Generalized Formable integral transform on Ψ-Hilfer-Prabhakar fractional derivatives and its applications

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Abstract: This work is focused on the study of the generalized forms of the fractional derivatives of Reimann-Liouville, Caputo and Hilfer, in terms of Ψ functions. The fractional derivatives of Ψ-Prabhakar, Ψ-Hilfer-Prabhakar, and its regularized form are also described in terms of Ψ-Mittage-Leffler type functions. These are then used to solve a number of Cauchy type equations involving Ψ-Hilfer-Prabhakar fractional derivatives and their regularized form, including the generalized fractional free electron laser equation.
Keywords and phrases: Ψ-Prabhakar integral, Ψ-Hilfer-Prabhakar derivatives, Ψ-Formable integral transform, Fourier transform, Ψ-Mittage-Leffler functions.

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1 Introduction

Recently, some researchers have focused on the generalization of integral transformations in the context of Ψ-fractional operators and fractional calculus, as seen in various studies such as Magar et al. [38], Hamoud [37], and Sousa Oliveira et al. [39]. The Hilfer-Prabhakar fractional derivative operator has been widely used by many scholars to model physical phenomena because of its special properties, particularly when combined with various integral transforms, including those from Fourier, Elzaki, Laplace, and others. These integral transform techniques are crucial as they provide a quick solution for a range of mathematical models and initial value problems that arise in differential equations. Ghadle et al. proposed a novel Sumudu-type integral transform in their study [40], which was then used to solve certain applications involving conformable derivative. Sousa and Oliviera presented the Ψ-Hilfer fractional derivative in [39] as a unique fractional derivative in the context of the Ψ-fractional operator. Magar et al. in [38] introduced a number of novel concepts of fractional derivatives in the context of Ψ-fractional operators, such as ”Ψ-Prabhakar integral”, ”Ψ-Prabhakar derivative", and "Ψ-Hilfer-Prabhakar fractional derivatives", and generalized integral transforms like "Laplace" and "Sumudu" to it.

Saadeh et al. [18] proposed a new integral transform named the Formable integral transform in their study in 2021. The main objective of the authors [18] was to use this transform to solve ordinary and partial differential equations. The Formable integral transform has stronger connections with Laplace, Sumudu, Elzaki and other transformations.

Recently, Sachin et al. [1] proposed a new integral transform called the Ψ-Shehu transform, which is a generalization of the Shehu integral transform that incorporates the advantages of the Ψ-function. They used this transform to solve various Cauchy type fractional differential equations involving the Ψ-Hilfer-Prabhakar fractional derivative and its regularized form.

The main objective of this study is to introduce a new generalization of the Formable transform known as the Ψ-Formable transform and to study its properties based on Ψ-functions, such as the Ψ-Riemann-Liouville, Ψ-Caputo, Ψ-Hilfer, Ψ-Prabhakar integral, derivative, and its regularized version in terms of the Ψ-Mittag-Leffler function. The study will then use the Ψ-Formable transform to solve various Cauchy-type problems involving the Ψ-Hilfer-Prabhakar fractional derivative and its regularized form, including the generalised fractional free electron laser equation and the space-time fractional advection-dispersion equation.

2 Definitions and preliminaries

Definition 2.1. *let ξ be an integrable function defined in $[a,b]$ and $\varrho \in \mathbb{R}^+$ such that $-\infty \leq a < b \leq \infty, n = \varrho + 1$ and $\Psi \in AC^1[a,b]$ be non decreasing function such that $\Psi'(t) \neq 0$ for all $t \in [a,b]$, consequently, the following are a few definitions of Ψ fractional integral and derivatives [1, 3, 17, 37, 41–44, 46].*

• *the Ψ-Reimann Liouville fractional integral of a function $\xi(t)$ is defined as*

$$\mathcal{I}_0^{\varrho,\Psi}\xi(t) = \frac{1}{\Gamma(\varrho)}\int_0^{\infty}(\Psi(t)-\Psi(r))^{\varrho-1}\Psi'(r)\xi(r)dr \tag{1}$$

• *the Ψ-Reimann Liouville fractional derivative of a function $\xi(t)$ is defined as*

$$D_0^{\varrho,\Psi}\xi(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_0^{n-\varrho,\Psi}\xi(r)dr \tag{2}$$

• *the Ψ-Caputo fractional derivative of a function $\xi(t)$ is defined as*

$${}^{C}\mathcal{D}_0^{\varrho,\Psi}\xi(t) = \mathcal{I}_0^{n-\varrho,\Psi}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi(r)dr \tag{3}$$

• *the Ψ Hilfer derivative of a function $\xi(t)$ is*

$$\mathcal{D}_0^{\varrho,\rho,\Psi}\xi(t) = \mathcal{I}_0^{n-\varrho,\Psi}\xi(t)\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \mathbf{I}_0^{(1-\varrho)(1-\rho),\Psi}\xi(t) \tag{4}$$

