GENERALIZED INTEGRAL TYPE HILBERT OPERATOR ACTING BETWEEN WEIGHTED BLOCH SPACE

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Abstract

Let μ be a finite Borel measure on [0 ,1). In this paper, we consider the generalized integral type Hilbert operator I μ α + 1 (f) (z) = [?] 0 1 f (t) (1 - tz) α + 1 d μ (t) (α > - 1) . The operator I μ 1 has been extensively studied recently. The aim of this paper is to study the boundedness(resp. compactness) of I μ α + 1 acting from the normal weight Bloch space into another of the same kind. As consequences of our study, we get completely results for the boundedness of I μ α + 1 acting between Bloch type spaces, logarithmic Bloch spaces among others.

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Abstract. Let μ be a finite Borel measure on $[0,1)$. In this paper, we consider the generalized integral type Hilbert operator

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z)=\int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}}d\mu(t) \quad (\alpha>-1).$$

The operator $\mathcal{I}_{\mu_1}$ has been extensively studied recently. The aim of this paper is to study the boundedness(resp. compactness) of $\mathcal{I}_{\mu_{\alpha+1}}$ acting from the normal weight Bloch space into another of the same kind. As consequences of our study, we get completely results for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch type spaces, logarithmic Bloch spaces among others.

1. Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$.

A positive continuous function ν on $[0,1)$ is called normal if there exist $0<a\leqslant b<\infty$ and $0\leqslant s_0<1$ such that $\dfrac{\nu(s)}{(1-s^2)^a}$ almost decreasing on $[s_0,1)$ and $\dfrac{\nu(s)}{(1-s^2)^b}$ almost increasing on $[s_0,1)$.

A function g is almost increasing if there exists $C>0$ such that $r_1<r_2$ implies $g(r_1)\leqslant Cg(r_2)$. Almost decreasing functions are defined in an analogous manner.

Functions such as

$$\nu(s)=(1-s^2)^t\log^\delta\frac{e}{1-s^2}(t>0,\delta\in\mathbb{R}) \text{ and } \nu(s)=\left(\sum_{k=1}^{\infty}\frac{ks^{2k-2}}{\log^3(k+1)}\right)^{-1}$$

are normal functions.

In this paper, we use $\mathcal{N}$ to denote the set of all normal functions on $[0,1)$ and let $s_0=0$. The letters a and b always represent the parameters in the definition of normal function.

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Let $\nu \in \mathcal{N}$, the normal weight Bloch space $\mathcal{B}_\nu$ consists of those functions $f \in H(\mathbb{D})$ for which

$$||f||_{\mathcal{B}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} \nu(|z|)|f'(z)| < \infty.$$

In particular, if $\nu(|z|) = (1-|z|^2)^\gamma (\gamma > 0)$, then $\mathcal{B}_\nu$ is the Bloch type space $\mathcal{B}^\gamma$. If $\nu(|z|) = (1-|z|^2)\log^{-\beta}\frac{e}{1-|z|^2} (\beta \in \mathbb{R})$, then $\mathcal{B}_\nu$ is just the logarithmic Bloch space $\mathcal{B}_{\log^\beta}$.

Let μ be a positive Borel measure on $[0,1)$, $0 \leqslant \gamma < \infty$ and $0 < s < \infty$. Then μ is a γ-logarithmic s-Carleson measure if there exists a positive constant C, such that

$$\mu([t,1))\log^\gamma \frac{e}{1-t} \leqslant C(1-t)^s, \quad \text{for all } 0 \leqslant t < 1.$$

In particular, μ is an s-Carleson measure if $\gamma = 0$. See [1] for more about logarithmic Carleson measure.

Let μ be a finite Borel measure on $[0,1)$ and $n \in \mathbb{N}$. We use μ_n to denote the sequence of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. Let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ induces an operator on $H(\mathbb{D})$ by its action on the Taylor coefficients : $a_n \to \sum_{k=0}^{\infty} \mu_{n,k} a_k, \quad n = 0, 1, 2, \cdots.$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, the generalized Hilbert operator defined as follows:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

It's known that the generalized Hilbert operator $\mathcal{H}_\mu$ is closely related to the integral operator

$$\mathcal{I}_\mu(f)(z) = \int_0^1 \frac{f(t)}{1-tz} d\mu(t)$$

If μ is the Lebesgue measure on $[0,1)$, then $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ reduce to the classic Hilbert operator $\mathcal{H}$ and $\mathcal{I}$.

The action of the operators $\mathcal{I}_\mu$ and $\mathcal{H}_\mu$ on distinct spaces of analytic functions have been studied in a number of articles (see, e.g., [2–8]). In this paper, we consider the generalized integral type Hilbert operator

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t), \quad (\alpha > -1).$$

If $\alpha = 0$, the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is just $\mathcal{I}_\mu$. The integral type operator $\mathcal{I}_{\mu_{\alpha+1}}$ is closely related to the Hilbert type operator

$$\mathcal{H}_\mu^\alpha(f)(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad (\alpha > -1),$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. The operator $\mathcal{H}_\mu^\alpha$ can be regarded as the fractional derivative of $\mathcal{H}_\mu$. If $\alpha = 1$, then $\mathcal{H}_\mu^\alpha$ called the Derivative-Hilbert operator which has been studied in [9, 10].

