4$.

Example 3. Concerning the nonlinear term $f : [0,1]\times\mathbb{R}_+ \to \mathbb{R}$ satisfying conditions **A2**, **A6** we propose

$$f(x) = \ln\left(x^2+1\right) + \sqrt{|x|}$$

where we take $\theta = \frac{3}{2}$ and $a = 4$ in order to have **A6**. Note that in this case we do not have **A4** satisfied.

2 | BACKGROUND AND AUXILIARY RESULTS

Our main tool pertains to the classical direct variational method, see[7] and[11] for the theoretical background. In what follows we assume E to be a real reflexive, separable Banach space and $J : E \to \mathbb{R}$.

Theorem 1. If $J \in C^1(E, \mathbb{R})$ is sequentially weakly lower semi-continuous and coercive, i.e. $\lim_{\|x\|\to\infty} J(x) = +\infty$, then there exist x_0 such that

$$\inf_{x\in E} J(x) = J(x_0)$$

and x_0 is also a critical point of J, i.e. $J'(x_0) = 0$.

The above theorem has also a version that originates from the application of the theory of monotone operators. Our approach relies on this theory. Thus we proceed to some other version of the above given theorem after some preparation. For the background results cited here we refer to[5]. Operator $A : E \to E^*$ is called:

i) *monotone* if for all $u, v \in E$

$$\langle A(u) - A(v), u - v\rangle \geq 0$$

and *strictly monotone* if the above inequality is strict for $u \neq v$;

ii) *demicontinuous* if $u_n \to u_0$ implies $A\left(u_n\right) \rightharpoonup A\left(u_0\right)$;

iii) *strongly continuous* if $u_n \rightharpoonup u_0$ implies $A\left(u_n\right) \to A\left(u_0\right)$;

iv) *potential if there exists a Gâteaux differentiable functional* $\mathcal{A} : E \to \mathbb{R}$, *called the potential of* A, *such that* $\mathcal{A}' = A$;

v) *satisfying condition (S) if*

$$u_n \rightharpoonup u_0 \textit{ in } E \textit{ and } \left\langle A\left(u_n\right) - A\left(u_0\right), u_n - u_0\right\rangle \to 0 \textit{ imply } u_n \to u_0 \textit{ in } E;$$

vi) *coercive if*

$$\lim_{\|v\|\to\infty} \frac{\langle A(v), v\rangle}{\|v\|} \to +\infty.$$

We need some version of *the Weierstrass-Tonelli Theorem* to be found in[4]*:*

Theorem 2. Assume that functional $\mathcal{J} : E \to \mathbb{R}$ is bounded from below, coercive, Gâteaux differentiable and that its derivative $\mathcal{J}' : E \to E^*$ satisfies condition (S). Then there is some $u_0 \in E$ such that

$$\mathcal{J}\left(u_0\right) = \inf_{u\in E} \mathcal{J}(u).$$

Now we introduce the multiplicity result. For $r > 0$ we put

$$B_r := \{x : \|x\| \leq r\}, \ S_r = \{x : \|x\| = r\}.$$

Theorem 3. Assume that $\mathcal{J} \in C^1(E)$ is sequentially weakly l.s.c., coercive and has a Gâteaux derivative $\mathcal{J}' : E \to E^*$ which satisfies condition (S). Let $\tilde{x} \in E$ and $r > 0$ be fixed. Assume further that conditions

B1 $\inf_{x\in E} \mathcal{J}(x) < \inf_{x\in B_r} \mathcal{J}(x)$;

B2 $\|\tilde{x}\| < r$ and $\mathcal{J}(\tilde{x}) < \inf_{x\in S_r} \mathcal{J}(x)$

are satisfied. Then functional $\mathcal{J}$ has at least three critical points in E, two of which are necessarily nontrivial.

