

# Survey of New Applications of Geometric Algebra

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## Abstract

This survey introduces 101 new publications on applications of Clifford's geometric algebras (GA) newly published during 2022 (until mid-January 2023). The selection of papers is based on a comprehensive search with Dimensions.ai, followed by detailed screening and clustering. Readers will learn about the use of GA for mathematics, computation, surface representations, geometry, image- and signal processing, computing and software, quantum computing, data processing, neural networks, medical science, physics, electric engineering, control and robotics.

## RESEARCH ARTICLE

## Survey of New Applications of Geometric Algebra

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## Summary

This survey introduces 101 new publications on applications of Clifford's geometric algebras (GA) newly published during 2022 (until mid-January 2023). The selection of papers is based on a comprehensive search with Dimensions.ai, followed by detailed screening and clustering. Readers will learn about the use of GA for mathematics, computation, surface representations, geometry, image- and signal processing, computing and software, quantum computing, data processing, neural networks, medical science, physics, electric engineering, control and robotics.

## KEYWORDS:

Clifford geometric algebra, mathematics, surface representations, geometry, image- and signal processing, computing, software, quantum computing, data processing, neural networks, medical science, physics, electric engineering, control, robotics

## 1 | INTRODUCTION

Applications<sup>1</sup> of Clifford's geometric algebras (Clifford algebras) are quickly increasing in numbers and diversity. In order to provide an up-to-date survey of the latest applications in mid-January of 2023, we used the Dimensions.ai search engine for the years of 2022 and 2023 with keywords *Geometric Algebra* and found 121 publications (articles, preprints, books, book chapters) with these keywords in title or abstract. After checking each item for its appropriateness we selected 101 of them for this survey. As survey authors we ourselves made quite a few *discoveries* of novel applications we were not aware of so far. We hope readers will have similar moments of surprise. One reason is that Dimensions.ai is not in any kind of theme- or information bubble, but delivers all results available.

We note that GA has become popularly used in applications dealing with geometry. It allows to reformulate and redefine problems involving geometry in a highly intuitive and general way. GA was defined thanks to the work of W. K. Clifford<sup>1</sup> to unify and generalize Grassmann algebra<sup>2</sup> and W.R. Hamilton's quaternions<sup>3</sup> into a universal algebraic framework by adding the inner product to H. G. Grassmann's outer product. One of the geometric algebras that is often applied is conformal geometric

<sup>0</sup>**Abbreviations:** GA, geometric algebra; CGA, conformal geometric algebra; PGA, projective geometric algebra; STA, spacetime algebra; GAC, geometric algebra of conics; QGA, quadric geometric algebra

<sup>1</sup>This paper is subject to the Creative Peace License, <https://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/>, accessed 17 Feb. 2023.

algebra. It became better known through<sup>4</sup>, is well described and illustrated in<sup>5</sup>, and in a brief illustrated form in<sup>6</sup>. For standard references on GA, we refer to the following textbooks:<sup>7,8,5</sup>. A brief introduction for engineers can be found in<sup>9</sup>, while a compact definition of GA is given in<sup>10</sup>, see also<sup>9,11</sup>.

Regarding the notation of Clifford geometric algebras, a certain variety can be found. A frequently used notation is  $Cl(p, q, r)$  for the Clifford geometric algebra of a space  $\mathbb{R}^{p,q,r}$ , with dimension  $n = p + q + r$ , with an orthonormal basis of  $p$  vectors squaring to  $+1$ ,  $q$  vector squaring to  $-1$ , and  $r$  vectors squaring to  $0$ . For example, projective geometric algebra (PGA) of three dimensions uses  $Cl(3, 0, 1)$ . Note that for  $r = 0$  it is customary to abbreviate  $\mathbb{R}^{p,q} = \mathbb{R}^{p,q,0}$ , and  $Cl(p, q) = Cl(p, q, 0)$ . Furthermore, for  $q = 0$ , many authors abbreviate  $\mathbb{R}^n = \mathbb{R}^{n,0}$ , and  $Cl(n) = Cl(n, 0)$ , e.g., the GA of three-dimensional Euclidean space  $\mathbb{R}^3$  is  $Cl(3, 0)$ , and conformal geometric algebra (CGA) for three-dimensional Euclidean space, extended by a Minkowski type plane  $\mathbb{R}^{1,1}$  is  $Cl(4, 1)$ . But note that particularly in the field of Clifford analysis, authors may instead use  $\mathbb{R}^n = \mathbb{R}^{0,n}$ , it is therefore advisable when reading a publication to first ascertain which notation the author uses. Moreover one often finds  $\mathcal{G}_{p,q,r} = \mathbb{G}_{p,q,r} = G_{p,q,r} = Cl_{p,q,r} = \mathbb{R}_{p,q,r}$ , etc.

The paper is structured as follows, revealing how we clustered and ordered the 101 publications thematically (which is of course somewhat subjective). First, Section 2 refers the reader to similar earlier survey projects conducted during the last ten years. Then, Section 3 provides an overview of applications of GA for mathematics and computations, while Section 4 introduces applications to higher order surfaces and geometry. Next, Section 5 shows the use of GA for image- and signal processing. This is followed by Section 6 on GA computing and GA software and Section 7 on quantum computing with GA. Data processing with GA is featured in Section 8, many applications to neural networks in Section 9, and in Section 10 to the medical field. After that, Section 11 explores applications in physics, i.e., mechanics, electrodynamics, gravity and quantum physics. The new pioneering applications in electric engineering are surveyed in Section 12, and control and robotics in Section 13. The paper is concluded with Section 14 and the list of references.

## 2 | PREVIOUS GA APPLICATION SURVEYS

In<sup>12</sup>, the authors present a survey including applications of Clifford geometric algebra (GA) in the past decade (mainly within 2013–2021), several of which were presented in the Applied Geometric Algebra for Computer Science and Engineering (AGACSE) conference series, as well as the annual Empowering Novel Geometric Algebra for Graphics and Engineering (ENGAGE) workshops, which are part of the conference Computer Graphics International.<sup>12</sup> can be seen as a continuation of the earlier survey<sup>13</sup> published in 2013, summarizes approximately 200 GA publications.