The connection between $\mathcal{I}_\mu$(or $\mathcal{H}_\mu$) and $\mathcal{I}_{\mu_{\alpha+1}}$(or $\mathcal{H}_\mu^\alpha$) motivates us to consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ in a unified manner. In [11](see also [5]), the authors have studied the boundedness of $\mathcal{I}_\mu$ acting on $\mathcal{B}$. Li and Zhou studied the operator $\mathcal{H}_\mu$ from Bloch type spaces to the BMOA and the Bloch space in [12]. Ye and Zhou investigated $\mathcal{I}_{\mu_2}$ acting on $\mathcal{B}$ in [9] and $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch-type space in [13]. But only partial results were obtained for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch-type spaces. The aim of this article is to deal with the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting from normal weight Bloch space into another of the same kind. As consequences of our study, we obtain complete results for the boundedness of $\mathcal{I}_{\mu_{\alpha+1}}$ acting between Bloch type spaces, logarithmic Bloch spaces among others.

Throughout the paper, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation "$P \lesssim Q$" if there exists a constant $C = C(\cdot)$ such that "$P \leqslant CQ$", and "$P \gtrsim Q$" is understood in an analogous manner. In particular, if "$P \lesssim Q$" and "$P \gtrsim Q$" , then we will write "$P \asymp Q$".

2. **Preliminary Results**

In [14], a sequence $\{V_n\}$ was constructed in the following way: Let ψ be a C^∞-function on $\mathbb{R}$ such that (1) $\psi(s) = 1$ for $s \leqslant 1$, (2) $\psi(s) = 0$ for $s \geqslant 2$, (3) ψ is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(s) = \psi(\frac{s}{2}) - \psi(s)$, and let $v_0 = 1 + z$, for $n \geqslant 1$,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi(\frac{k}{2^{n-1}}) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi(\frac{k}{2^{n-1}}) z^k.$$

The polynomials V_n have the properties:

(1) $f(z) = \sum_{n=0}^{\infty} V_n * g(z)$, for $f \in H(\mathbb{D})$;

(2) $||V_n * f||_p \lesssim ||f||_p$, for $f \in H^p, p > 0$;

(3) $||V_n||_p \asymp 2^{n(1-\frac{1}{p})}$, for all $p > 0$, where $*$ denotes the Hadamard product and $||\cdot||_p$ denotes the norm of Hardy space H^p.

Lemma 2.1. *Let $\nu \in \mathcal{N}$ and $f \in H(\mathbb{D})$, then $f \in \mathcal{B}_\nu$ if and only if*

$$\sup_{n \geqslant 0} \nu(1 - 2^{-n}) 2^n ||V_n * f||_\infty < \infty.$$

Moreover,

$$||f||_{\mathcal{B}_\nu} \asymp \sup_{n \geqslant 0} \nu(1 - 2^{-n}) 2^n ||V_n * f||_\infty.$$

The proof of this Lemma is similar to that Theorem 3.1 in [15], we leave it to the interested readers.

Lemma 2.2. *Let $\nu \in \mathcal{N}$ and*

$$g(\zeta) = 1 + \sum_{s=1}^{\infty} 2^s \zeta^{n_s} \quad (\zeta \in \mathbb{D}),$$

where n_s is the integer part of $(1-r_s)^{-1}$, $r_0 = 0$, $\nu(r_s) = 2^{-s}(s = 1, 2, \cdots)$. Then $g(r)$ is strictly increasing on $[0,1)$ and there exist two positive constants N_1 and N_2 such that

$$\inf_{[0,1)} \nu(r)g(r) = N_1 > 0, \quad \sup_{\zeta \in \mathbb{D}} \nu(|\zeta|)|g(\zeta)| = N_2 < +\infty.$$

This result is originated from Theorem 1 in [16].

Lemma 2.3. *If $\nu \in \mathcal{N}$, then*

$$\frac{\nu(|z|)}{\nu(|w|)} \lesssim \left(\frac{1-|z|^2}{1-|w|^2}\right)^a + \left(\frac{1-|z|^2}{1-|w|^2}\right)^b \quad \textit{for all} \quad z, w \in \mathbb{D}.$$

This result comes from Lemma 2.2 in [17].

Lemma 2.4. *Let $\nu \in \mathcal{N}$, $0 < \delta < \frac{1}{e^2}$, then*

$$\int_e^\infty \frac{e^{-\delta t} dt}{t\nu(1-\frac{1}{t})} \lesssim \frac{1}{\nu(1-\delta)}.$$

Proof.

$$\int_e^\infty \frac{e^{-\delta t} dt}{t\nu(1-\frac{1}{t})} = \int_e^{\frac{1}{\delta}} \frac{e^{-\delta t} dt}{t\nu(1-\frac{1}{t})} + \int_{\frac{1}{\delta}}^\infty \frac{e^{-\delta t} dt}{t\nu(1-\frac{1}{t})} = I_1 + I_2.$$

By the definition of normal function, we have

$$I_1 \leqslant \int_e^{\frac{1}{\delta}} \frac{dt}{t\nu(1-\frac{1}{t})} \lesssim \frac{\delta^a}{\nu(1-\delta)} \int_e^{\frac{1}{\delta}} t^{a-1} dt \lesssim \frac{1}{\nu(1-\delta)}.$$