• *the Ψ-Prabhakar fractional integral and derivative of $\xi(t)$ are defined as follows:*

$$\begin{aligned}(\mathcal{I}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi)(t) &= \int_0^t (\Psi(t)-\Psi(r))^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi(\Psi(t)-\Psi(r))^{\varrho})\xi(r)dr \\ &= (e^{\gamma}_{\varrho,\rho,\varpi}*_{\Psi})(t)\end{aligned} \tag{5}$$

where $*_\Psi$ denote the convolution operator for $\varrho, \rho, \gamma \in (C), Re(\varrho), Re(\rho) > 0$

$$\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(t) \tag{6}$$

• *the Ψ-regularised Prabhakar fractional derivative of $\xi(t)$ is*

$$^{C}\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t) = \mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi(t) \tag{7}$$

• *the Ψ-Hilfer Prabhakar fractional derivative of $\xi(t)$ is*

$$\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t) = \left(\mathcal{I}^{-\gamma\nu,\Psi}_{\varrho,\nu(n-\rho),\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n (\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi)\right)(t) \tag{8}$$

• *the regularised version of Ψ-Hilfer Prabhakar fractional derivative of $\xi(t)$ is*

$$\begin{aligned} ^{C}\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t) &= \left(\mathcal{I}^{-\gamma\nu,\Psi}_{\varrho,\nu(n-\rho),\varpi,0^+}\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi\right)(t) \\ &= \mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi(t) \end{aligned} \tag{9}$$

Definition 2.2. *[4] For $\varrho, \rho, \gamma \in \mathbb{C}$, three parameter Mittage-Leffler function of Prabhakar is*

$$E^{\gamma}_{\varrho,\rho}(s) = \sum_{m=0}^{\infty}\frac{(\gamma)_m}{\Gamma(\varrho m+\rho)}\frac{(s)^m}{m!} \tag{10}$$

where $(\gamma)_m$ is the Pochhammer symbol

$$(s)_0 = 1, (s)_n = s(s+1)(s+2) - - - - - - - - - (s+m-1), \ \ m \in \mathbb{N}$$

Definition 2.3. *[1]: Let $\xi(t)$ be a real valued function defined from $[0,\infty)$ to $\mathbb{R}$ and Ψ is the non decreasing function such that $\Psi(0) = 0$, then the Ψ-Shehu transform of the function $\xi(t)$ is denoted by $\mathcal{S}_\Psi[\xi(t)]$ and defined by*

$$\mathcal{S}_\Psi[\xi(t)] = \mathcal{V}_\Psi(X,v) = \int_0^\infty \xi(t)e^{\left(\frac{-X\Psi(t)}{v}\right)}\Psi'(t)dt, \tag{11}$$

Definition 2.4. *[18]: The Formable integral transform denoted by $\mathcal{B}(X,v)$ for the function $\xi(t)$, which is given as:*

$$\begin{aligned} \mathcal{F}[\xi(t)] = \mathcal{B}(X,v) = X\int_0^\infty \xi(vt)exp(-Xt)dt \\ = \frac{X}{v}\int_0^\infty exp\left(\frac{-Xt}{v}\right)\xi(t)dt, X \in (\lambda_1,\lambda_2), \end{aligned} \tag{12}$$

over the set of functions

$$\mathcal{Z} = \left\{\xi(t) \ : \ \exists\, N, \ 0 < \lambda_1, \lambda_2, 0 < k, |\xi(t)| \leq N\, e^{\left(\frac{t}{\lambda_j}\right)}, if \ t \in (-1)^j \times [0,\infty)\right\},$$

The integral transform (12) exists for all $\xi(t) > k$.

- **Formable-Sumudu duality [18]:** Let $G(v)$ be the Sumudu transform of $\xi(t)$ then

$$\mathcal{B}(1, v) = G(v) \tag{13}$$

- **Formable-Shehu duality [18]:** Let $\mathcal{V}(X, v)$ be the Shehu transform of $\xi(t)$ then

$$\mathcal{B}(X, v) = \frac{X}{v}\mathcal{V}(X, v) \tag{14}$$

Definition 2.5. *[1]: Suppose $\mathcal{SH}_{\Psi}(X, v)$ is the Ψ-Shehu transform of $\xi(t)$, then the Ψ-Shehu transform of n^{th} derivative $\xi^{(n)}(t)$ is denoted by $\mathcal{SH}_{\Psi n}(X, v)$ and*

$$\begin{aligned}\mathcal{SH}_{\Psi n}(X, v) &= \mathcal{SH}_{\Psi}[\xi^{(n)}(t)] \\ &= \left(\frac{X}{v}\right)^{n} \mathcal{SH}_{\Psi}(X, v) - \sum_{k=0}^{n-1} \left(\frac{X}{v}\right)^{n-(k+1)} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{k} \xi(0), n \geq 0\end{aligned} \tag{15}$$

Definition 2.6. *[19]: The Ψ-convolution of ξ and χ defined over $[0, T]$ is*

$$(\xi *_{\Psi} \chi) = \int_{0}^{\Psi^{-1}\Psi(t)=t} \xi(\Psi^{-1}(\Psi(t) - \Psi(r)))\chi(r)\Psi'(r)dr \tag{16}$$

where ξ and χ are the piecewise continuous functions over $[0, T]$

Definition 2.7. *[45]: Let a function f from $[0, \infty)$ to $\mathbb{R}$ of Ψ-exponential order c, if $\exists$ a positive constant N such that $\forall\ t > T$, then*

$$|\xi(t)| \leq Ne^{c\Psi(t)} \tag{17}$$

Definition 2.8. *[19]: For $0 < \varrho < 1$ and $\varpi \in \mathbb{C}$ such that $Re(\varrho) > 0,\ Re(\rho) > 0),\ Re(\gamma) > 0$. The Shehu transform of Mittage-Leffler function $t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi t^{\varrho})$ is given by*

$$SH\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi t^{\varrho})\right](X, v) = \left(\frac{X}{v}\right)^{-\rho} \left(1 - \varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma} \tag{18}$$