It is further supplemented by a similar amount of publications on GA applications in<sup>14</sup> of early 2022, surveying publications in the years 2019 to 2022. These surveys include GA applications related to engineering, electric engineering, optical fibers, geographic information systems, geometry, molecular geometry, protein structure, neural networks, artificial intelligence, encryption, physics and software as well as signal, image, and video processing, etc.

## 3 | GA FOR MATHEMATICS AND COMPUTATION

This section is divided into three topical parts: the mathematical structure of GA, application of GA to mathematical problems and GA-based computing.

The first part consists of papers where the authors study the objects of GA, i.e. geometric entities or transformations and functions. Thus, Eduardo Bayro-Corrochano et al.<sup>15</sup> revisit a proposal for the formulation of objects and geometric relations and constraints in CGA and discuss its application to various engineering disciplines. Representations of transformations, rotors in particular, are then searched for by Anthony Lasenby et al. in<sup>16</sup> in the form of their reconstruction from initial and final frames. A series of papers by Acus Arturas et al. is handling the closed form exponentials of multivectors in general dimension<sup>17</sup> with additional conditions on the GA signature, both in low dimensions<sup>18</sup> and generally in<sup>19</sup>, respectively. In algebras of dimension  $n$  less than six, more maps such as normalization, square roots, exponential and logarithmic maps are treated by Steven De Keninck et al.<sup>20</sup>. As for the operations on GA, in<sup>21</sup> Tomáš Stejskal et al. provide a comprehensive description of relations between objects in 2D space using the matrix product of vectors, the geometric product, and the dot product of complex numbers. Finally, structural issues of the embedding of octonions in GA of all signatures in three and four dimensions are treated by Eckhard Hitzer in<sup>22</sup>.

The second part is perfectly represented by a paper of Sira–Ramírez et al.<sup>23</sup> where Lyapunov stability theory for smooth nonlinear autonomous dynamical systems is presented in terms of GA, or the paper by Kamron Abdulkhaev et al.<sup>24</sup> where basis-free formulae for characteristic polynomial coefficients are derived in terms of GA. Structural mathematical concepts, such as the generalization of Lipschitz and spin groups, or twisted algebras of GA are targeted by Ekaterina Filimoshina et al. in<sup>25</sup> and Masaki Matsuno in<sup>26</sup>, respectively.

In the last category the paper by Stephane Breuils et al.<sup>27</sup> discusses the complexity of products in GA, more precisely the number of operations required to compute a product, in a dedicated program for example, and the complexity of enumerating these operations.

## 4 | GA FOR SURFACES AND GEOMETRY

In his new book *Mathematics for Computer Graphics*<sup>28</sup>, Vince explains a wide range of mathematical techniques and problem-solving strategies associated with computer games, computer animation, virtual reality, CAD and other areas of computer graphics. Among the list of worked examples and colour illustrations revolving around the mathematics required for computer graphics, the author dedicates a chapter to introduce the basic notation and functionality of GA, as an extension of the complex numbers and quaternions that were described in detail in previous sections. The introduction to GA is performed in the author's characteristic descriptive manner, and along with the numerical examples lure the reader to this exciting new world of multivectors.

The GA  $\mathbb{G}_{6,3} = Cl(6,3)$ , also denoted as *Quadric Geometric Algebra* (QGA), is a known generalization of CGA. In<sup>29</sup>, Esquivel provides the description of common geometric entities, e.g., points, planes, spheres, but also (hyperbolic and parabolic) cylinders, elliptic cones or ellipsoids. The rotation and revolution of points and quadratic primitives as well as the vanishing coordinate frame  $\{e_{\infty x}, e_{\infty y}, e_{\infty z}\}$ , are described in detail, providing better insight of QGA.

As the interpolation of the trajectory of points and geometric entities remains an important problem for kinematics (e.g., movement for robots), several algorithms exist to describe such trajectories, often involving the use of matrices, quaternions, dual-quaternions and the Study quadric. In<sup>30</sup>, the authors exploit CGA to represent motors as 8D vectors in projective space  $\mathbb{P}^7$ , thus reducing the interpolation of rotations and translations to a linear problem. A CPU and a GPU (CUDA) implementation were tested to obtain performance metrics and the methodology was applied to interpolate trajectories in medical robotics for kidney surgery.

Towards an effective generation of molecular surfaces, Alfarraj and Wei<sup>31</sup> employ Clifford Fourier transforms (CFT), a generalization of the classical FT. Using the CFT in  $\mathbb{R}_3 = Cl(3,0)$  allows solving partial differential equations and specifically the ones involved in the mode decomposition process. After setting the theoretical background, authors apply the proposed method to small molecules and proteins, generating their surfaces and comparing their output with other definitions. The importance of their work is further highlighted, as their methodology can be applied for protein electro-static surface potentials and solvation free energy, as well as other biological sciences.

Using quaternion operators on orbits (curves or surfaces) in the Euclidean space  $\mathbb{E}^3$  and its GA  $Cl(3,0)$ ,<sup>32</sup> demonstrates how to generate these orbits from points, curves or surfaces. An explicit form of the motions that achieves this is proved and provided as 1- or 2-parameter homothetic motions, along with detailed numerical examples.

Identifying sphere intersections is a simply-to-state yet important core problem of many applications, among a variety of scientific fields, e.g., data sciences and 3D protein structure determination. When the radii of the spheres are not known, the notion of a *spherical shell* can be employed, by replacing the precise radii with interval values. Considering the intersection of spheres and/or spherical shells in higher dimensions, Lavor et al.<sup>33</sup> provide a methodology to identify and characterize them. A comparison of the theoretical approaches via linear algebra or CGA is provided and illustrated with numerical examples. The result of this comparison further highlights the ability of CGA to naturally preserve the geometric intuition of the problem, even in dimensions higher than three.