If $t > \frac{1}{\delta}$, then $1 - \frac{1}{t} > 1 - \delta$. The definition of normal function shows that

$$\frac{\nu(1-\delta)}{[1-(1-\delta)]^b} \lesssim \frac{\nu(1-\frac{1}{t})}{[1-(1-\frac{1}{t})]^b}.$$

Hence, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{\delta}}^\infty \frac{\nu(1-\delta)}{\nu(1-\frac{1}{t})} \frac{e^{-\delta t} dt}{t\nu(1-\delta)} \\ &\lesssim \int_{\frac{1}{\delta}}^\infty \frac{\delta^b t^{b-1} e^{-\delta t}}{\nu(1-\delta)} dt = \frac{1}{\nu(1-\delta)} \int_1^\infty e^{-s} s^{b-1} ds \\ &\lesssim \frac{1}{\nu(1-\delta)}. \end{aligned}$$

The proof is complete. $\square$

Lemma 2.5. *Let μ be a positive Borel measure on $[0,1)$, $\beta > 0$, $\gamma > 0$. Let τ be the Borel measure on $[0,1)$ defined by*

$$d\tau(t) = \frac{d\mu(t)}{(1-t)^{\gamma}}.$$

Then, the following two conditions are equivalent.
(a) τ is a β-Carleson measure.
(b) μ is a $\beta+\gamma$-Carleson measure.

Proof. $(a) \Rightarrow (b)$. Assume (a). Then there exists a positive constant $C > 0$ such that

$$\int_t^1 \frac{d\mu(r)}{(1-r)^{\gamma}} \leqslant C(1-t)^{\beta}, \quad t \in [0,1).$$

Using this and the fact that the function $x \to \frac{1}{(1-x)^{\gamma}}$ is increasing in $[0,1)$, we obtain

$$\frac{\mu([t,1))}{(1-t)^{\gamma}} \leqslant \int_t^1 \frac{d\mu(r)}{(1-r)^{\gamma}} \leqslant C(1-t)^{\beta}, \quad t \in [0,1).$$

This shows that μ is a $\beta+\gamma$-Carleson measure.

$(b) \Rightarrow (a)$. Assume (b). Then there exists a positive constant $C > 0$ such that

$$\mu(t) \leqslant C(1-t)^{\beta+\gamma}, \quad t \in [0,1).$$

For $0 < x < 1$, let $h(x) = \mu([0,x)) - \mu([0,1)) = -\mu([x,1))$. Integrating by parts and using the inequality above , we obtain

$$\begin{aligned}
\tau([t,1)) &= \int_t^1 \frac{d\mu(x)}{(1-x)^{\gamma}} \\
&= \frac{1}{(1-t)^{\gamma}}\mu([t,1)) - \lim_{x\to 1} \frac{1}{(1-x)^{\gamma}}\mu([x,1)) + \gamma \int_t^1 \frac{\mu([x,1))}{(1-x)^{\gamma+1}} dx \\
&= \frac{1}{(1-t)^{\gamma}}\mu([t,1)) + \gamma \int_t^1 \frac{\mu([x,1))}{(1-x)^{\gamma+1}} dx \\
&\lesssim (1-t)^{\beta} + \int_t^1 (1-x)^{\beta-1} dx \lesssim (1-t)^{\beta}.
\end{aligned}$$

Thus, τ is an β-Carleson measure. □

Lemma 2.6. *Let $\omega, \nu \in \mathcal{N}$. If T is a bounded operator from $\mathcal{B}_\omega$ into $\mathcal{B}_\nu$, then T is compact operator from $\mathcal{B}_\omega$ into $\mathcal{B}_\nu$ if and only if for any bounded sequence $\{h_n\}$ in $\mathcal{B}_\omega$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have* $\lim_{n\to\infty} ||T(h_n)||_{\mathcal{B}_\nu} = 0$.

The proof is similar to that of Proposition 3.11 in [18], we omit the details.

3. **Nonnegative Coefficients of normal weight Bloch functions**

First, we give a characterization of the functions $f \in H(\mathbb{D})$ whose sequence of Taylor coefficients is non-negative which belongs to $\mathcal{B}_\nu$.

Theorem 3.1. *Let $\nu \in \mathcal{N}$ and $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \geqslant 0$ for all $n \geqslant 0$. Then $f \in \mathcal{B}_\nu$ if and only if*

$$S(f) := \sup_{n \geqslant 1} \nu(1 - \frac{1}{n}) \sum_{k=1}^{n} k a_k < \infty.$$

Moreover,

$$||f||_{\mathcal{B}_\nu} \asymp S(f) + a_0.$$

Proof. If $f \in \mathcal{B}_\nu$, then for each $n \in \mathbb{N}$,

$$\begin{aligned} ||f||_{\mathcal{B}_\nu} &\geqslant \sup_{z = 1 - \frac{1}{n}} \nu(|z|)|f'(z)| \\ &\geqslant \nu(1 - \frac{1}{n}) \left| \sum_{k=1}^{\infty} k a_k (1 - \frac{1}{n})^{k-1} \right| \\ &\gtrsim \nu(1 - \frac{1}{n}) \sum_{k=1}^{n} k a_k, \end{aligned}$$

and hence $S(f) \lesssim ||f||_{\mathcal{B}_\nu}$. Since $a_0 = |f(0)| \leqslant ||f||_{\mathcal{B}_\nu}$, we may obtain

$$S(f) + a_0 \lesssim ||f||_{\mathcal{B}_\nu}.$$

On the other hand, if $S(f) < \infty$, then

$$\nu(1 - 2^{-j}) \sum_{k=2^j}^{2^{j+1}-1} k a_k \lesssim S(f), \quad j \in \mathbb{N}.$$