We need some additional technical results about monotonicity and potentiality of the perturbation operator. We assume that

Aφ $\varphi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ is a Carathéodory function for which there is a constant $M > 0$ such that

$$|\varphi(t,x)| \leq M \text{ for a.e. } t \in [0,1] \text{ and all } x \in \mathbb{R}_+.$$

Under some additional growth assumption on function φ we will consider the monotonicity of an operator

$$A : L^2(0,1) \to L^2(0,1),$$

given by

$$\langle A(u), v\rangle = \int_0^1 \varphi(t, |u(t)|)\, u(t)\, v(t)\, dt \tag{3}$$

Theorem 4. Assume that condition $\mathbf{A}\varphi$ is satisfied. Operator A given by (3) is potential with the potential $F : L^2(0,1) \to \mathbb{R}$ defined by

$$F(u) = \int_0^1 \int_0^{|u(t)|} \varphi(t,s)\, s ds dt \text{ for } u \in L^2(0,1).$$

If

$$\varphi(t,x)\, x - \varphi(t,y)\, y \geq 0$$

for all $x \geq y \geq 0$ and a.e. $t \in [0,1]$, then A is monotone.

Now we introduce some technical tools which will be used further on:

Lemma 1. Assume that $A : E \to E^*$ is potential, demicontinuous, bounded and coercive. Then its potential $F : E \to \mathbb{R}$ is coercive.

Lemma 2. Assume that operator $A : E \to E^*$ fulfills property (S) and that $T : E \to E^*$ is strongly continuous. Then $A+T$ also has property (S).

Lemma 3. Assume that operator $A : E \to E^*$ is strongly continuous and bounded. Then it is demicontinuous.

Lemma 4. Assume that $A : E \to E^*$ is potential and monotone. Then A is demicontinuous.

3 | TRUNCATED PROBLEM - EXISTENCE AND UNIQUENESS

We see from condition **A3** that if any solution to (1) exists, it is necessarily bounded. Let us define

$$\overline{\lambda} = \frac{\pi^2\left(\pi^2 - g_0\right)}{\|a\|_{L^\infty}}.$$

Lemma 5. Assume that conditions **A1, A2,A3** are satisfied. Then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ there is some $R > 0$ such that $\|u\|_{H_0^2} \leq R$ and $\|\dot{u}\|_C \leq R$ for every $u \in H_0^2(0,1)$ that solves problem (1).

Proof. Let us fix $\lambda \in \left(0, \overline{\lambda}\right)$. Assume that $u \in H_0^1(0,1)$ solves problem (1). Testing it with $v = u$ we have

$$\|u\|_{H_0^2}^2 + \int_0^1 g\left(t, \left|\frac{d}{dt}u(t)\right|\right)\left|\frac{d}{dt}u(t)\right|^2 dt = \lambda \int_0^1 f(t,u(t))u(t)\, dt. \tag{4}$$

Then we obtain concerning the left hand side of (4)

$$\begin{gathered}\|u\|_{H_0^2}^2 + \int_0^1 g\left(t, \left|\tfrac{d}{dt}u(t)\right|\right)\left|\tfrac{d}{dt}u(t)\right|^2 dt \geq \\ \|u\|_{H_0^2}^2 + g_0\|u\|_{H_0^1}^2 \geq \|u\|_{H_0^2}^2 - \tfrac{g_0}{\pi^2}\|u\|_{H_0^2}^2.\end{gathered} \tag{5}$$

Estimating the right hand side of (4) we have what follows

$$\begin{gathered}\lambda\int_0^1 f(t,u(t))u(t)\, dt \leq \lambda\int_0^1 a(t)|u(t)|^2 dt + \lambda\int_0^1 b(t)\, dt \leq \\ \tfrac{\lambda}{\pi^2}\|a\|_{L^\infty}\|u\|_{H_0^1}^2 + \lambda\|b\|_{L^1} \leq \lambda\tfrac{\|a\|_{L^\infty}}{\pi^4}\|u\|_{H_0^2}^2 + \lambda\|b\|_{L^1}\end{gathered}$$

Summing up we arrive at

$$\left(1 - \frac{g_0}{\pi^2} - \lambda\frac{\|a\|_{L^\infty}}{\pi^4}\right)\|u\|_{H_0^2}^2 - \lambda\|b\|_{L^1} \leq 0$$

which implies the assertion $\|u\|_{H_0^2} \le R$ since $1 - \frac{g_0}{\pi^2} - \lambda \frac{\|a\|_{L^\infty}}{\pi^4} > 0$. We see that we can take

$$R^2 := \frac{\lambda \|b\|_{L^1}}{1 - \frac{g_0}{\pi^2} - \lambda \frac{\|a\|_{L^\infty}}{\pi^4}}.$$