Lemma 2.1. *Let $0 < \varrho < 1$ and $\varpi \in \mathbb{C}$ such that $Re(\varrho) > 0,\ Re(\rho) > 0),\ Re(\gamma) > 0$. The Formable transform of Mittage-Leffler type function $t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi t^{\varrho})$, is given by*

$$\mathcal{F}\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi t^{\varrho})\right](X, v) = \left(\frac{X}{v}\right)^{1-\rho} \left(1 - \varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma}, \tag{19}$$

Proof. Using equation (18) and the duality of Formable-Shahu transform (14) we got the desired result

$$\mathcal{F}\left[t^{\rho-1}E^{\gamma}_{\varrho,\rho}(\varpi t^{\varrho})\right](X, v) = \left(\frac{v}{X}\right)^{\rho-1} \left(1 - \varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma}$$

□

3 Main Results:

Definition 3.1. *let Ψ be a non negative rising function with the property that $\Psi(0) = 0$, defined from $[0, \infty)$ to $\mathbb{R}$ and let ξ be a real valued function. The exponential order Ψ-Formable transform of ξ thus denoted by $\mathcal{F}_{\Psi}[\xi(t)]$ is defined as*

$$\mathcal{F}_{\Psi}[\xi(t)] = \mathcal{B}_{\Psi}(X, v) = \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\xi(t)\Psi'(t)dt \tag{20}$$

Remark: If we take the function $\Psi(t) = t$, we obtain the Formable transform, which was studied in the publication [18].

Theorem 3.1. *Suppose $\xi(t)$ be a piecewise continuous function in evry interval $\rho \geq t \geq 0$ of Ψ-exponential order, then the Ψ-Formable transform of $\xi(t)$ exist for $0 < c < t$.*

Proof. Using the Ψ-Formable transform definition and equation (17) for any non negative number c, we obtain

$$\begin{aligned}
|\mathcal{F}_{\Psi}\xi(t)| &= \left|\frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\xi(t)\Psi'(t)dt\right| \\
&\leq \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)|\xi(t)|\Psi'(t)dt \\
&\leq N\frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)e^{c\Psi(t)}\Psi'(t)dt \\
&= N\frac{X}{v}\int_0^{\infty} exp\left(-\frac{X-vc}{v}\Psi(t)\right)\Psi'(t)dt \\
&= \frac{X}{v}\left[\frac{-v}{X-vc}(0-1)\right] \\
&= \frac{XN}{X-vc}
\end{aligned}$$

□

Property 1: Suppose a and b are non-zero arbitrary constants in $\mathbb{R}$ and the functions $a\xi(t)$ and $b\chi(t)$ are in $\mathcal{Z}$ then $a\xi(t) + b\chi(t) \in \mathcal{Z}$, such that

$$\mathcal{F}_{\Psi}[a\xi(t) + b\chi(t)] = a\mathcal{F}_{\Psi}[\xi(t)] + b\mathcal{F}_{\Psi}[\chi(t)] \tag{21}$$

Proof. Using the Ψ-Formable transform definition (20), we obtain

$$\begin{aligned}
&\mathcal{F}_{\Psi}[a\xi(t) + b\chi(t)] \\
&\quad= \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)(a\xi(t) + b\chi(t))\Psi'(t)dt \\
&\quad= \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)(a\xi(t))\Psi'(t)dt + \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)(b\chi(t))\Psi'(t)dt \\
&\quad= a\frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\xi(t)\Psi'(t)dt + b\frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\chi(t)\Psi'(t)dt \\
&\quad= a\mathcal{F}_{\Psi}[\xi(t)] + b\mathcal{F}_{\Psi}[\chi(t)]
\end{aligned} \tag{22}$$

□

Property 2: Suppose $\xi(t) = \Psi(t)^{\rho-1}$, then the Ψ-Formable transform of $\Psi(t)^{\rho-1}$ is provided by

$$\mathcal{F}_{\Psi}[\xi(t)] = \left(\frac{v}{X}\right)^{\rho-1} \Gamma(\rho) \tag{23}$$

Proof. By using the Ψ-Formable transform definition (20), we obtain

$$\begin{aligned}
\mathcal{F}_{\Psi}[\xi(t)] &= \mathcal{F}[\Psi(t)^{\rho-1}] \\
&= \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\Psi(t)^{\rho-1}\Psi'(t)dt \\
&= \frac{X}{v}\frac{v}{X}(\rho-1)\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\Psi(t)^{\rho-2}\Psi'(t)dt \\
&= \frac{v}{X}(\rho-1)(\rho-2)\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\Psi(t)^{\rho-3}\Psi'(t)dt \\
&= \left(\frac{v}{X}\right)^2(\rho-1)(\rho-2)(\rho-3)\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)\Psi(t)^{\rho-4}\Psi'(t)dt \\
&\vdots \\
&= \left(\frac{v}{X}\right)^{\rho-1}(\rho-1)! \\
&= \left(\frac{v}{X}\right)^{\rho-1}\Gamma(\rho)
\end{aligned}$$