To view fractal-based devices not as a problem only related to high-tech innovation, but also as a marketing problem,<sup>34</sup> tracks the development of Fractal Information Theory using a universal geometric musical language and a 12-dimensional GA. It explores how Fractal decision making can be used in the fields of business analytics, security for risk mitigation, and healthcare.

MAGES 4.0: Accelerating the world's transition to VR training and democratizing the authoring of the medical metaverse

## 5 | GA FOR IMAGE AND SIGNAL PROCESSING

Eckhard Hitzer<sup>35</sup> generalizes the spacetime Fourier transform (SFT) of<sup>36</sup> to a special affine Fourier transform (SASFT, also known as offset linear canonical transform) for 16-dimensional spacetime multivector  $Cl(3, 1)$ -valued signals over the domain of spacetime (Minkowski space)  $\mathbb{R}^{3,1}$ . This includes computation in terms of the SFT, its properties of multivector coefficient linearity, shift and modulation, inversion, Rayleigh (Parseval) energy theorem, partial derivative identities, a directional uncertainty principle and its specialization to coordinates.

Zihao Zhang et al.<sup>37</sup> observe that the complex-valued random Fourier GA mapping (CRFGAM) method can solve the over-coupling issue of real and imaginary parts for complex-valued signals. In order to improve the accuracy of nonlinear mapping in the CRFGAM method, they propose a *multi-dimensional* complex-valued random Fourier GA mapping (MDCRFGAM) method by expanding CRFGAM to a multi-dimensional mapping space and extend this to a multi-dimensional complex-valued random Fourier GA least mean square (MDCRFGALMS) algorithm. Related to this Gangyi Huang et al.<sup>38</sup> propose a novel *fixed dimensional* adaptive filter for complex-valued signals named complex-valued random Fourier GA least mean square (CRFGALMS). With GA adaptive filtering, real and imaginary parts of complex-valued signals are mapped to random Fourier features space (RFFS) to improve the efficiency of the nonlinear mapping for complex-valued signals. The proposed GA based mapping has superior presentation abilities in the complex-valued domain, and decouples the nonlinear mapping of real and imaginary parts of complex-valued signals. In both cases simulations on nonlinear channel equalization are conducted for validation.

Xiangyang Wang et al.<sup>39</sup> present two novel GA methods to estimate two-dimensional (2D) directions-of-arrival (DOA) of non-circular (NC) signals for uniform rectangular arrays (URA). Traditional long vector methods inevitably lose orthogonality inside each electromagnetic vector sensor (EMVS) and miss information of second-order statistical properties and increase computational complexity. New GA-based estimating of signal parameters via rotational invariance techniques (ESPRIT) and propagation method (PM) algorithms are proposed. GA maintains the relationship among multidimensional signals. The six EMVS components are represented as a GA multivector, and a GA-based extended covariance matrix utilizes the signal information more completely. The new GA estimation of signal parameters via rotational invariance techniques for NC signal processing (GANC-ESPRIT) yields the DOA with high accuracy. The new GA propagation method for NC signal estimation (GANC-PM) uses linear transformations to calculate angle parameters. Memory requirement is greatly reduced versus long vector methods. Simulations validate good angular resolution and complexity analysis shows better performance with reduced computation.

Rui Wang et al.<sup>40</sup> propose two novel GA-based adaptive filtering algorithms based on the minimum error entropy (MEE) criterion and the joint criterion (MSEMEE) of MEE and the mean square error (MSE). Simulation results show that for the mean square deviation (MSD) learning curve, the GA-based MEE (GA-MEE) algorithm has faster convergence rate and better steady-state accuracy compared to the GA-based maximum correntropy criterion algorithm (GA-MCC) under the same generalized signal-to-noise ratio (GSNR). The GA-MEE algorithm reduces the convergence rate, but improves the steady-state accuracy by 10–15 dB compared to adaptive filtering algorithms based on GA and second-order statistics. When GA-MSEMEE and the adaptive filtering algorithms based on GA and second-order statistics, respectively, keep the same convergence rate, its steady-state accuracy is improved by 10–15 dB, and when GA-MSEMEE and GA-MEE maintain approximately steady-state accuracy, its convergence rate is improved by nearly 100 iterations. For noise cancellation, the average recovery error of the two new algorithms improves over other GA-based adaptive filtering algorithms. This provides new methods to deal with multi-channel interference in wireless networks.

Wenyuan Wang and Jiaolong Wang<sup>41</sup> observe that GA based adaptive filters have been applied to fields like 3D wind speed, computer vision and fusion prediction of dynamic pressure. To further improve performance they propose GA adaptive algorithms convexly combining two different step size GA least mean square algorithms (CGA-LMS), and provide detailed steady state performance analysis. To address the phenomenon that the slow filter may lag considerably behind the fast filter, which slows down the overall convergence of the combined GA filter, they add a novel instantaneous transfer strategy, creating a CGA-LMS algorithm with transfer strategy (CGA-LMS-TS). To process the non-circular 3D and 4D signals, they employ a convex combination of a widely linear GA-LMS (CWL-GA-LMS) algorithm with a CWL-GA-LMS with transfer strategy (CWL-GA-LMS-TS). Simulations validate performance and correctness.

Anna Derevianko and Pavel Loucka<sup>42</sup> introduce a search for the similarity transformation of two-dimensional point clouds using GA for conics (GAC). They represent image objects by ellipses fitted into contour points. This speeds up consequent similarity search and saves memory. Examples with real object images are included.

Haishun Liu et al.<sup>43</sup> propose a GA product expression associated with geometric relationships of vectorized THz refractive index and absorption coefficients. From this expression, candidate characteristic parameters are extracted for liquids discrimination presenting abundant second order correlation information of optical parameters with rising dimensions. Three groups of liquids, containing C-reactive protein calibrators and alpha fetoprotein calibrators, were used for validation. Comparing with traditional THz parameters of refractive index, absorption coefficient, and complex permittivity, the novel approach is superior in differentiation with the evaluation of statistical differences and effect size.