The remaining assertion follows by the Sobolev inequality. □

We define the following continuous function $g_R : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$

$$g_R(t,x) = \left\{ \begin{array}{ll} g(t,x), & 0 \le x \le R \\ g(t,R), & x > R. \end{array} \right\} \tag{6}$$

We see that for $t \in [0,1]$, and all $x \in \mathbb{R}_+$

$$g_0 \le g_R(t,x) \le \max_{t \in [0,1], 0 \le x \le R} g(t,x)$$

and moreover

$$g_R(t,x)\,x - g_R(t,y)\,y \ge 0$$

for all $t \in [0,1]$, and all $x \ge y \ge 0$ in case **A5** holds. With the above result we can define the following truncated problem for a numerical parameter $\lambda > 0$

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right) \frac{d}{dt}u(t) \right) = \lambda f(t,u(t)), \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{7}$$

Now problem (7) is variational so we can apply the direct method in order to investigate its solvability. Again we will look for weak solutions whose further regularity will be investigated in what follows. The meaning of the weak solution to (7) is now obvious, in fact it stems from equating (8) provided below.

We need some preparation as well as some auxiliary results from the theory of monotone operators. We define the following operator

$$A : H_0^2(0,1) \to \left(H_0^2(0,1)\right)^*$$

by

$$\begin{aligned} \langle A(u), v\rangle = &\int_0^1 \frac{d^2}{dt^2}u(t)\frac{d^2}{dt^2}v(t)\,dt + \int_0^1 g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\frac{d}{dt}v(t)\,dt \\ &-\lambda \int_0^1 f(t,u(t))v(t)\,dt \end{aligned} \tag{8}$$

and we put $A_1, A_2, A_3 : H_0^2(0,1) \to \left(H_0^2(0,1)\right)^*$ by

$$\langle A_1(u), v\rangle = \int_0^1 \frac{d^2}{dt^2}u(t)\frac{d^2}{dt^2}v(t)\,dt$$

$$\langle A_2(u), v\rangle = \int_0^1 g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\frac{d}{dt}v(t)\,dt$$

$$\langle A_3(u), v\rangle = -\int_0^1 f(t,u(t))v(t)\,dt.$$

Then by (8)

$$A = A_1 + A_2 + \lambda A_3.$$

We obtain the following technical lemmas which will be used in both the existence and uniqueness results.

Lemma 6. Operator A_1 is continuous, bounded, coercive strongly monotone and satisfies condition (S). Moreover, A_1 is potential with the potential $\mathcal{A}_1 : H_0^2(0,1) \to \mathbb{R}$ defined by

$$\mathcal{A}_1(u) = \frac{1}{2}\int_0^1 \left|\frac{d^2}{dt^2}u(t)\right|^2 dt.$$

Proof. The continuity of A is obvious. We calculate for any $u, v \in H_0^2(0,1)$ that

$$\langle A_1(u) - A_1(v), u - v\rangle = \int_0^1 \left|\frac{d^2}{dt^2}u(t) - \frac{d^2}{dt^2}v(t)\right|^2 dt$$

which provides the strong monotonicity and this implies the remaining assertions. □

Lemma 7. Assume condition **A2**. Then operator A_3 is potential, bounded and strongly continuous. If additionally **A4** holds, then A_3 is monotone. The potential $\mathcal{A}_3 : H_0^2(0,1) \to \mathbb{R}$ is defined by

$$\mathcal{A}_3(u) = \int_0^1 F(t, u(t))\, dt.$$

Proof. Operator A_3 is obviously potential. The monotonicity in case **A4** holds is also obvious. Operator A_3 is bounded since f is an L^2-Carathéodory function. We prove that it is strongly continuous. Take a sequence $(u_n)_{n=1}^{\infty}$ which is weakly convergent to some u_0 in $H_0^2(0,1)$. Then $(u_n)_{n=1}^{\infty}$ converges uniformly on $[0,1]$ to u_0. Thus there is some $d > 0$ that $\|u_n\|_C \leq d$. Since F is L^2-Carathéodory function we find a function $f_d \in L^2(0,1)$ such that for a.e. $t \in [0,1]$ and all $n \in \mathbb{N}$ it holds

$$\left|f\left(t, u_n(t)\right)\right| \leq f_d(t)$$

Hence we can apply the Lebesgue Dominated Convergence Theorem. □

Lemma 8. Assume condition **A1**. Then operator A_2 is bounded, potential and strongly continuous and demicontinuous. If additionally **A5** holds, then A_2 is monotone. The potential $\mathcal{A}_2 : H_0^2(0,1) \to \mathbb{R}$ is defined by

$$\mathcal{A}_2(u) = \int_0^1 \int_0^{\left|\frac{d}{dt}u(t)\right|} g_R(t,s)\, s\, ds\, dt.$$