□

Property 3: Suppose $\xi(t) = exp(\rho\Psi(t))$, then the Ψ- Formable transform of $exp(\rho\Psi(t))$, provided by

$$\mathcal{F}_{\Psi}[\xi(t)] = \frac{X}{X-\rho v} \tag{24}$$

Proof. By using the Ψ-Formable transform definition (20), we obtain

$$\begin{aligned}
\mathcal{F}_{\Psi}[\xi(t)] &= \mathcal{F}_{\Psi}[exp(\rho\Psi(t))] \\
&= \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X}{v}\Psi(t)\right)exp(\rho\Psi(t))\Psi'(t)dt \\
&= \frac{X}{v}\int_0^{\infty} exp\left(-\frac{X-\rho v}{v}\Psi(t)\right)\Psi'(t)dt \\
&= \frac{X}{v}\frac{v}{X-\rho v} \\
&= \frac{X}{X-\rho v}
\end{aligned}$$

□

Property 4: Suppose $\xi(t) = \Psi(t)exp(\rho\Psi(t))$, then the Ψ-Formable transform of $\Psi(t)exp(\rho\Psi(t))$, provided by

$$\mathcal{F}_{\Psi}[\xi(t)] = \frac{Xv}{(X-\rho v)^2} \tag{25}$$

Proof. By using the Ψ-Formable transform definition (11), we obtain

$$\begin{aligned}
\mathcal{F}_\Psi[\xi(t)] &= \mathcal{F}_\Psi[\Psi(t)exp(\rho\Psi(t))] \\
&= \frac{X}{v}\int_0^\infty exp\left(-\frac{X}{v}\Psi(t)\right)(\Psi(t)exp(\rho\Psi(t)))\Psi'(t)dt \\
&= \frac{X}{v}\int_0^\infty \Psi(t)exp\left(-\frac{X-\rho v}{v}\Psi(t)\right)\Psi'(t)dt \\
&= \frac{X}{v}\left[\frac{-v}{X-\rho v}\Psi(t)exp\left(-\frac{X-\rho v}{v}\Psi(t)\right)\right]_0^\infty \\
&- \frac{X}{v}\left[\frac{-v^2}{(X-\rho v)^2}exp\left(-\frac{X-\rho v}{v}\Psi(t)\right)\right]_0^\infty \\
&= \frac{X}{v}\left[\frac{v^2}{(X-\rho v)^2}exp\left(-\frac{X-\rho v}{v}\Psi(t)\right)\right]_0^\infty \\
&= \frac{X}{v}\frac{v^2}{(X-\rho v)^2} \\
&= \frac{Xv}{(X-\rho v)^2}
\end{aligned}$$

□

Property 5: Suppose $\xi(t) = Sin(\lambda\Psi(t))$, then the Ψ-Formable transform of $Sin(\lambda\Psi(t))$, provided by

$$\mathcal{F}_\Psi[\xi(t)] = \frac{Xv\lambda}{X^2+\lambda^2v^2} \tag{26}$$

Proof. By using the Ψ-Formable transform definition (11), we obtain

$$\begin{aligned}
\mathcal{F}_\Psi[\xi(t)] &= \mathcal{F}_\Psi[Sin(\lambda\Psi(t))] \\
&= \frac{X}{v}\int_0^\infty exp\left(-\frac{X}{v}\Psi(t)\right)Sin(\lambda\Psi(t))\Psi'(t)dt \\
&= \frac{X}{v}\frac{e^{-\frac{X}{v}\Psi(t)}}{\left(\frac{X}{v}\right)^2+\lambda^2}\left[\frac{-X}{v}Sin(\lambda\Psi(t))-\lambda Cos(\lambda\Psi(t))\right]_0^\infty \\
&= \frac{Xv^2}{v(X^2+\lambda^2v^2)}(0+\lambda) \\
&= \frac{Xv\lambda}{X^2+\lambda^2v^2}
\end{aligned}$$

□

Property 6: Suppose $\xi(t) = Cos(\lambda\Psi(t))$, then the Ψ-Formable transform of $Cos(\lambda\Psi(t))$, provided by

$$\mathcal{F}_\Psi[\xi(t)] = \frac{X^2}{X^2+\lambda^2v^2} \tag{27}$$

Proof. By using the Ψ-Formable transform definition (11), we obtain

$$\begin{aligned}\mathcal{F}_\Psi[\xi(t)] &= \mathcal{F}_\Psi[Cos(\lambda\Psi(t))]\\ &= \frac{X}{v}\int_0^\infty exp\left(-\frac{X}{v}\Psi(t)\right)Cos(\lambda\Psi(t))\Psi'(t)dt\\ &= \frac{X}{v}\frac{e^{-\frac{X}{v}\Psi(t)}}{\left(\frac{X}{v}\right)^2+\lambda^2}\left[\frac{-X}{v}Cos(\lambda\Psi(t))+\lambda Sin(\lambda\Psi(t))\right]_0^\infty\\ &= \frac{Xv^2}{v(X^2+\lambda^2v^2)}\frac{X}{v}\\ &= \frac{X^2}{X^2+\lambda^2v^2}\end{aligned}$$

