Voss-Weyl divergence formula based field equations

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Abstract

Field equations that have sensible classic limes for three types of symmetries are obtained from simple divergence formula for curvilinear coordinates.

Motivation

Einstein field equations [(Einstein, 1915)] were tremendous work and Schwarzschild soon solved them for most important case. But soon after they were solved for axially and plane symmetric problems and solutions had no classical limes.
On the other hand even more complicated field equations were investigated since. One should try to find simple solution if such exist.
We know that Poisson equation is classical limes of field equations. In Scwarzschild-de Sitter solution of Einstein field equation for g_{00} for vacuum is just Poisson equation with constant source. Classical limes suggests that same should be valid for all problems disregarding their symmetry
Several other difficulties are also present but out of scope of this paper.

Divergence law

For special relativity we have four-divergence law:

$$div_{\mu} F^{\mu} = \frac{1}{\sqrt{h}} \partial_{\mu} \sqrt{h} F^{\mu\nu} \dot{x}_{\nu} = \rho$$

(1)

that is equivalent to Gauss law- first filed equation someone made. In general relativity where space is curvilinear one should expect:

$$div_{\mu} F^{\mu} = \rho$$

(2)

But in this case we need to employ curvilinear divergence.Voss-Weyl formula for divergence gives:

$$div_\mu F^\mu = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} F^\mu = \rho$$

(3)

where g is absolute value of metric determinant. Since contra-variant four-force F^μ is expressed with Chrystoffel symbols we get:

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \Gamma^\mu{}_{\beta\alpha} \dot{x}^\alpha \dot{x}^\beta = \rho$$

(4)

This is field scalar equation and from it can be obtained field vector and field tensor equation by variation with respect to $\dot{x}^\nu$, but it turns out that they don't agree with Schwarzschild solution without inserting $g^{\alpha\beta}$. One tensor equation that agrees with Schwarzschild solution is:

$$\sum_\mu \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\nu\nu} G^\mu{}_{\nu\nu} = \sum_\mu \frac{1}{\sqrt{h}} \partial_\mu \sqrt{h} h^{\nu\nu} H^\mu{}_{\nu\nu}$$

(5)

Summation only apply to μ, but if it is done over one more index we get field vector and in case of two indexes field scalar equation.Chrystoffel symbols for metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ are written as $G^\sigma{}_{\mu\nu}$ and $H^\sigma{}_{\mu\nu}$ describing curved and flat space. In spherically symmetric problems h_{22} and h_{33} are not constants even in flat space, they are responsible for centrifugal force.

If we take as field tensor equation following:

$$\sum_\mu \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\nu\nu} G^\mu{}_{\nu\nu} - \sum_\mu \frac{1}{\sqrt{h}} \partial_\mu \sqrt{h} h^{\nu\nu} H^\mu{}_{\nu\nu} = \Lambda$$

(6)

we get de Sitter-Schwarzschild solution.

Spherical symmetry,stationary,diagonal metric

In this problem, if we have spherical symmetry preserved we have three nontrivial differential equations:

Solution that satisfies them is:

$$g_{tt} = -g^{rr} = 1 - \frac{r_s}{r} + \frac{\Lambda r^2}{3}$$

(7)

Axial symmetry,stationary, diagonal metric

In this problem, if we have axial symmetry preserved we have three nontrivial differential equations:

Solution that satisfies them is:

$$g_{tt} = -g^{\rho\rho} = 1 - \lambda \ln \frac{\rho}{\rho_0} + \frac{\Lambda \rho^2}{2}$$

(8)

Planar symmetry,stationary, diagonal metric

In this problem, if we have axial symmetry preserved we have three nontrivial differential equations:

Solution that satisfies them is:

$$g_{tt} = -g^{zz} = 1 - \sigma z + \Lambda z^2$$

(9)

More general approach

Former approach was based on inserting inverse metric and eliminating 4-velocities. Let us start again trying to justify that step.
Let us start with geodesic equation and subtract and add same term describing fictitious forces present in absence of gravitation:

$$\ddot{x}^{\mu} = [G^{\mu}{}_{\beta\alpha}\dot{x}^{\alpha}\dot{x}^{\beta} - H^{\mu}{}_{\beta\alpha}\mathring{x}^{\alpha}\mathring{x}^{\beta}] + H^{\mu}{}_{\beta\alpha}\mathring{x}^{\alpha}\mathring{x}^{\beta}$$