Proof. Using same arguments as in the proof of Lemma 7 and noting that a sequence $(u_n)_{n=1}^{\infty}$ which is weakly convergent to some u_0 in $H_0^2(0,1)$ is such that $(\dot{u}_n)_{n=1}^{\infty}$ converges uniformly on $[0,1]$ to $\dot{u}_0$, we see that A_2 is strongly continuous. Since g_R is bounded, it is also bounded. Applying Theorem 4 and Lemma 4 we see the remaining part of the assertion. □

With the above preparations we obtain the following:

Lemma 9. Assume that conditions **A1-A3** are satisfied. Then operator A is monotone, demicontinuous, potential, bounded and coercive. Moreover, A satisfies condition (S). If we assume additionally **A4-A5** then A is strictly monotone.

Proof. Using Lemma 3 we see that A_3 is demicontinuous. We now use Lemmas 6, 8, 7 to conclude that A is monotone, demicontinuous, potential, bounded. By arguments contained in Lemma 5 we see that A is coercive. By Lemma 6 operator A_1 satisfies condition (S), so by Lemma 2 we see that so is true for A. The strict monotonicity of A follows since A_1 being strongly monotone is strictly monotone. □

Now we define the following action functional $\mathcal{J} : H_0^2(0,1) \to \mathbb{R}$ by

$$\mathcal{J} = \mathcal{A}_1 + \mathcal{A}_2 + \lambda \mathcal{A}_3 \tag{9}$$

We see that functional $\mathcal{J}$ satisfies the assumption of Theorem 2 and we can propose the following result:

Theorem 5. Assume that conditions **A1-A3** are satisfied. Then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (7) has at least one weak solution. If we assume additionally **A4-A5** then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (7) has exactly one weak solution.

We have also the following obvious result required later on:

Lemma 10. Assume that conditions **A1-A2** are satisfied. Then operator $A_1 + A_2$ is strongly monotone, continuous, coercive and satisfies condition (S). Moreover, $A_1 + A_2$ is invertible.

4 | ORIGINAL PROBLEM -REGULARITY, EXISTENCE AND UNIQUENESS

We start with some observation concerning the regularity of the weak solution. We introduce an auxiliary result pertaining to the higher order du Bois-Reymond Lemma provided to match problem which we consider. Some related issues were considered in [12] however for different type of problems. From [11], Proposition 4.5 we get the following result:

Lemma 11. If $h \in L^2(0,1)$ and if

$$\int_0^1 h(t) \frac{d^2}{dt^2} v(t)\, dt = 0$$

for all $v \in H_0^2(0,1)$, then there exist constants $c_0, c_1 \in \mathbb{R}$ such that $h(t) = c_0 + c_1 t$ a.e. on $[0,1]$.

Now we obtain from the above lemma the regularity result about the weak solution:

Proposition 1. Assume that conditions **A1, A2,A3** are satisfied. Fix $\lambda \in \left(0, \overline{\lambda}\right)$. Then any $u \in H_0^2(0,1)$ which is weak solution to (7) is such that $u, \frac{d}{dt}u, \frac{d^2}{dt^2}u, \frac{d^3}{dt^3}u$ are absolutely continuous and $\frac{d^4}{dt^4}u \in L^2(0,1)$ and that u satisfies (7) a.e. on $[0,1]$.

Proof. We will apply Lemma 11. Since $u \in H_0^2(0,1)$ is a weak solution to (7), we see that $u, \frac{d}{dt}u$ are absolutely continuous. Next using the definition of the weak solution to (7) and integrating by parts twice we see that the following holds for any $v \in H_0^2(0,1)$

$$\int_0^1 \left(\frac{d^2}{dt^2}u(t) - \int_0^t \left(g_R\left(s, \left|\frac{d}{ds}u_R(s)\right|\right) \frac{d}{ds}u(s)\right) ds \right) \frac{d^2}{dt^2}v(t)\, dt -,$$

$$\lambda \int_0^1 \int_0^t \left(\int_0^s f(\tau, u(\tau)) d\tau\right) ds \frac{d^2}{dt^2} v(t)\, dt = 0.$$