□

Lemma 3.2. *Soppose the Ψ-Formable transform of $\xi(t)$ and $\chi(t)$ are $M(X,v)$ and $N(X,v)$ respective, then the Ψ-Formable convolution of $(\xi *_\Psi \chi)$, provided by*

$$\mathcal{F}_\Psi[(\xi *_\Psi \chi)] = \frac{v}{X}M(X,v)N(X,v) \tag{28}$$

Proof. By using the Ψ-Formable transform definition (11) and equation (16), we obtain

$$\begin{aligned}&\mathcal{F}_\Psi[(\xi *_\Psi \chi)]\\ &\quad= \frac{X}{v}\int_0^\infty exp\left(-\frac{X}{v}\Psi(t)\right)(\xi *_\Psi \chi)\Psi'(t)dt\\ &\quad= \frac{X}{v}\int_0^\infty exp\left(-\frac{X}{v}\Psi(t)\right)\left(\int_0^t \xi(\Psi^{-1}(\Psi(t)-\Psi(r)))\chi(r)\Psi'(r)dr\right)\Psi'(t)dt\end{aligned}$$

$$\begin{aligned}\mathcal{F}_\Psi[(\xi *_\Psi \chi)] = \frac{X}{v}\int_0^\infty & exp\left(-\frac{X\Psi(t)-\Psi(t)+\Psi(t)}{v}\right)\\ &\times\left(\int_0^\infty \xi(\Psi^{-1}(\Psi(t)-\Psi(r)))\Psi'(r)dr\right)\chi(r)\Psi'(t)dt\end{aligned}$$

by reversing the integration order and replacing the formula above, we obtain

$$\begin{aligned}\mathcal{F}_\Psi[(\xi *_\Psi \chi)] &= \frac{X}{v}\int_0^\infty e^{\frac{-X}{v}\Psi(t)}\chi(r)\Psi'(r)dr\int_0^\infty e^{\frac{-X}{v}\Psi(\eta)}\Psi'(\eta)\xi(\eta)d\eta\\ &= \frac{X}{v}\frac{v}{X}\mathcal{F}_\Psi[\chi(t)]\times\frac{v}{X}\mathcal{F}_\Psi[\xi(t)]\\ &= \frac{v}{X}N(x,v)M(X,v)\end{aligned}$$

□

Lemma 3.3. *Suppose $\mathcal{B}_\Psi(X,v)$ is the Ψ-Formable transform of $\xi(t)$, then the Ψ-Formable transform of n^{th} derivative $\xi^{(n)}(t)$ is denoted by $\mathcal{B}_{\Psi n}(X,v)$ and*

$$\mathcal{F}_{\Psi n}(X,v) = \mathcal{F}_\Psi[\xi^{(n)}(t)] = \left(\frac{X}{v}\right)^n\mathcal{B}_\Psi(X,v) - \sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{n-k}\xi^{(k)}(0), n \geq 0 \tag{29}$$

or equivalently,

$$\mathcal{F}_{\Psi n}(X,v)=\mathcal{F}_{\Psi}[\xi^{(n)}(t)]=\left(\frac{v}{X}\right)^{-n}\mathcal{B}_{\Psi}(X,v)-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\xi^{(k)}(0) \tag{30}$$

Proof. Using equation (15) and the duality of Formable-Shahu transform (14) we got the desired result

$$\mathcal{F}_{\Psi n}(X,v)=\mathcal{F}_{\Psi}[\xi^{(n)}(t)]=\left(\frac{v}{X}\right)^{-n}\mathcal{B}_{\Psi}(X,v)-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\xi^{(k)}(0)$$

□

Definition 3.2. *Ψ-Formable integral transform on Ψ-**Riemann Liouville fractional integral** defined by*

$$\mathcal{F}_{\Psi}[(\mathcal{I}_0^{\varrho,\Psi}\xi(t))]=\left(\frac{v}{X}\right)^{\varrho}\mathcal{F}_{\Psi}[\xi(t)] \tag{31}$$

Proof. By using the definition of Ψ-Formable transform on Ψ-Reimann Liouville fractional integral (1) and equations (28), (23), we obtain

$$\begin{aligned}\mathcal{F}_{\Psi}\left[(\mathcal{I}_0^{\varrho,\Psi}\xi(t))\right]&=\mathcal{F}_{\Psi}\left[\frac{1}{\Gamma(\varrho)}(\Psi(t))^{\varrho-1}*_{\Psi}\xi(t)\right]\\&=\frac{1}{\Gamma(\varrho)}\frac{v}{X}\mathcal{F}_{\Psi}[\Psi(t)^{\varrho-1}]\mathcal{F}_{\Psi}[xi(t)]\\&=\frac{1}{\Gamma(\varrho)}\frac{v}{X}\left(\frac{v}{X}\right)^{\varrho-1}\Gamma(\varrho)\mathcal{F}_{\Psi}[\xi(t)]\\&=\left(\frac{v}{X}\right)^{\varrho}\mathcal{F}_{\Psi}[\xi(t)]\end{aligned}$$