For each $z \in \mathbb{D}$ with $\frac{1}{2} \leqslant |z| < 1$, we have

$$\begin{aligned} |f'(z)| = \left| \sum_{j=0}^{\infty} \sum_{k=2^j}^{2^{j+1}-1} k a_k z^{k-1} \right| &\leqslant \sum_{j=0}^{\infty} \left(\sum_{k=2^j}^{2^{j+1}-1} k a_k |z|^{k-1} \right) \\ &\lesssim S(f) \sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu(1 - 2^{-j})}. \end{aligned}$$

To finish the proof, it suffices to prove that

$$\sum_{j=0}^{\infty} \frac{|z|^{2^j}}{\nu(1 - 2^{-j})} \lesssim \frac{1}{\nu(|z|)} \quad \text{for all } \frac{1}{2} \leqslant |z| < 1. \tag{3.1}$$

For each $\frac{1}{2} \leqslant |z| = r < 1$, by choosing $m \geqslant 2$ such that $r_{m-1} \leqslant r \leqslant r_m$, where $r_m = 1 - 2^{-m}$. Then

$$\sum_{j=0}^{\infty} \nu^{-1}(1-2^{-j})r^{2^j} \leqslant \sum_{j=0}^{m} \nu^{-1}(1-2^{-j}) + \sum_{j=m+1}^{\infty} \nu^{-1}(1-2^{-j})r^{2^j} = S_1 + S_2.$$

Using Lemma 2.3 we have

$$\begin{aligned} S_1 &\lesssim \nu^{-1}(1-2^{-m}) \sum_{j=0}^{m} \left((\frac{1}{2})^{(m-j)a} + (\frac{1}{2})^{(m-j)b} \right) \\ &\lesssim \nu^{-1}(1-2^{-m}). \end{aligned}$$

On the other hand,

$$\begin{aligned} S_2 = \sum_{j=m+1}^{\infty} \nu^{-1}(1-2^{-j})r^{2^j} &\leqslant \sum_{j=m+1}^{\infty} \nu^{-1}(1-2^{-j})r_m^{2^m \cdot 2^{j-m}} \\ &\leqslant \sum_{j=m+1}^{\infty} \nu^{-1}(1-2^{-j})e^{-2^{(j-m)}} = \sum_{l=1}^{\infty} \nu^{-1}(1-2^{-(l+m)})e^{-2^l} \\ &\lesssim \nu^{-1}(1-2^{-m}) \sum_{l=1}^{\infty} e^{-2^l} 2^{lb} \lesssim \nu^{-1}(1-2^{-m}). \end{aligned}$$

Since $\nu^{-1}(1-2^{-m}) \asymp \nu^{-1}(r)$, it follows that (3.1) is valid for all $\frac{1}{2} \leqslant |z| < 1$.

Therefore,

$$|f(0)| + \sup_{z \in \mathbb{D}} \nu(|z|)|f'(z)| \lesssim a_0 + S(f).$$

The proof is complete. $\square$

Corollary 3.2. *Let $\gamma > 0$ and $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \geqslant 0$ for all $n \geqslant 0$. Then $f \in \mathcal{B}^\gamma$ if and only if*

$$\sup_{n \geqslant 1} n^{-\gamma} \sum_{k=1}^{n} k a_k < \infty.$$

If $f \in \mathcal{B}_\nu$ has nonnegative and non-increasing coefficients, then the result of Theorem 3.1 can be state as follows.

Theorem 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ with a_n nonnegative and non-increasing. Then $f \in \mathcal{B}_\nu$ if and only if*

$$\sup_{n \geqslant 1} n^2 \nu(1 - \frac{1}{n}) a_n < \infty.$$

Moreover,

$$||f||_{\mathcal{B}_\nu} \asymp a_0 + \sup_{n \geqslant 1} n^2 \nu(1 - \frac{1}{n}) a_n.$$

Proof. If a_n nonnegative and non-increasing, then $\sum_{k=1}^n ka_k \gtrsim n^2 a_n$. The proof of the necessity follows from Theorem 3.1 immediately.

On the other hand, if $M := \sup_{n\geqslant 1} n^2\nu(1-\frac{1}{n})a_n < \infty$, then

$$a_n \lesssim \frac{M}{n^2\nu(1-\frac{1}{n})} \quad \text{for all } n \geqslant 1.$$

For every $z \in \mathbb{D}$ and $\frac{1}{2} < |z| < 1$,

$$|f'(z)| \leqslant \sum_{n=1}^{\infty} na_n|z|^{n-1} \lesssim M \sum_{n=1}^{\infty} \frac{|z|^n}{n\nu(1-\frac{1}{n})}.$$

Let

$$h_x(t) = \frac{x^t}{t\nu(1-\frac{1}{t})} \quad x \in (0,1),$$

then h_x is decreasing in t, for sufficiently large t and each $x \in (0,1)$. So, by Lemma 2.4 we have

$$\sum_{n=1}^{\infty} \frac{|z|^n}{n\nu(1-\frac{1}{n})} \asymp \int_e^{\infty} \frac{e^{-t\log\frac{1}{|z|}}}{t\nu(1-\frac{1}{t})}dt \lesssim \frac{1}{\nu(1-\log\frac{1}{|z|})} \asymp \frac{1}{\nu(|z|)}.$$