Now using Lemma 11 and differentiating twice, we obtain the assertion. □

With Proposition 1 we can introduce another notion of a solution to (7), namely the classical solution. We say that a function $u \in H_0^2(0,1)$ is a **classical solution** to (7), if it is a weak solution, satisfies (7) a.e. on $[0,1]$ and if $u, \frac{d}{dt}u, \frac{d^2}{dt^2}u, \frac{d^3}{dt^3}u$ are absolutely continuous while $\frac{d^4}{dt^4}u \in L^2(0,1)$.

From Proposition 1 and Theorem 5 we immediately obtain that:

Theorem 6. Assume that conditions **A1-A3** are satisfied. Then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (7) has at least one classical solution. If we assume additionally **A4-A5** then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (7) has exactly one classical solution.

We can state the following main existence result:

Theorem 7. Assume that conditions **A1-A3** are satisfied. Then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (1) has at least one classical solution. If we assume additionally **A4-A5** then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ problem (1) has exactly one classical solution.

Proof. Due to Lemma 5 solutions to (1) and (7) coincide since the estimate R which we obtain is independent of a solution. From Lemma 5 it also follows that problem (1), due to the coercivity, has only bounded solutions. Now the result follows from Theorem 6. □

From the above result we have an immediate corollary which we provide directly:

Corollary 1. Assume that conditions **A1-A3** are satisfied. Then for each fixed $\lambda \in \left(0, \overline{\lambda}\right)$ solutions to problems (1) and (7) coincide.

Now we proceed to the case when instead of **A3** we assume **A6.**

Lemma 12. Assume that conditions **A1, A2, A6** are satisfied. Then for each fixed $\lambda > 0$ there is some $R > 0$ such that $\|u\|_{H_0^2} \le R$ and $\|\dot{u}\|_C \le R$ for every $u \in H_0^2(0,1)$ that solves problem (1).

Proof. We follow the proof of Lemma 5, so we provide only some minor results. Let us fix $\lambda > 0$. Assume that $u \in H_0^2(0,1)$ solves problem (1). Since $\nu = \frac{2}{2-\theta}$, we see that $\nu' = \frac{2}{\theta}$ $\left(\left(\nu'\right)^{-1} + \nu^{-1} = 1\right)$. Then we proceed as in the proof of Lemma 5 using the Hölder Inequality and obtain

$$\|u\|^2_{H^2_0} - \tfrac{g_0}{\pi^2}\|u\|^2_{H^2_0} \leq \lambda \int_0^1 f(t,u(t))u(t)\,dt \leq$$

$$\lambda\|a\|_{L^\nu}\|u\|^\theta_{L^2} + \lambda\int_0^1 b(t)\,dt \leq$$

$$\tfrac{\lambda}{\pi^{4\theta}}\|a\|_{L^\nu}\|u\|^\theta_{H^2_0} + \lambda\|b\|_{L^1}.$$

Therefore we obtain the following inequality

$$\left(1 - \frac{g_0}{\pi^2}\right)\|u\|^2_{H^2_0} - \frac{\lambda}{\pi^{4\theta}}\|a\|_{L^\nu}\|u\|^\theta_{H^2_0} \leq \lambda\|b\|_{L^1}$$

and the assertion follows since $\theta \in (0,2)$. □

From the above lemma and results of this section we now obtain:

Theorem 8. Assume that conditions **A1, A2, A6** are satisfied. Then for each fixed $\lambda > 0$ problem (1) has at least one classical solution. If we assume additionally **A4-A5** then for each fixed $\lambda > 0$ problem (1) has exactly one classical solution. Moreover, for each fixed $\lambda > 0$ solutions to problems (1) and (7) coincide.

5 | MULTIPLICITY

Now we state the main multiplicity result.

Theorem 9. Assume that conditions **A1, A2, A6-A8** are satisfied. Then there is $\bar{\lambda} > 0$ such that for $\lambda > \bar{\lambda}$ problem (1) has at least three classical solutions.