□

Definition 3.3. *Ψ-Formable integral transform on Ψ-**Riemann Liouville fractional derivative** defined by*

$$\mathcal{F}_{\Psi}[(\mathcal{D}_0^{\varrho,\Psi}\xi(t))]=\left(\frac{v}{X}\right)^{-\varrho}\mathcal{F}_{\Psi}[\xi(t)]-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}(\mathcal{I}_0^{n-\varrho-k}\xi)(t)|_{t=0} \tag{32}$$

Proof. By using the definition of Ψ-Formable transform on Ψ-Reimann Liouville fractional derivative (2) and equations (29), (31), we obtain

$$\begin{aligned}\mathcal{F}_{\Psi}[(\mathcal{D}_0^{\varrho,\Psi}\xi(t))]&=\mathcal{F}_{\Psi}\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}_0^{n-\varrho,\Psi}\xi(t)dt\right]\\&=\left(\frac{v}{X}\right)^{-n}\mathcal{F}_{\Psi}(\mathcal{I}_0^{n-\varrho,\Psi}\xi)(t)-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{k}(\mathcal{I}_0^{n-\varrho,\Psi}\xi)(0)\\&=\left(\frac{v}{X}\right)^{-n}\left(\frac{v}{X}\right)^{n-\varrho}\mathcal{F}_{\Psi}[\xi(t)]-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\mathcal{D}_0^{k,\Psi}(\mathcal{I}_0^{n-\varrho,\Psi}\xi)(0)\\&=\left(\frac{v}{X}\right)^{-\varrho}\mathcal{F}_{\Psi}[\xi(t)]-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}(\mathcal{I}_0^{n-\varrho-k,\Psi}\xi)(0)\end{aligned}$$

□

Definition 3.4. Ψ*-Formable transform on* Ψ***-Cputo fractional derivative*** *defined by*

$$\mathcal{F}_\Psi[({}^C\mathcal{D}_0^{\varrho,\Psi}\xi(t))] = \left(\frac{v}{X}\right)^{-\varrho}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}(\mathcal{D}_0^{k,\Psi}\xi)(t)|_{t=0} \tag{33}$$

Proof. By using the definition of Ψ-Formable transform on Ψ-Caputo fractional derivative (3) and equations (31), (29), we get

$$\begin{aligned}
\mathcal{F}_\Psi[({}^C\mathcal{D}_0^{\varrho,\Psi}\xi(t))] &= \mathcal{F}_\Psi\left[\mathcal{I}_0^{n-\varrho,\Psi}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\xi(t)dt\right]\\
&= \left(\frac{v}{X}\right)^{n-\varrho}\mathcal{F}_\Psi\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\xi(t)dt\right]\\
&= \left(\frac{v}{X}\right)^{n-\varrho}\left[\left(\frac{v}{X}\right)^{-n}\mathcal{F}_\Psi[\xi(t)]\right] - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}(\mathcal{D}^{k,\Psi}\xi)(0)\\
&= \left(\frac{v}{X}\right)^{-\varrho}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}(\mathcal{D}^{k,\Psi}\xi)(0)
\end{aligned}$$

□

Definition 3.5. Ψ*-Formable transform on* Ψ***-Hilfer fractional derivative*** *defined by*

$$\mathcal{F}_\Psi[(\mathcal{D}_0^{\rho,\nu,\Psi}\xi(t))] = \left(\frac{v}{X}\right)^{-\rho}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{\nu(n-\rho)+k-n}(\mathcal{I}_0^{(1-\nu)(n-\rho)-k,\Psi}\xi)(t)|_{t=0} \tag{34}$$

Proof. By using the definition of Ψ-Formable transform on Ψ-Hilfer fractional derivative (4) and equations (29), (31), we get

$$\begin{aligned}
\mathcal{F}_\Psi[(\mathcal{D}_0^{\rho,\nu,\Psi}\xi(t))] &= \mathcal{F}_\Psi\left[\mathcal{I}_0^{\nu(n-\rho),\Psi}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}_0^{(1-\nu)(n-\rho),\Psi}\xi(t)\right]\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\mathcal{F}_\Psi\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}_0^{(1-\nu)(n-\rho),\Psi}\xi(t)\right]\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\left[\left(\frac{v}{X}\right)^{-n}\mathcal{F}_\Psi(\mathcal{I}_0^{(1-\nu)(n-\rho),\Psi}\xi)(t)\right]\\
&- \left(\frac{v}{X}\right)^{\nu(n-\rho)}\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k\mathcal{I}_0^{(1-\nu)(n-\rho),\Psi}\xi(0)\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\left[\left(\frac{v}{X}\right)^{-n}\left(\frac{v}{X}\right)^{(1-\nu)(n-\rho)}\mathcal{F}_\Psi\xi(t)\right]\\
&- \left(\frac{v}{X}\right)^{\nu(n-\rho)}\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\mathcal{D}^{k,\Psi}\mathcal{I}_0^{(1-\nu)(n-\rho),\Psi}\xi(0)\\
&= \left(\frac{v}{X}\right)^{-\rho}\mathcal{F}_\Psi\xi(t) - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{\nu(n-\rho)+k-n}\mathcal{I}_0^{(1-\nu)(n-\rho)-k,\Psi}\xi(0)
\end{aligned}$$