Alexandre Calado et al.<sup>44</sup> compare two new geometric model-based approaches to gesture recognition which support the visualization and geometrical interpretation of the recognition process, with two classical ML algorithms,  $k$ -nearest neighbor ( $k$ -NN) and support vector machine (SVM), and two state-of-the-art (SotA) deep learning (DL) models, bidirectional long short-term memory (BiLSTM) and gated recurrent unit (GRU), on an experimental Italian Sign Language (LIS) data set. They achieve a compromise between high recognition rates ( $> 90\%$ ) and fast recognition times ( $< 0.1s$ ) adequate for human-computer interaction.

## 6 | GA FOR COMPUTING AND SOFTWARE

Aiming to educate both GA experts and people that are not yet acquainted with the concept of multivectors, Hildenbrand and Rockwood<sup>45</sup> gave a SIGGRAPH course presentation<sup>2</sup> touching various aspects of GA. After a motivating introduction to the work of Grassman and Clifford, they provided the deep connection between algebra and geometry that holds for the entities of such mathematical frameworks, as well as basic notation and properties. After describing in more detail the two dominant GAs, namely projective GA (PGA) and CGA, the GAALOP software framework is used to perform various operations. Specifically, GAALOP and its web version, GAALOPWeb, is a well known software dedicated to optimize GA files. Its usage is demonstrated in a straightforward way for both PGA and CGA, to perform numerical operations as well as to obtain visual results. This work concludes with the presentation of GAC, an algebra that has become increasingly popular due to a diverse range of applications.

By analysing the majority of approaches used by logicians to support a mathematical claim, one may deduce that there are two ways that a computer can help establish a claim: it can either help find a proof in the first place (*automated theorem proving*) or it can help verify that a purported proof is correct (*interactive theorem proving*). The *Lean Theorem Prover* aims to bridge the gap between interactive and automated theorem proving, by situating automated tools and methods in a framework that supports user interaction and the construction of fully specified axiomatic proofs. Wieser and Song<sup>46</sup> recently presented the partial formalization of GA in Lean via describing multivectors as the quotient of a tensor algebra by a suitable relation, in a basis-agnostic manner. Although not complete yet, their work<sup>3</sup> is applicable to GA of various dimensions and sets the bases for a complete GA formalization within the promising `mathlib` library, in a future proof and easy-to-extend way.

In the past decades, quaternions have been the dominant representation form for rotations in the computer graphics (CG) pipeline, along with the use of matrices and vectors to store translations and scalings. Although there is concrete evidence that the use of more advanced forms such as dual quaternions and multivectors (for motors, rotors and dilators in PGA and CGA) can alleviate commonly appearing rendering artifacts, such as the candy-wrapper effect, these forms are usually not incorporated in the modern CG curriculum, mainly due to the lack of support from the frameworks used. In<sup>47</sup> the authors present the *Elements* project, a pythonic framework that is based on the Entity-Component-System principle on top of a Scenegraph-based implementation. Using *Elements*, users may create and render a 3D scene, using transformation data that is either of the traditional form, i.e., a TRS matrix, or any other potential form, such as quaternions, dual-quaternions or multivectors.

## 7 | GA FOR QUANTUM COMPUTING

In<sup>48</sup>, Hrdina et al. investigate the representation of  $n$ -qubits and quantum gates acting on them as elements of a complex GA defined on a complex vector space of dimension  $2n$ . In such spaces the Dirac formalism can be realized straightforwardly. Aiming to establish GA as a major language for quantum computing, the authors introduce the non-complex Quantum Register Algebra (QRA)<sup>49</sup> and exploit the GAALOP (Geometric Algebra Algorithms Optimizer) framework to perform numerical operations.

<sup>2</sup>The complete presentation is available at <https://dl.acm.org/doi/10.1145/3532720.3535655>.

<sup>3</sup>This work can be found at <https://github.com/pygae/lean-ga>.

Their latter work proves advantageous for quantum computing beginners since they only have to know GA in order to intuitively describe the objects and operations of quantum computing.

In the same context, Soiguine<sup>50</sup> demonstrates how the the double split experiment results can be resolved with diffraction patterns inherent to wave diffraction. In his work, he exploits the GA formalism along with generalization of complex numbers and subsequent lift of the two-dimensional Hilbert space valued qubits to geometrically feasible elements of an even GA subalgebra.

## 8 | GA IN DATA PROCESSING

Addressing the demand for multi-level declassification of geographic vector field data (GVFD), the authors of<sup>51</sup> employ GA to uniformly express it as a GA object and then sequentially apply a rotor and a perturbation operator. Declassifying the final output and comparing it to the input shows that despite the alteration of the original vector field some general geo-spatial features are retained. This approach offers effective multi-level controls and has good randomness, it suggests a viable solution for data disclosure, secure transmission and encapsulation storage of such types of data.

In<sup>52</sup>, the authors conducted an analysis of the spatial geometric similarity computation based on CGA, aiming to shed light on spatial analysis and data retrieval. By (a) developing a unified expression model for spatial geometric scenes, (b) integrating shapes of objects and spatial relations between them, and (c) establishing a model for spatial geometric similarity computation, they are able to derive measurement information and topological relations of spatial objects. Their method involves simple inner, outer and geometric product operations and can be applied to retrieve spatial scenes with objects of different types, yielding satisfying results.

As the world is evolving from a binary space to a ternary space, a suitable framework to capture diverse geographic information is required. The seven-dimensional framework proposed in<sup>53</sup> exploits a form of GA representation that is suitable to represent characteristics such as semantics, attribute, interrelationship, and interplay mechanisms, which are usually unstructured and cannot be represented as plain geometries or simple algebraic equations.