Proof. By Theorem 8 we see that it suffices to consider multiple solvability of (7) instead of (1). We apply Theorem 3 to action functional $\mathcal{J}$ defined by (9) which is coercive by Lemma 1. We see that $E := H^2_0(0,1)$. Since all other assumptions are satisfied by Lemma 9, we see that we need to show that conditions **B1** and **B2** are satisfied. Using **A7** we see that there is a function $\overline{u} \in H^2_0(0,1)$ such that

$$\mathcal{A}_3(\overline{u}) < 0.$$

Since $\mathcal{A}_1(\bar{u}) + \mathcal{A}_2(\bar{u}) > 0$ we define

$$\bar{\lambda} := -\frac{\mathcal{A}_1(\bar{u}) + \mathcal{A}_2(\bar{u})}{\mathcal{A}_3(\overline{u})} > 0$$

and we fix any $\lambda > \bar{\lambda}$. Then for $\not{f} > \bar{\lambda}$ it holds that $\mathcal{J}(\bar{u}) < 0$. Reasoning similarly as with (5) we obtain that it holds for any u

$$\mathcal{A}_1(u) + \mathcal{A}_2(u) \geq \frac{1}{2}\left(1 - \frac{g_0}{\pi^2}\right)\|u\|^2_{H^2_0}.$$

Following the known technique applied in checking the mountain pass geometry we see from **A8** that for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|x| \leq \delta$ it holds

$$|F(t,x)| \leq \varepsilon\frac{|x|^2}{2} \text{ for a.e. } t \in [0,1].$$

Using the above we have that for $\varepsilon \in \left(0, \frac{\pi^2 - g_0}{\lambda}\right)$ there is a constant $r \leq \varepsilon$ such that for $\|u\|_{H^2_0} \leq r$ it holds

$$\mathcal{J}(u) \geq \frac{1}{2}\left(1 - \frac{g_0}{\pi^2} - \frac{\varepsilon\lambda}{\pi^2}\right)\|u\|^2_{H^2_0}.$$

Since $\mathcal{J}(0) = 0$ and since $\mathcal{J}(u) \geq 0$ for $u \in B_r$ we see that in fact

$$0 = \inf_{u\in B_r}\mathcal{J}(u) < \inf_{u\in S_r}\mathcal{J}(u).$$

Hence we obtain condition **B2.** Since $\mathcal{J}(\bar{u}) < 0$ we see that

$$\inf_{u\in H^2_0}\mathcal{J}(u) < \inf_{u\in B_r}\mathcal{J}(u)$$

Therefore condition **B1** is satisfied. The assertion now follows. □

Example 4. In the example of a nonlinear term satisfying our assumptions we drop the dependence on t. We put $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sqrt{|x|}, & x < -1 \\ -|x|\, x, & |x| \le 1 \\ -\sqrt{|x|}, & x > 1. \end{cases}$$

6 | DEPENDENCE ON PARAMETERS

We conclude this paper with results pertaining to dependence on functional parameters for weak solutions. Note that we do not include numerical parameter $\lambda > 0$ here in order to simplify the considerations. We will need the following definition: $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function if the following conditions are satisfied (the first two meaning it is Carathéodory):

(i). $t \mapsto f(t,x,y)$ is measurable on $[0,1]$ for each fixed pair $(x,y) \in \mathbb{R} \times \mathbb{R}$,

(ii). $(x,y) \mapsto f(t,x,y)$ is continuous on $\mathbb{R} \times \mathbb{R}$ for a.e. $t \in [0,1]$,

(iii). for each $d \in \mathbb{R}^+$ function

$$t \mapsto \max_{x^2+y^2 \le d^2} |f(t,x,y)|$$

belongs to $L^2(0,1)$.

The problem under consideration and the assumptions which employ now read:

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\right) = f(t,u(t),w(t)), \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{10}$$

where g satisfies **A1** and where f is subject to the following condition:

A9 $f : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *is an* L^2-*Carathéodory function satisfying for* $a \in L^\infty\left(0,1;\mathbb{R}_+\right)$, $\|a\|_{L^\infty} < \pi^4 - \pi^2 g_0$, $b \in L^1(0,1)$ *the following growth condition*

$$f(t,x,y)x \le a(t)\,|x|^2 + b(t) \text{ for a.e } t \in [0,1] \text{ and all } x,y \in \mathbb{R}.$$

Let $M > 0$ be fixed. The parameter is as follows

$$w \in L_M := \left\{ w \in L^2(0,1) : |w(t)| \le M \text{ for a.e } t \in [0,1] \right\}.$$