□

Definition 3.6. Ψ*-Formable transform on* Ψ***- Prabhakar fractional integral*** *defined by*

$$\mathcal{F}_\Psi[(\mathcal{I}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t))] = \left(\frac{v}{X}\right)^{\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma}\mathcal{F}_\Psi[\xi(t)] \tag{35}$$

Proof. By using the Ψ-Formable transform definition on Ψ-Prabhakar fractional integral (5) and equation (28), we obtain

$$\begin{aligned}
\mathcal{F}_\Psi[(\mathcal{I}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t))] &= \mathcal{F}_\Psi\left[\int_0^x (\Psi(x)-\Psi(t))^{\rho-1}E^{\gamma,\Psi}_{\varrho,\rho}[\varpi(\Psi(x)-\Psi(t)))^{\varrho}]\right]\xi(t)\Psi'(t)dt \\
&= \frac{v}{X}.\mathcal{F}_\Psi\left[\Psi(t)^{\rho-1}E^{\gamma,\Psi}_{\varrho,\rho}(\varpi(\Psi(t))^{\varrho})\right]\times\mathcal{F}_\Psi[\xi(t)] \\
&= \frac{v}{X}\times\left(\frac{v}{X}\right)^{\rho-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma}\mathcal{F}_\Psi[\xi(t)] \\
&= \left(\frac{v}{X}\right)^{\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma}\mathcal{F}_\Psi[\xi(t)],
\end{aligned}$$

□

Definition 3.7. Ψ*-Formable transform on* Ψ***- Prabhakar fractional derivative*** *defined by*

$$\mathcal{F}_\Psi[\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}] = \left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\mathcal{I}^{-\gamma-k,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(t)|_{t=0} \tag{36}$$

Proof. By using the Ψ-Formable transform definition on Ψ-Prabhakar fractional derivative (6) and equations (29), (35) we get

$$\begin{aligned}
&\mathcal{F}_\Psi[\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t)](X,v) \\
&\quad= \mathcal{F}_\Psi\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(t)\right](X,v) \\
&\quad= \mathcal{F}_\Psi\left[\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n g(t)\right](X,v), \ \ where \ \ g(t) = \mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(t) \\
&\quad= \left(\frac{X}{v}\right)^n\mathcal{F}_\Psi[g(t)](X,v) - \sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{n-k}g^{(k)}(0), \ \ g^{(k)}(0) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(0) \\
&\quad= \left(\frac{X}{v}\right)^n\mathcal{F}_\Psi\left[(\xi * e^{\gamma,\Psi}_{\varrho,(n-\rho),\varpi})(t)\right](X,v) - \sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{n-k}g^{(k)}(0) \\
&\quad= \left(\frac{X}{v}\right)^n\left(\frac{v}{X}\right)^{n-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{n-k}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\xi(t)\Big|_{t=0} \\
&\quad= \left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{B}_\Psi(X,v) - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\left[\mathcal{D}^{\gamma,\Psi}_{\varrho,k-n+\rho,\varpi,0^+}\xi(t)\right]_{t=0}
\end{aligned}$$

□

Definition 3.8. Ψ*-Formable transform on* Ψ***-regularized Prabhakar fractional derivative*** *defined by*

$$\begin{aligned}
\mathcal{F}_\Psi[{}^C\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t)] &= \left(\frac{X}{v}\right)^{\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{B}_\Psi(X,v) \\
&\quad - \sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{\rho-k}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{D}^{k,\Psi}\xi(t)|_{t=0}
\end{aligned} \tag{37}$$

Proof. By using the Ψ-Formable transform definition on Ψ-regularised Prabhakar fractional derivative(7) and equations (35), (29), we obtain

$$\begin{aligned}
&\mathcal{F}_\Psi[{}^C\mathcal{D}^{\gamma,\Psi}_{\varrho,\rho,\varpi,0^+}\xi(t)](X,v)\\
&\quad= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi(t)\right](X,v)\\
&\quad= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}h(t)\right](X,v), \quad where \quad h(t) = \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \xi(t)\\
&\quad= \mathcal{F}_\Psi\left[(h * e^{\gamma,\Psi}_{\varrho,(n-\rho),\varpi})(t)\right](X,v)\\
&\quad= \left(\frac{v}{X}\right)^{n-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left[\left(\frac{v}{X}\right)^{-n}\mathcal{F}_\Psi[\xi(t)] - \sum_{k=0}^{m-1}\left(\frac{v}{X}\right)^{k-n}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k \xi(0)\right]\\
&\quad= \left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{B}_\Psi(X,v) - \sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k \xi(0^+)
\end{aligned}$$

□

Definition 3.9. *Ψ-Formable transform on Ψ-**Hilfer Prabhakar fractional derivative** defined by*

$$\begin{aligned}
\mathcal{F}_\Psi[\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t)] = &\left(\frac{X}{v}\right)^{\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{B}_\Psi(X,v)\\
&-\sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{\nu(\rho-n)+n-k}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}\mathcal{I}^{-\gamma(1-\nu)-k,\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(t)|_{t=0^+}
\end{aligned} \tag{38}$$