In<sup>54</sup>, Wu et al. propose a GA-based Collaborative Filtering model for recommendation, based on representation learning. Their methodology involves the usage of multivectors to represent users and items, while plain geometric and inner product operations are used to indicate the historical interaction between users and items. Training the model with the resulting multivectors, interdependencies between components of multivectors can be predicted, allowing more complex interactions between users and items to be captured.

A quite intriguing area where GA proves to yield enhanced results involves Knowledge Graph Embedding (KGE), which introduces important challenges for knowledge representation learning such as the management of time-evolving data nodes. Link prediction tasks in KGE have shown promising performance, especially due to complex or hypercomplex representation forms. In<sup>55</sup>, Xu et al. present a GA-based approach that achieves or even surpasses the state-of-the-art performance threshold. Their approach involves data representation as multivectors of  $\mathbb{G}^2 = Cl(2, 0)$  or  $\mathbb{G}^3 = Cl(3, 0)$ , as well as a GA-based model that consumes the obtained data set. This novel, yet effective, approach for KG, manages to successfully handle the Temporal KGE completion problem, by exploiting the geometric meaning of the time embedding.

A variant of the well-known threshold secret sharing scheme proposed by Adi Shamir in 1979 is presented in<sup>56</sup>. Its methodology involves using multivector objects to be secrets in a secret sharing scenario without incurring any additional overhead in comparison with the reference scheme. Fundamental computations such as addition and multiplication can be performed over random shares, making this approach ideal to use the context of multi-secret sharing. As both secret and random shares of members are members of the same space, the fundamental property of Shamir's scene, *idealness*, is also preserved.

## 9 | GA APPLIED TO NEURAL NETWORKS

GA seems to be well suited regarding the representation of multi-dimensional data. Expressing such data in multivector form, Li et al.<sup>57</sup> were able to capture the inherent structures and preserve the correlation of multiple dimensions in the context of a *long and short term time series* network that processes multivariate time series. Their method outperforms traditional techniques with higher prediction accuracy.

Following a similar approach,<sup>58</sup> proposes a GA-based mapping of each spectrum in a hyperspectral image (HSI), the multivectors derived are then used in a Convolutional Neural Network (CNN) that deals with multi-channel HSIs. In such an

approach, increased performance, less overfitting risks and better information preservation was achieved with respect to identical real-valued CNNs.

As Graph Neural Networks become increasingly popular, the authors of<sup>59</sup> proved that graph feature embedding in GA can improve the quality of graph feature presentation. Using a *few shot cross domain* classification task as application, their proposed approach yields improved results over metric-based methods as it uses the high algebraic dimensions of GA to reduce the distortion of feature information despite the increased hidden layers.

Robust and efficient transmission of data over networks is crucial for collaborative AR/VR applications. In such a context, the authors of<sup>60</sup> compare traditional representation forms (vectors and matrices) with GA-based forms (quaternions, dual quaternions and PGA/CGA multivectors) for transmission, recording and replay purposes. Their work suggests that, regarding transmission, GA is the only viable solution for poor network conditions and yields better or on par results when no such conditions occur. It is also shown that, using GA forms, less transformation data per second can be recorded without impacting the interpolated keyframes during replay.

Regarding computer-vision oriented deep learning techniques, GA is a great means to capture rotational data that constantly appear, as many of the related tasks such as pose estimation from images or point clouds can be formulated as a regression on rotations. Overcoming limitations posed by commonly used representations, the authors of<sup>61</sup> exploit multivectors to reduce errors in high-noise datasets, while learning fewer parameters.

With the development of cities and the increased demand for traffic management, predicting traffic data has become an increasingly researched yet complicated task due to changeable and complex traffic conditions. By capturing related traffic information as multivectors, the multidimensionality of the data can be maintained and complex features can also be extracted in the context of spatio-temporal attention neural networks<sup>62</sup>, multi-channel residual networks<sup>63</sup>, generative adversarial networks<sup>64</sup>, or graph attention networks<sup>65</sup>. These works suggest an improvement of performance when GA-based forms are used compared with real-data based implementations and are suitable to predict even long-term features such as the traffic speed for a whole day.

Alongside algorithmic advances in neural networks, new technological paradigms are developed where machine- and deep learning techniques are incorporated in dedicated hardware. Such an example is described in<sup>66</sup>, where a Hypersphere Neural Network for energy consumption monitoring, based on geometric algebra representation of points and hyperspheres, is implemented in an IoT device, consisting of a NodeMCU board and an Esp8266 microcontroller.

Towards solving the problem that derives from non-orthogonal data attributes in conventional machine learning applications,<sup>67</sup> proposes a shift-invariant  $k$ -means methodology. The authors suggest the use of complex, hypercomplex and GA based approaches and encoding schemes, to better capture invariance under rotation or general transformations, via convolution operators that preserve spatio-temporal information.

GA based neural networks also find applications in the medical field.

## 10 | GA IN MEDICAL SCIENCE

GA is able to positively impact the majority of scientific areas, including the medical sciences. A good example is provided in<sup>68</sup> where the most common neurodegenerative disorder, the Alzheimer's disease (AD), is considered. Introducing a GA-based multimodal feature transformation and fusion model, a fast-convergent Artificial Neural Network framework can provide a highly accurate AD diagnosis.

Further proving that high-dimensional information can be more efficiently handled via GA, Wang et al.<sup>69</sup> propose a multi-modal medical image fusion algorithm based on a discrete GA cosine transform. In their work, they conduct fusion experiments on four groups of brain medical color images, by considering the connection between the color image channels and using multivectors to represent the source image. Results indicate improved performance, but only marginal advantage compared to traditional algorithms, thus leaving room for future improvements.

A similar technique is exploited in<sup>70</sup> where GA is employed for (a) a multi-modal medical image fusion algorithm, (b) an orthogonal matching pursuit algorithm and (c) a  $K$ -means clustering singular value decomposition algorithm. All these components are involved into an effective algorithm that avoids losing the correlation of color channels of medical images and surpasses the state-of-the-art approaches in terms of subjective and objective quality evaluation.