This means that

$$||f||_{\mathcal{B}_\nu} \lesssim a_0 + \sup_{n\geqslant 1} n^2\nu(1-\frac{1}{n})a_n.$$

The proof is complete. □

Corollary 3.4. *Let $\gamma > 0$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ with a_n nonnegative and non-increasing. Then $f \in \mathcal{B}^\gamma$ if and only if*

$$\sup_{n\geqslant 1} n^{2-\gamma} a_n < \infty.$$

4. Generalized integral type Hilbert operator acting on weighted Bloch space

Let $\omega \in \mathcal{N}$, we write $\widetilde{\omega}(t) = \int_0^t \frac{1}{\omega(s)}ds$. We begin with characterizing those measure μ for which the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is well defined on $\mathcal{B}_\omega$.

Proposition 4.1. *Let μ be a positive Borel measure on $[0,1)$ and $\alpha > -1$. For any given $f \in \mathcal{B}_\omega$, $\mathcal{I}_{\mu_{\alpha+1}}(f)$ uniformly converges on any compact subset of $\mathbb{D}$ if and only if*

$$\int_0^1 (\widetilde{\omega}(t)+1)d\mu(t) < \infty. \tag{4.1}$$

Proof. Let $f \in \mathcal{B}_\omega$, it is easy to verify that

$$|f(z)| \lesssim (\widetilde{\omega}(|z|)+1)||f||_{\mathcal{B}_\omega} \quad \text{for all} \quad z \in \mathbb{D}. \tag{4.2}$$

If (4.1) holds, then for each $0 < r < 1$ and $z \in \mathbb{D}$ with $|z| \leqslant r$, we have

$$\begin{aligned}|\mathcal{I}_{\mu_{\alpha+1}}(f)(z)| &\leqslant \int_0^1 \frac{|f(t)|}{|1-tz|^{\alpha+1}} d\mu(t) \\ &\lesssim \frac{||f||_{\mathcal{B}_\omega}}{(1-r)^{\alpha+1}} \int_0^1 (\widetilde{\omega}(t)+1) d\mu(t) \\ &\lesssim \frac{||f||_{\mathcal{B}_\omega}}{(1-r)^{\alpha+1}}.\end{aligned}$$

This implies that $\mathcal{I}_{\mu_{\alpha+1}}(f)$ uniformly converges on any compact subset of $\mathbb{D}$ and hence analytic in $\mathbb{D}$.

Suppose that the operator $\mathcal{I}_{\mu_{\alpha+1}}$ is well defined in $\mathcal{B}_\omega$. Considering the function

$$f(z) = \int_0^z g(s) ds + 1$$

where g is the function in Lemma 2.2 with respect to ω. Then Lemma 2.2 implies that $f \in \mathcal{B}_\omega$. Since $\mathcal{I}_{\mu_{\alpha+1}}(f)(z)$ is well defined for every $z \in \mathbb{D}$, we have

$$|I_{\mu_{\alpha+1}}(f)(0)| = \left| \int_0^1 f(t) d\mu(t) \right| < \infty.$$

Since μ is a positive measure and $g(s) > 0$ for all $s \in [0, 1)$, it follows from Lemma 2.2 that

$$f(t) = \int_0^t g(s) ds + 1 \asymp \widetilde{\omega}(t) + 1. \tag{4.3}$$

Therefore,

$$\int_0^1 (\widetilde{\omega}(t) + 1) d\mu(t) < \infty.$$

The proof is complete. $\square$

The sublinear generalized integral type Hilbert operator $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}$ defined by

$$\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{|f(t)|}{(1-tz)^{\alpha+1}} d\mu(t), \quad (\alpha > -1).$$

It is obvious that Proposition 4.1 is remain valid if $\mathcal{I}_{\mu_{\alpha+1}}$ is replaced by $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}$. By mean of Lemma 2.1, Theorem 3.1 and the sublinear integral type Hilbert operator $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}$, we have the following results.

Theorem 4.2. *Let $\omega, \nu \in \mathcal{N}$ and $\alpha > -1$. Suppose μ is a positive Borel measure on $[0, 1)$ and satisfies (4.1). Then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded;
(b) $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded;
(c) $\sup_{n \geqslant 1} n^{\alpha+2} \nu(1 - \frac{1}{n}) \int_0^1 t^n (\widetilde{\omega}(t) + 1) d\mu(t) < \infty$.

Proof. $(a) \Rightarrow (c)$: If $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded. For each $f \in \mathcal{B}_\omega$, Proposition 4.1 implies that $\mathcal{I}_{\mu_{\alpha+1}}(f)$ converges absolutely for every $z \in \mathbb{D}$ and

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n f(t) d\mu(t) \right) z^n, \quad z \in \mathbb{D}.$$

Take

$$f(z) = \int_0^z g(s)ds + 1,$$

where g is the function in Lemma 2.2 with respect to ω. Then $f \in \mathcal{B}_\omega$ and

$$\mathcal{I}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} b_n z^n$$

where

$$b_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n \left(\int_0^t g(s)ds + 1 \right) d\mu(t).$$