The continuous dependence on parameters is understood as follows: given a convergent sequence of parameters $(w_n) \subset L_M$ with a limit w_0 there exists a corresponding bounded sequence $(u_n) \subset H_0^2(0,1)$ of solutions to (10) such that any of its weakly convergent subsequences converges weakly in $H_0^2(0,1)$ to a solution of (10) with $w = w_0$. Note that we do not need uniqueness here. In case the solution were unique, we could apply the abstract results from[3] to reach the assertion and this is why we do not pursue this further. Instead we modify the method indicated in[6] which works also in case when the solutions are not unique. As is the case with problem (1) we see that here as well due to the coercivity which is uniform with respect to the parameter we introduce the auxiliary problem and next work with it in order to reach our assertion. We obtain the following counter part of Lemma 5:

Lemma 13. Assume that conditions **A1, A9** are satisfied. Then there is some $R > 0$ such that $\|u\|_{H_0^2} \le R$ and $\|\dot{u}\|_C \le R$ for every $u \in H_0^2(0,1)$ that solves problem (10) with any fixed $w \in L_M$.

Using Lemma 13 we define g_R by formula (6) and introduce the following auxiliary problem

$$\begin{cases} \frac{d^4}{dt^4}u(t) - \frac{d}{dt}\left(g_R\left(t, \left|\frac{d}{dt}u(t)\right|\right)\frac{d}{dt}u(t)\right) = f(t,u(t),w(t)), \text{ for a.e. } t \in (0,1), \\ u(0) = u(1) = 0, \ \dot{u}(0) = \dot{u}(1) = 0, \end{cases} \tag{11}$$

for which we have the following existence result which is obtained exactly as was for the case of problem (7):

Lemma 14. Assume that conditions **A1, A9** are satisfied. Then for each fixed $w \in L_M$ problem (11) has at least one classical solution which is also a solution to (10).

With the above preparations we can formulate the main result of this section:

Theorem 10. Assume that conditions **A1, A9** are satisfied. Let $(w_n) \subset L_M$ be a convergent sequence of parameters with a limit $w_0 \in L_M$. Then there is a constant $R > 0$ such that for any $n \in \mathbb{N}$ there is at least one classical solution u_n to (10) such that $\|u_n\|_{H_0^2} \leq R$. Moreover, there is a subsequence (u_{n_k}) of the sequence (u_n) such that $u_{n_k} \rightharpoonup u_0$ weakly in $H_0^2(0,1)$ and where u_0 is a solution to (10) corresponding to w_0.

Proof. From Lemma 14 we learn that we can work with problem (11) which is solvable for any fixed $n \in \mathbb{N}$. The existence of a constant $R > 0$ such that $\|u_n\|_{H_0^2} \leq R$ for all $n \in \mathbb{N}$ follows from Lemma 13. Hence the sequence (u_n) admits a weakly convergent subsequence (u_n) which we do not renumber for simplicity with a weak limit $u_0 \in H_0^2(0,1)$ and which can be chosen so that it converges strongly in $H_0^1(0,1)$ and also uniformly on $[0,1]$. Take a corresponding subsequence of parameters (w_n) which we again do not renumber. Using the definition of a weak solution of (11) we now obtain for any fixed $v \in H_0^2(0,1)$

$$\begin{aligned}&\int_0^1 \tfrac{d^2}{dt^2}u_n(t)\tfrac{d^2}{dt^2}v(t)\,dt + \int_0^1 g_R\left(t,\left|\tfrac{d}{dt}u_n(t)\right|\right)\tfrac{d}{dt}u_n(t)\tfrac{d}{dt}v(t)\,dt = \\ &\int_0^1 f(t,u_n(t),w_n(t))v(t)\,dt.\end{aligned} \tag{12}$$

Now we investigate the convergences in (12). Since (u_n) is weakly convergent in $H_0^2(0,1)$ we see that

$$\int_0^1 \frac{d^2}{dt^2}u_n(t)\frac{d^2}{dt^2}v(t)\,dt \to \int_0^1 \frac{d^2}{dt^2}u_0(t)\frac{d^2}{dt^2}v(t)\,dt.$$