GA is also employed in the context of modern applications that impact the medical training landscape. In<sup>71</sup>, the authors propose MAGES 4.0, a novel Software Development Kit (SDK) to accelerate the creation of collaborative medical training



applications in VR/AR. Their solution is essentially a low-code metaverse authoring platform for developers to rapidly prototype high-fidelity and high-complexity medical simulations. Among the variety of novelties it incorporates, this Unity3D-based framework exploits GA-based representation forms, namely dual quaternions and multivectors, to efficiently transmit user actions over the network in multi-user collaborative scenarios. Exploiting the mechanisms described in<sup>60</sup>, its under-the-hood GA interpolation engine achieves optimal performance compared to state-of-the-art. The same engine and all-in-one GA framework is also employed to effectively record and replay VR/AR sessions.

See also the discussion of<sup>31</sup> and<sup>30</sup> in Section 4.

## 11 | GA IN PHYSICS

### 11.1 | Mechanics

Steward Jensen and Jack Poling<sup>72</sup> describe angular momentum with  $CI(3,0)$  bivectors, visualized as *tiles* with area and orientation whose components form an antisymmetric matrix. Bivectors have historically been considered mostly in specialized contexts like spacetime classification or GA, but are no more complicated than cross products. Teaching rotational physics in this language is ultimately viewed more fundamental, and helps to understand rotations in relativity and extra dimensions.

According to Sylvain Brechet<sup>73</sup> in  $CI(3,0)$  the Poisson formula for the time derivative of unit vectors of a moving frame is expressed by the angular velocity bivector and applied to cylindrical and spherical frames. The rotational dynamics of a point particle and a rigid body are fully determined by the time evolution of  $CI(3,0)$  rotors. The mapping of the angular velocity bivector onto the angular momentum bivector is the inertia map. It is characterised by symmetric coefficients (moments of inertia) in the (rigid body) principal axis frame. The Huygens-Steiner theorem, the kinetic energy of a rigid body and the Euler equations are expressed in terms of bivector components. The rotational dynamics of a gyroscope provides an example.

Hitoshi Ikemori et al<sup>74</sup> formulate the Runge-Lenz vector in the Kepler problem as a three-dimensional GA projection of a  $SO(4)$  moment map that acts on the phase space of a four-dimensional particle motion. The Runge-Lenz vector originates from geometric symmetry of  $\mathbb{R}^4 \times \mathbb{R}^4$  phase space.

### 11.2 | Electrodynamics

Sylvain Brechet<sup>75</sup> treats electrodynamics of electric charges and currents in vacuum and dielectric and magnetic material media, using  $CI(3,0)$  and STA  $CI(1,3)$ . With a polarisation multivector and an auxiliary electromagnetic field multivector, Maxwell's equation is formulated in a material medium in  $CI(3,0)$ , and in STA with an extra bound current vector. The wave equation in a material medium is obtained from the gradient of the Maxwell equation. For a uniform electromagnetic medium of induced electric and magnetic dipoles, the stress-energy momentum vector is formulated with the electromagnetic force density vector as inhomogeneity, and the Maxwell equation in a material medium is written in STA as vector potential wave equation.

According to Debashis Sen<sup>76</sup>, STA provides an invariant description of electromagnetic theory, without reference to any inertial system. Using elementary geometric calculus, STA allows the direct analytical introduction of magnetic monopoles and renders the equations for both constituent fields, symmetric and inhomogeneous. STA unifies the Lorentz force equation and the electromagnetic power equation.

### 11.3 | Gravity

Noticing that the even subalgebras of  $CI(3,1)$  and  $CI(1,3)$  can algebraically not be distinguished, Bofeng Wu<sup>77</sup> provides a signature invariant treatment of general Lorentz boosts and general spatial rotations in arbitrary planes. For a massive particle, the spacetime splits of the velocity, acceleration, momentum, and force four-vectors with the normalized four-velocity of the fiducial observer, at rest in the coordinate system of the spacetime metric, are given, where the proper time of the fiducial observer is identified, and the contribution of the bivector connection is considered, and with these results, a three-dimensional analogue of Newton's second law for this particle in curved spacetime is achieved. As example, in Lense-Thirring spacetime, the signature invariant precessional angular velocity of a gyroscopic spin is derived in STA.

Jianfei Xu<sup>78</sup> discusses in STA gravitational wave solutions of the gauge theory of gravity field equations with a negative cosmological constant, and shows that these solutions are of Petrov type- $N$ . He also discusses the velocity memory effect by calculating the velocity change of an initially free falling massive particle due to the presence of gravitational waves.

## 11.4 | Quantum Physics

Because geometric algebra of spacetime (STA) is isomorphic to the  $4 \times 4$  complex Dirac matrix algebra of Dirac's equation for relativistic electrons, applications of geometric algebra to quantum mechanics have a long and rich tradition.

Andrew Hamilton shows in<sup>79</sup> how it is possible to start with row- and column spinors and their products which include GA multivectors to construct a *super geometric algebra* for both spinors and GA. Thus viewing fermions (spinors) as truly fundamental and expressing the exclusion principle as elementary spinor product rule. Anthony Lasenby<sup>80</sup> embeds octonions<sup>4</sup> in STA such that the octonion product norm corresponds to the preservation of the timelike part of a particle Dirac current. This allows to embed and geometrically interpret earlier work of particle physics based on octonions. Furthermore  $SU(3)$  of six Euclidean dimensions can also be embedded in STA based on bivector norm preservation. Finally interesting connections to  $G_2$  and  $SU(8)$  are considered.