It is clear that $\{b_n\}_{n=1}^{\infty}$ is a nonnegative sequence. Using Theorem 3.1, (4.3) and Stirling's formula we have

$$\begin{aligned}
||\mathcal{I}_{\mu_{\alpha+1}}(f)||_{\mathcal{B}_\nu} &\gtrsim \sup_{n \geqslant 1} \nu(1-\frac{1}{n}) \sum_{k=1}^{n} k b_k \\
&\gtrsim \sup_{n \geqslant 1} \nu(1-\frac{1}{n}) \int_0^1 t^n \left(\widetilde{\omega}(t) + 1\right) d\mu(t) \sum_{k=1}^{n} k^{\alpha+1} \\
&\asymp \sup_{n \geqslant 1} n^{\alpha+2} \nu(1-\frac{1}{n}) \int_0^1 t^n (\widetilde{\omega}(t) + 1) d\mu(t).
\end{aligned}$$

Therefore,

$$\sup_{n \geqslant 1} n^{\alpha+2} \nu(1-\frac{1}{n}) \int_0^1 t^n (\widetilde{\omega}(t) + 1) d\mu(t) < \infty.$$

$(c) \Rightarrow (b)$: Assume (c). Then for each $n \in \mathbb{N}$, we have

$$\int_0^1 t^n (\widetilde{\omega}(t) + 1) d\mu(t) \lesssim \frac{1}{n^{\alpha+2} \nu(1-\frac{1}{n})}. \tag{4.4}$$

For a given $0 \not\equiv f \in \mathcal{B}_\omega$,

$$\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)(z) = \int_0^1 \frac{|f(t)|}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n |f(t)| d\mu(t).$$

Obviously, $\{c_n\}_{n=1}^{\infty}$ is a nonnegative sequence. Using (4.2), (4.4), and the definition of normal weight, we deduce that

$$\begin{aligned}
&|c_0| + \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{k=1}^{n} kc_k \\
&\lesssim ||f||_{\mathcal{B}_\omega} + ||f||_{\mathcal{B}_\omega} \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{k=1}^{n} (k+1)^{\alpha+1} \int_0^1 t^k(\widetilde{\omega}(t)+1)d\mu(t) \\
&\lesssim ||f||_{\mathcal{B}_\omega} + ||f||_{\mathcal{B}_\omega} \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{k=1}^{n} \frac{1}{k\nu(1-\frac{1}{k})} \\
&\lesssim ||f||_{\mathcal{B}_\omega} + ||f||_{\mathcal{B}_\omega} \sup_{n\geqslant 1} \frac{1}{(n+1)^a} \sum_{k=1}^{n} (k+1)^{a-1} \\
&\lesssim ||f||_{\mathcal{B}_\omega}.
\end{aligned}$$

Hence $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded by Theorem 3.1.

$(b) \Rightarrow (a)$: If $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded, then for each $f \in \mathcal{B}_\omega$, by Lemma 2.1 we have

$$\sup_{n\geqslant 1} \nu(1-2^{-n})2^n ||V_n * \widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)||_\infty \asymp ||\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)||_{\mathcal{B}_\nu} \lesssim ||f||_{\mathcal{B}_\omega} ||\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}||.$$

Since the coefficients of $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)$ are non-negative, it is easy to check that

$$M_\infty(r, V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)) \leqslant M_\infty(r, V_n * \widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)) \ \text{ for all } 0 < r < 1.$$

Therefore,

$$||V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)||_\infty = \sup_{0<r<1} M_\infty(r, V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)) \leqslant ||V_n * \widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f)||_\infty.$$

Consequently,

$$||\mathcal{I}_{\mu_{\alpha+1}}(f)||_{\mathcal{B}_\nu} \asymp \sup_{n\geqslant 1} \nu(1-2^{-n})2^n ||V_n * \mathcal{I}_{\mu_{\alpha+1}}(f)||_\infty \lesssim ||f||_{\mathcal{B}_\omega} ||\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}||.$$

This implies that $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is bounded. □

Theorem 4.3. *Let $\omega, \nu \in \mathcal{N}$ and $\alpha > -1$. Suppose μ is a finite positive Borel measure on $[0,1)$ and $\widetilde{\omega}(1) < \infty$. Then the following statements are equivalent.*

(a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ *is bounded;*
(b) $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ *is bounded;*
(c) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ *is compact;*
(d) $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ *is compact;*
(e) $\sup_{n\geqslant 1} n^{\alpha+2} \nu(1-\frac{1}{n})\mu_n < \infty.$

Proof. The equivalence of $(a) \Leftrightarrow (b) \Leftrightarrow (e)$ follows from Theorem 4.2 immediately and the implications of $(d) \Rightarrow (c) \Rightarrow (a)$ are obvious. Therefore, we only need to prove that $(e) \Rightarrow (d)$.