Since (u_n) is norm convergent in $H_0^1(0,1)$ we see that

$$\begin{aligned}&\int_0^1 g_R\left(t,\left|\tfrac{d}{dt}u_n(t)\right|\right)\tfrac{d}{dt}u_n(t)\tfrac{d}{dt}v(t)\,dt \to \\ &\int_0^1 g_R\left(t,\left|\tfrac{d}{dt}u_0(t)\right|\right)\tfrac{d}{dt}u_0(t)\tfrac{d}{dt}v(t)\,dt.\end{aligned}$$

Due to assumption **A9** we can employ the Lebesgue Dominated Convergence Theorem in order to obtain that

$$\int_0^1 f(t,u_n(t),w_n(t))v(t)\,dt \to \int_0^1 f(t,u_0(t),w_0(t))v(t)\,dt.$$

Summarizing for any $v \in H_0^2(0,1)$ it holds

$$\begin{aligned}&\int_0^1 \tfrac{d^2}{dt^2}u_0(t)\tfrac{d^2}{dt^2}v(t)\,dt + \int_0^1 g_R\left(t,\left|\tfrac{d}{dt}u_0(t)\right|\right)\tfrac{d}{dt}u_0(t)\tfrac{d}{dt}v(t)\,dt = \\ &\int_0^1 f(t,u_0(t),w_0(t))v(t)\,dt.\end{aligned}$$

But this means that u_0 is a solution to (11) corresponding to w_0. Now the assertion of the theorem follows. □

We conclude with some suggestions how to tackle other conditions on a parameter.

Remark 1. If one assumes that the parameter is in $L^2(0,1)$ without the bound imposed in the definition of L_M then we need to impose some additional sublinear growth on f. If one assumes that the parameter is in $L^2(0,1)$ but that the sequence is merely weakly convergent, that some special structure on f must be imposed, namely the following

$$f(t,x,u) = f_1(t,x) + f_2(t,x)\,u$$

with f_1, f_2 being L^2-Carathéodory functions.

References

1. P. Amster, P.P. Cárdenas Alzate, *Existence of solutions for some nonlinear beam equations.* Port. Math. (N.S.) **63** (2006), 113–125.

2. G. Bonanno and B. Di Bella, *A boundary value problem for fourth-order elastic beam equation,* J. Math. Anal. Appl. **342** (2008), 1166-1176.

3. M. Bełdzinski, M. Galewski and I. Kossowski, *Dependence on parameters for nonlinear equations—Abstract principles and applications*, Math. Methods Appl. Sci. **45** (2022), (3) 1668–1686

4. J. Diblik, M. Galewski, V. Radulescu and Z. Smarda, *Multiplicity of solutions for nonlinear coercive problems*, submitted.

5. M. Galewski, *Basic Monotonicity Methods with Some Applications.* Compact Textbooks in Mathematics; Birkhäuser: Basel, Switzerland; SpringerNature: Basingstoke, UK, 2021; ISBN: 978-3-030-75308-5., 389–394.

6. U. Ledzewicz, H. Schättler and S. Walczak, *Optimal control systems governed by second-order ODEs with Dirichlet boundary data and variable parameters.* Ill. J. Math. **47**, (2003), 1189–1206.

7. J. Mawhin, *Problèmes de Dirichlet Variationnels non Linéaires, Les Presses de l'Université de Montréal*, Montréal, 1987.

8. D. Motreanu and E. Tornatore, *Nonhomogeneous degenerate quasilinear problems with convection*, Nonlinear Anal., Real World Appl. **71**, (2023), 103800

9. D. Motreanu, *Nonhomogeneous Dirichlet Problems with Unbounded Coefficient in the Principal Part*, Axioms 2022, 11(12), 739; https://doi.org/10.3390/axioms11120739.

10. B. Ricceri, *On a three critical points theorem*, Arch. Math. **75**, (2000), (3), 220-226.

11. J.L. Troutman, *Variational Calculus with Elementary Convexity*, Springer-Verlag New York Inc 1983.

12. S. Walczak, *On some generalization of the fundamental lemma and its applications to differential equations*, Bull. Soc. Math. Belgique, **45** (1993), Ser. B., 237-243.

13. X. Yang and K. Lo, *Existence of a positive solution to a fourth-order boundary value problem*, *Nonlinear Anal.* **69** (2008), 2267–2273.

14. Q. Yao, *Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right*, *Nonlinear Anal.* **69** (2008), 2267–2273.

$\square$