Václav Zatloukal<sup>81</sup> examines the minimal coupling procedure in Hestenes' STA Dirac equation, where spinors are identified with even multivectors, and finds a non-Abelian generalisation of the electromagnetic gauge potential. Dennis Marks<sup>82</sup> observes that real geometric algebras  $Cl(n; s) = Cl(p, q)$ ,  $s = p - q$ , are periodic in  $s$ , i.e.  $Cl(n; s) \cong Cl(n; s + 8k)$  for  $p + 4k \geq 0$ ,  $p - 4k \geq 0$ , and recursive in  $n$ , i.e.  $Cl(n; s) \otimes Cl(2; 0) = Cl(n + 2; s)$ . Moreover, in GA two-dimensional Euclidean planes and spacetime Minkowski planes have isomorphic GAs. Their direct product gives STA. A further product algebraically describes strong forces and then generates a lattice with standard model physics at each node. This implies Noether's theorem conservation laws from uniformity of spacetime based on the recursive generation.

Alexander Soiguine<sup>83</sup> begins by replacing complex numbers by real GA multivectors, achieving clear explanations of atoms as a kind of planetary system. The three-sphere  $S^3$  hosts torsion like states eliminating abstract Hilbert space vectors.  $S^3$  points evolve, governed by the updated Schrödinger equation, and act in measurements on observables as operators. In<sup>84</sup> he furthermore explains double split experiments in GA, where particles create diffraction patterns inherent to wave diffraction, eliminating a key difficulty in the interpretation of quantum mechanics.

In<sup>85</sup> Sokol Andoni uses the spin–position decoupling approximation, to substitute the Pauli vector–matrix spin model by a vector with a phase in 3D orientation space endowed with GA. He explains the resulting properties, including measurement irreversibility, entanglement and 2D in 3D spin space embedding. The formalism appears in two complementary “spinor” or “vector” forms, providing a clear geometric picture of spin correlations and transformations entirely in the 3D physical orientation space.

Erik Trell<sup>86</sup> examines Marius Sophus Lie's 1871 PhD thesis *Over en Classe geometriske Transformationer*, relates it to Hermann Grassmann's *Ausdehnungslehre* (1844/1862) and William Kingdom Clifford's *Space-theory of Matter* (1876), and develops a concrete cellular automaton building kit of the Standard Model organized as structural  $\mathbb{R}^3 \times SO(3)$  wave-packets, both inwards from the elementary particles and outwards via the periodic table of the atoms over further hierarchical growth in molecular and crystal stages to an isotropic space-filling of the whole classical Euclidean Universe in harmonic exchange with its relativistic spherical moiety, dark mass and energy.

Joy Christian<sup>87</sup> tries to further explain his model for local origins of quantum correlations, assumed to be rooted in GA, and to defend it against some criticism.

## 12 | GA FOR ELECTRIC ENGINEERING

Ahmad Eid and Francisco Montoya<sup>88</sup> explore the concept of generalized geometrical frequency in electrical systems with an arbitrary number of phases by using GA and differential geometry. With the *Darboux bivector* concept they can find a bivector that encodes the invariant geometrical properties of a spatial curve named electrical curve. The traditional concept of instantaneous frequency in power networks can be intimately linked to the Darboux bivector. Several examples illustrate this result.

Eid and Montoya<sup>89</sup> then present a new GA framework for a systematic generalization of well-known instantaneous transformations used in electrical engineering for power system analysis and computing through geometric principles. By introducing the Kirchhoff vector and Kirchhoff subspace, a new generalized transformation is presented that unifies the Clarke, Park or hyper-space vector transformations (widely used in electrical engineering) as particular cases. A generalization to an arbitrary number of phases is achieved. All the underlying ideas are presented by means of space-like conceptualizations, substantiated

<sup>4</sup>See the further expansion of this embedding of octonions in other GAs in<sup>22</sup> discussed on page 2.

by their corresponding algebraic formulation. This proposal has wide ranging power system applications such as to electrical machines, current compensation, power quality, electronic converters or transmission lines. Preliminary results show superior efficiency compared to matrix methods. Some real-world examples are included.

Francisco Montoya et al.<sup>90</sup> establish an alternative physical formulation for the harmonic power flow in electrical systems provided by GA, the Poynting Vector (PV) and the Poynting Theorem (PT). Given the traditional definition of PV (Abraham approach) as cross product of electric and magnetic field vectors, they exploit the duality of the cross product to the much more powerful wedge product of exterior algebra. Using vector space concepts they develop a completely GA-based approach founded on top of the isomorphism among periodic time-domain signals and Euclidean spaces. This sheds light on the long-running discussion of electric power flow in non-sinusoidal and non-linear electrical power systems. Then Francisco Montoya et al.<sup>91</sup> for the first time model power flows in electrical circuits in a mixed time-frequency domain by using GA and the Hilbert transform. They overcome some of the limitations of the existing methodologies, in which the so-called *active current* may not lead to the lowest Root Mean Square (RMS) current under distorted supply or unbalanced load. Moreover, this current may contain higher levels of harmonic distortion compared to the supply voltage. The proposed method can be used for sinusoidal and non-sinusoidal power supplies, non-linear loads and single- and multi-phase electrical circuits, and it provides meaningful engineering results with a compact formulation. It can also serve as an advanced tool for developing algorithms in the power electronics field. Several examples verify the approach.

The circuit analysis approach based on GA and  $\mathbf{M}$ , the power definition based on the geometric product between the voltage and the current multivectors, are used by Milton Castro-Nunez et al.<sup>92</sup> to demonstrate shortcomings of the traditional definition of the non-sinusoidal apparent power  $S$ . The shortcomings of  $S$  are illustrated: Firstly, by showing an example of how the norm of  $\mathbf{M}$  contains  $S$ . Secondly, through six experiments that involve compliance with: Kirchhoff's circuit laws, Tellegen's theorem, the principle of conservation of energy, the equivalency of two terminal networks and the concept of reactive power compensation. Lastly, by showing how the use of  $S$  leads the current's physical component power theory astray. The experiments show contradictions between the aforementioned circuit theory fundamentals and the results attained with  $S$  but a compelling harmony with the results attained with  $\mathbf{M}$ . The evidence reveals: (1) that mathematical models aimed at explaining energy flow in non-sinusoidal circuits should not be based on the traditional decomposition of  $S$ , and (2) the inappropriateness of extrapolating definitions from sinusoidal to non-sinusoidal settings.