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $\mathcal{B}_\omega$ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Since $\widetilde{\omega}(1) < \infty$, arguing as the proof of Lemma 2.5 in [19], we have that

$$\lim_{k\to\infty} \sup_{z\in\mathbb{D}} |f_k(z)| = 0.$$

For each $k \in \mathbb{N}$, we have

$$\widetilde{\mathcal{I}}_{\mu_{\alpha+1}}(f_k)(z) = \int_0^1 \frac{|f_k(t)|}{(1-tz)^{\alpha+1}} d\mu(t) = \sum_{n=0}^{\infty} c_{n,k} z^n,$$

where

$$c_{n,k} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)} \int_0^1 t^n |f_k(t)| d\mu(t).$$

It is obvious that $\{c_{n,k}\}_{n=1}^{\infty}$ is a nonnegative sequence for each $k \in \mathbb{N}$. To prove that $\widetilde{\mathcal{I}}_{\mu_{\alpha+1}} : \mathcal{B}_\omega \to \mathcal{B}_\nu$ is compact, it is sufficient to prove that

$$\lim_{k\to\infty} \left(c_{0,k} + \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{j=1}^{n} j c_{j,k} \right) = 0$$

by using Theorem 3.1 and Lemma 2.6. If $\sup_{n\geqslant 1} n^{\alpha+2}\nu(1-\frac{1}{n})\mu_n < \infty$, then

$$\mu_n \lesssim \frac{1}{n^{\alpha+2}\nu(1-\frac{1}{n})} \text{ for all } n \in \mathbb{N}.$$

By Stirling's formula and the above inequality, we have

$$\begin{aligned}
&|c_{0,k}| + \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{j=1}^{n} j c_{j,k} \\
\lesssim& \int_0^1 |f_k(t)| d\mu(t) + \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{j=1}^{n} j^{\alpha+1} \int_0^1 t^j |f_k(t)| d\mu(t) \\
\lesssim& \sup_{t\in[0,1)} |f_k(t)| + \sup_{t\in[0,1)} |f_k(t)| \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{j=1}^{n} j^{\alpha+1}\mu_j \\
\lesssim& \sup_{t\in[0,1)} |f_k(t)| + \sup_{t\in[0,1)} |f_k(t)| \sup_{n\geqslant 1} \nu(1-\frac{1}{n}) \sum_{j=1}^{n} \frac{1}{j\nu(1-\frac{1}{j})} \\
\lesssim& \sup_{t\in[0,1)} |f_k(t)| \to 0, \quad (k\to\infty).
\end{aligned}$$

Hence (d) holds. $\square$

5. **Some Applications**

As a direct application of the above results, we first consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting from $\mathcal{B}^\beta$ to $\mathcal{B}^\gamma$. If $\gamma \geqslant \alpha+2$, then it is easy to see that $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}^\beta \to \mathcal{B}^\gamma$ is always a bounded operator under the condition (4.1). Therefore, we only need to consider the case $0<\gamma<\alpha+2$.

Corollary 5.1. *Let μ be a positive Borel measure on $[0,1)$ and satisfies $\int_0^1 \log\frac{e}{1-t}d\mu(t)<\infty$, $\alpha>-1$. If $0<\gamma<\alpha+2$, then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}\to\mathcal{B}^\gamma$ is bounded;
(b) μ is a 1-logarithmic $\alpha+2-\gamma$-Carleson measure;
(c) $\displaystyle\int_0^1 t^n\log\frac{e}{1-t}d\mu(t)=O(\frac{1}{n^{\alpha+2-\gamma}})$.

Proof. Let $d\lambda(t)=\log\frac{e}{1-t}d\mu(t)$, then Lemma 2.5 in [11] shows that μ is a 1-logarithmic $\alpha+2-\gamma$-Carleson measure if and only if λ is an $\alpha+2-\gamma$-Carleson measure. By Theorem 2.1 in [20], λ is an $\alpha+2-\gamma$-Carleson measure if and only if

$$\int_0^1 t^n d\lambda(t)=O(\frac{1}{n^{\alpha+2-\gamma}}).$$

The desired result follows from Theorem 4.2 immediately. □

Remark 5.2. *If $\gamma=1$ and $\alpha=0$, the result of Theorem 5.1 have been obtained in [11](or [5]). In addition, if $\gamma=1$ and $\alpha=1$, the result have been given in [9].*

Corollary 5.3. *Let μ be a positive Borel measure on $[0,1)$ and satisfies $\int_0^1\frac{d\mu(t)}{(1-t)^{\beta-1}}<\infty$, $\alpha>-1$. If $0<\gamma<\alpha+2$ and $\beta>1$, then $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}^\beta\to\mathcal{B}^\gamma$ is bounded if and only if μ is an $\alpha+1+\beta-\gamma$-Carleson measure.*

Proof. It follows from Theorem 4.2 that $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}^\beta\to\mathcal{B}^\gamma$ is bounded if and only if

$$\int_0^1 t^n\frac{d\mu(t)}{(1-t)^{\beta-1}}=O(\frac{1}{n^{\alpha+2-\gamma}}).$$

This is equivalent to saying that $\frac{d\mu(t)}{(1-t)^{\beta-1}}$ is an $\alpha+2-\gamma$-Carleson measure. The proof can be done by using Lemma 2.5. □

Corollary 5.4. *Let μ be a finite positive Borel measure on $[0,1)$ and $\alpha>-1$. If $0<\gamma<\alpha+2$ and $0<\beta<1$, then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}^\beta\to\mathcal{B}^\gamma$ is bounded;
(b) $\mathcal{I}_{\mu_{\alpha+1}}:\mathcal{B}^\beta\to\mathcal{B}^\gamma$ is compact;
(c) μ is an $\alpha+2-\gamma$-Carleson measure.

Proof. This is a direct consequence of Theorem 4.3. □

Remark 5.5. *It should be mentioned that Ye and Zhou [13] have obtained some results of Corollary 5.1-5.4 by using the duality theorem. In fact, they dealt with $\gamma=\alpha$ and $\alpha\geqslant 1$.*

In what follows, we consider the operator $\mathcal{I}_{\mu_{\alpha+1}}$ acting between logarithmic Bloch spaces.