(10)

where $\dot{x}$ derivative over self-times in presence and $\mathring{x}$ in absence of gravity.
In classical mechanics gravity is velocity independent. In circular orbits radial acceleration is zero and any increase from zero increases circular velocity, keeping gravitational force same.Since divergence of gravitational field is zero outside source, divergence of two other terms is zero two, although they are not same.

$$\nabla \cdot [ma - F_{cf}] = 0$$

(11)

Same way we could expect:

$$0 = div_{\mu}[\ddot{x}^{\mu} - H^{\mu}{}_{\beta\alpha}\mathring{x}^{\alpha}\mathring{x}^{\beta}] = \frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g}[\ddot{x}^{\mu} - H^{\mu}{}_{\beta\alpha}\mathring{x}^{\alpha}\mathring{x}^{\beta}]$$

(12)

thus inserting condition:

$$\frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g}[G^{\mu}{}_{\beta\alpha}\dot{x}^{\alpha}\dot{x}^{\beta} - H^{\mu}{}_{\beta\alpha}\mathring{x}^{\alpha}\mathring{x}^{\beta}] = 0$$

(13)

Now we can look on surfaces when only one component of four velocity is not zero:

end come to:

$$\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}[\frac{G^\mu{}_{\nu\nu}}{g_{\nu\nu}} - \frac{H^\mu{}_{\nu\nu}}{h_{\nu\nu}}] = 0$$

(14)

Since this equation doesn't depend on velocities it is valid for all velocities.It doesn't require summation over ν and is equivalent with equation (5) if metric determinants are equal $g = h$ and coordinates are orthogonal so inverse metric element $g^{\nu\nu}$ is inverse scalar to metric element $g_{\nu\nu}$.

Cosmological constant

In contrast to equation (16) is equation (6) that can be rewritten as:

$$\frac{1}{\sqrt{\tilde{g}}}\tilde{\partial}_\mu\sqrt{\tilde{g}}[\frac{\tilde{G}^\mu{}_{\nu\nu}}{\tilde{g}_{\nu\nu}} - \frac{H^\mu{}_{\nu\nu}}{h_{\nu\nu}}] = \Lambda$$

(15)

We can then try to find divergence free solution by adding scalar function $\tilde{f}$ into lagrangian that satisfies:

$$\frac{1}{\sqrt{\tilde{g}}}\tilde{\partial}_\mu\sqrt{\tilde{g}}\tilde{\partial}^\mu\tilde{f} = -\Lambda$$

(16)

but remembering that it alters (14):

so we have:

$$\frac{1}{\sqrt{\tilde{g}}}\tilde{\partial}_\mu\sqrt{\tilde{g}}[\tilde{\partial}^\mu\tilde{f} + \frac{(1-2\tilde{f})\tilde{G}^\mu{}_{\nu\nu}}{\tilde{g}_{\nu\nu}} - \frac{H^\mu{}_{\nu\nu}}{h_{\nu\nu}}] = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}[\frac{G^\mu{}_{\nu\nu}}{g_{\nu\nu}} - \frac{H^\mu{}_{\nu\nu}}{h_{\nu\nu}}] = 0$$

(17)

or alternatively abandon divergence-free condition.

Maxwell-type field equations

Equations of section 7. give physical solutions to gravitational static problems are just D'Alamberian acting on g_{tt} but one would expect Maxwell-type field vector equations, something like:

Field vector equation is:

$$\frac{\dot{x}^{\nu}}{\sqrt{g}}\partial_{\mu}\sqrt{g}[\frac{G^{\mu}{}_{\nu\nu}}{g_{\nu\nu}} - \frac{H^{\mu}{}_{\nu\nu}}{h_{\nu\nu}}] := \frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g}G^{\mu\nu} = \rho\dot{x}^{\nu}$$

(18)

Main difference between this equation and Maxwell's is that gauge term is twice larger, which is due fact that starting lagrangian for motion of particle in electromagnetic field is two body problem, and here we started from one body lagrangian analogue of releative particle in center of mass problem.

Conclusion

In this paper simplified alternative for field equations for gravity were given, that give physical solution for axial and planar problems.Some assertions as how Maxwell type equation for gravity can be obtained are discussed.

Acknowledgement

To my family and friends.

References

Die Feldgleichungen der Gravitation. (German) [The Field Equations of Gravitation]. (1915). *j-S-B-PREUSS-AKAD-WISS-2*, 844–847.