Finally, Nitin Sundriyal et al.<sup>93</sup> show that for linear and non-linear non-sinusoidal circuit conditions, a consensus can be reached on norms that comply with well-known, established standards. They compare the use of the harmonic domain and GA in circuits with disturbances for sinusoidal and non-sinusoidal excitations in order to demonstrate the accuracy of GA in power flow calculations.

## 13 | GA FOR CONTROL AND ROBOTICS

In previous years, GA has been used mostly to unify various frameworks of screw theory, Lie algebra and dual quaternions. Even now, papers on specific aspects of motion descriptions may be found, see e.g. Ben Cross et al.<sup>94</sup>. The simplicity of motion, i.e. Euclidean transformations and their interpretation have been heavily used in robot kinematics. Indeed, papers providing description of robotic motion planning are, e.g. Jesús Medrano-Hermosilo et al.<sup>95</sup> for 6-DOF serial robot's forward kinematics, or Edgar Macias-Garcia et al.<sup>96</sup> solving the inverse position kinematics for  $n$ -degrees-of-freedom kinematic chains with revolving joints and, similarly, Lechuga-Gutierrez et al.<sup>97</sup> present a set of generalized iterative algorithms for these mechanisms. A planar three-revolute (3R) serial chain motion generation by establishing the relative kinematics model based on CGA is introduced in Lei Wang et al.<sup>98</sup>. Finally, a new coordinate-invariant geometric constraint equation for 3-RPR planar parallel mechanisms is elaborated by Ganmin Zhu et al. in<sup>99</sup>.

Recently, specific GA algorithms focusing on various robotic mechanisms have been developed. Thus, for instance, Ge Li et al.<sup>100</sup> propose a novel 6-DOF platform for pose adjustment of heavy equipment. Also, kinematics of various types of serial robots have been designed in GA, see Zhipeng Tong et al.<sup>101</sup> for a study of spherical 2R mechanism in order to design a seedling pick-up mechanism, or Lingmin Xu<sup>102</sup> for a design of a new parallel manipulator with two rotational and one translational motion. Dynamics of novel robotic mechanisms is also solved in GA, see e.g. Zchieng Song et al.<sup>103</sup> for dynamic modeling and generalized force analysis of a three-(rotation pair)-(prismatic pair)-spherical pair (3-RPS) parallel mechanism.

In combination with questions of kinematics, associated problems are solved in GA too. For instance, singularities of serial robots are tackled by Isiah Zaplana et al. in<sup>104</sup> and binocular visual control for a 6-DOF robotic manipulator is described by

Marek Stodola et al. in <sup>105</sup>. Apart from the control for specific robots or associated problems, there is a group of papers that generalize the concept of kinematic chains in terms of GA. Thus Isiah Zaplana et al. search for closed-form solutions for the inverse kinematics of serial robots in <sup>106</sup>, and Bahar Kalkan et al. in <sup>107</sup> introduce the Study variety of conformal (CGA) kinematics, i.e. a generalization of the Study quadric model of rigid body kinematics.

To spread the information about GA, authors integrate new methods and models with standard approaches. Thus, the connection between GA and control on Lie groups is discussed by Jaroslav Hrdina et al. in <sup>108</sup> and Eduardo Bayro-Corrochano et al. in <sup>109</sup> show the importance of the Hamiltonian in control theory in terms of GA.

A relatively new type of kinematics studied by means of GA is the one of machining with three- or five axis. Although the algorithms are not too complicated, the spread of GA to this area is interesting. In Yongxue Chen et al. <sup>110</sup> the authors propose an approach that can generate a smooth tool-path that passes through the discrete cutter locations given in the original linear segments analytically and discuss the impact on accuracy. Another approach to volumetric accuracy of a five-axis machine tool can be found in Barbora Navrátilová et al. <sup>111</sup>.

Apart from controlling algorithms, hardware and computational aspects are also treated. Thus, Tarik Uzinović et al. <sup>112</sup> present a combination of two methods that can be effectively combined for control of electrical machines. For computational acceleration, Salvatore Vitabile et al. <sup>113</sup> propose a novel embedded coprocessor for accelerating CGA operations in robotic tasks. Furthermore, Hebet Sira Ramírez et al. in <sup>114</sup> present a general method, based on GA, for the synthesis of sliding mode controllers in SISO switched nonlinear systems, and the same authors in <sup>115</sup> examine the sliding mode existence conditions, the switching policy, the invariance conditions, the associated equivalent control, and the characterization of ideal sliding dynamics of sliding mode controllers from a GA viewpoint.

In <sup>116</sup> the authors exploit GA for conics ability to represent transformed objects such as rotated ellipses to propose a novel algorithm for finding the optimal control of a switched dynamical systems with purely imaginary eigenvalues. Such a geometric approach guarantees the optimality of the switching path and eliminates the need for any solver, thus yielding results with minimal numerical errors.

## 14 | CONCLUSION

We hope to provide with our survey a comprehensive insight for the reader on new applications of Clifford's geometric algebras published during the last 12 months, prior to writing it (mid-January 2023). The use of the search engine Dimensions.ai ensures a relatively complete overview of the relevant literature, where apart from scientific journal articles, preprints, book chapters (proceedings) and books are also included.

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## Author contributions

E.H. did the Dimensions.ai search, clustering, wrote abstract, introduction, conclusion and the sections on Physics, Image and Signal Processing, and Electric Engineering. P.V. wrote the sections on Mathematics and Computing and on Control and Robotics. M.K. wrote all other sections under the guidance and supervision of G.P.

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## Conflict of interest

The authors declare no potential conflict of interests.

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