Corollary 5.6. *Let $\alpha > -1$, $\beta > -1$, $\gamma \in \mathbb{R}$. Suppose μ is a positive Borel measure on $[0,1)$ and satisfies $\int_0^1 \frac{\log^\beta \frac{e}{1-t}}{1-t} d\mu(t) < \infty$. Then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^\beta} \to \mathcal{B}_{\log^\gamma}$ is bounded;
(b) $\sup_{n\geqslant 1} n^{\alpha+1} \log^{-\gamma}(n+1) \int_0^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) < \infty$;
(c) $\sup_{t\in[0,1)} \dfrac{\mu([t,1))(\log \frac{e}{1-t})^{\beta+1-\gamma}}{(1-t)^{\alpha+1}} < \infty$.

Proof. It follows from Theorem 4.2 that $(a) \Leftrightarrow (b)$. We only need to show that $(b) \Leftrightarrow (c)$. The implication $(b) \Rightarrow (c)$ follows from the inequalities

$$\mu\left([1-\frac{1}{n},1)\right) \log^{\beta+1}(n+1) \lesssim \int_{1-\frac{1}{n}}^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) \lesssim \frac{\log^\gamma(n+1)}{n^{\alpha+1}}.$$

$(c) \Rightarrow (b)$. Assume (c). Then there exists a positive constant C such that

$$\mu([t,1)) \left(\log \frac{e}{1-t}\right)^{\beta+1-\gamma} \leqslant C(1-t)^{\alpha+1}, \quad 0 \leqslant t < 1.$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 t^n \log^{\beta+1} \frac{e}{1-t} d\mu(t) \\
&= n \int_0^1 t^{n-1} \mu([t,1)) \log^{\beta+1} \frac{e}{1-t} dt + (\beta+1) \int_0^1 t^n \mu([t,1)) \log^\beta \frac{e}{1-t} \frac{dt}{1-t} \\
&\lesssim n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t} dt + \int_0^1 t^n (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t} dt.
\end{aligned}$$

Note that

$$\phi_1(t) = (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t}, \quad \phi_2(t) = (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t}$$

are regular in the sense of [21]. Then, using Lemma 1.3 and (1.1) in [21], we have

$$n \int_0^1 t^{n-1} (1-t)^{\alpha+1} \log^\gamma \frac{e}{1-t} dt \asymp \frac{\log^\gamma(n+1)}{n^{\alpha+1}}$$

and

$$\int_0^1 t^n (1-t)^\alpha \log^{\gamma-1} \frac{e}{1-t} dt \asymp \frac{\log^{\gamma-1}(n+1)}{n^{\alpha+1}}.$$

These two estimates imply that

$$\int_0^1 t^n \log^{\beta+1} \frac{2}{1-t} d\mu(t) \lesssim \frac{\log^\gamma(n+1)}{n^{\alpha+1}}.$$

Thus, (b) holds. $\square$

Arguing as the proof of previous theorem, one can obtain the following theorems.

Corollary 5.7. *Let $\alpha > -1$, $\beta = -1$, $\gamma \in \mathbb{R}$. Suppose μ is a positive Borel measure on $[0,1)$ and satisfies $\int_0^1 \log\log\frac{e}{1-t}d\mu(t) < \infty$. Then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^{-1}} \to \mathcal{B}_{\log^{\gamma}}$ *is bounded;*
(b) $\sup\limits_{n\geqslant 1} n^{\alpha+1}\log^{-\gamma}(n+1)\int_0^1 t^n \log\log\dfrac{e}{1-t}d\mu(t) < \infty$*;*
(c) $\sup\limits_{t\in[0,1)} \dfrac{\mu([t,1))\log\log\frac{e}{1-t}}{(1-t)^{\alpha+1}\log^{\gamma}\frac{e}{1-t}} < \infty$*.*

Corollary 5.8. *Let $\alpha > -1$, $\beta < -1$, $\gamma \in \mathbb{R}$. Suppose μ is a finite positive Borel measure on $[0,1)$, then the following statements are equivalent.*
(a) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^{\beta}} \to \mathcal{B}_{\log^{\gamma}}$ *is bounded;*
(b) $\mathcal{I}_{\mu_{\alpha+1}} : \mathcal{B}_{\log^{\beta}} \to \mathcal{B}_{\log^{\gamma}}$ *is compact;*
(c) $\sup\limits_{n\geqslant 1} n^{\alpha+1}\log^{-\gamma}(n+1)\mu_n < \infty$*;*
(d) $\sup\limits_{t\in[0,1)} \dfrac{\mu([t,1))\log^{-\gamma}\frac{e}{1-t}}{(1-t)^{\alpha+1}} < \infty$*.*

It is known that $\mathcal{H}$ maps $\mathcal{B}_{\log^{\beta}}$ into $\mathcal{B}_{\log^{\beta+1}}$ for all $\beta \in \mathbb{R}$(see e.g., [22]). If μ is Lebesgue measure on $[0,1)$, then Corollary 5.6-5.8 show that the integral type Hilbert operator $\mathcal{I} : \mathcal{B}_{\log^{\beta}} \to \mathcal{B}_{\log^{\beta+1}}$ is bounded if and only if $\beta > -1$.

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