Proof. By using the Ψ-Formable transform definition on Ψ-Hilfer Prabhakar fractional derivative (8) and equations (29), (35), we obtain

$$\begin{aligned}
&\mathcal{F}_\Psi[\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t)](X,v)\\
&= \mathcal{F}_\Psi\left[\left(\mathcal{I}^{-\gamma\nu,\Psi}_{\varrho,\nu(n-\rho),\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n(\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi)\right)(t)\right](X,v)\\
&= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma\nu,\Psi}_{\varrho,\nu(n-\rho),\varpi,0^+}k(t)\right](X,v),\;\; where\;\; k(t)=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(t)\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma\nu}\mathcal{F}_\Psi[k(t)](X,v)\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma\nu}\\
&\times\left[\left(\frac{v}{X}\right)^{-n}\mathcal{F}_\Psi[\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(t)](X,v)-\left(\frac{v}{X}\right)^{k-n}\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(0^+)\right]\\
&= \left(\frac{v}{X}\right)^{\nu(n-\rho)}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma\nu}\left[\left(\frac{v}{X}\right)^{(1-\nu)(n-\rho)-n}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma(1-\nu)}\mathcal{F}_\Psi[\xi(t)]\right]\\
&-\left(\frac{v}{X}\right)^{\nu(n-\rho)}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma\nu}\left[\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^k\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(0^+)\right]\\
&= \left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma}\mathcal{B}_\Psi(X,v)\\
&-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{\nu(n-\rho)+k-n}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^{\gamma\nu}\mathcal{D}^{k,\Psi}\,\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\xi(t)|_{t=0^+}
\end{aligned}$$

□

Definition 3.10. *Ψ-Formable transform on regularized version of Ψ-**Hilfer Prabhakar fractional derivative** defined by*

$$\begin{aligned}\mathcal{F}_\Psi[{}^C\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t)] = \left(\frac{X}{v}\right)^\rho\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^\gamma\mathcal{B}_\Psi(X,v)\\
-\sum_{k=0}^{n-1}\left(\frac{X}{v}\right)^{\rho-k}\left(1-\varpi\left(\frac{v}{X}\right)^\varrho\right)^\gamma\mathcal{D}^{k,\Psi}\xi(t)|_{t=0}\end{aligned} \tag{39}$$

Proof. By using the Ψ-Formable transform definition on Ψ-regularized Hilfer Prabhakar fractional derivative (9) and equations (29), (35), we obtain

$$
\begin{aligned}
&\mathcal{F}_\Psi[{}^C\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(t)](X,v)\\
&\quad= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma\nu,\Psi}_{\varrho,\nu(n-\rho),\varpi,0^+}\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(n-\rho),\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\xi(t)\right](X,v)\\
&\quad= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\xi(t)\right](X,v)\\
&\quad= \mathcal{F}_\Psi\left[\mathcal{I}^{-\gamma,\Psi}_{\varrho,n-\rho,\varpi,0^+}z(t)\right](X,v),\ \ z(t)=\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n\xi(t)\\
&\quad= \left(\frac{v}{X}\right)^{n-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{F}_\Psi[z(t)](X,v)\\
&\quad= \left(\frac{v}{X}\right)^{n-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left[\left(\frac{v}{X}\right)^{-n}\mathcal{F}_\Psi[\xi(t)]-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-n}\mathcal{D}^{k,\Psi}\xi(0^+)\right]\\
&\quad= \left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{B}_\Psi(X,v)-\sum_{k=0}^{n-1}\left(\frac{v}{X}\right)^{k-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\mathcal{D}^{k,\Psi}\xi(0^+)
\end{aligned}
$$

□

4 Applications

In this section, the applications of the Ψ-Formable transform on Ψ-Hilfer-Prabhakar and Ψ-regularized Hilfer-Prabhakar fractional derivatives are discussed for solving Cauchy type fractional differential equations.

Theorem 4.1. *The solution for the generalized Cauchy type problem of the fractional advection dispersion equation*

$$\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(x,t)=-w\mathcal{D}_x\xi(x,t)+\vartheta\,\eta^{\frac{\lambda}{2}}\xi(x,t) \tag{40}$$

subjects to below constraints

$$\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(1-\rho),\varpi,0^+}\xi(x,0^+)=g(x),\ \ \varpi,\ \gamma,\ x\in\mathbb{R},\ \varrho>0, \tag{41}$$

$$\lim_{x\to\infty}\xi(x,t)=0,\ \ t\geq 0, \tag{42}$$

is provided

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})^n t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\varpi(\Psi(t))^{\varrho})dk \tag{43}$$

where $\eta^{\frac{\lambda}{2}}$ is the fractional generalized Laplace operator of order λ, $\lambda\in(0,2),\ \rho\in(0,1),\ \nu\in[0,1]:\ x\in\mathbb{R},\ t\in\mathbb{R}^+,\ \gamma>0$ *Fourier transform of $\eta^{\frac{\lambda}{2}}$ is $-|k|^{\lambda}$ discussed in [31]*

Proof. Applying the Fourier and Ψ-Formable transforms on equation (40) by using the equation (38). First we will use Fourier transform on (40)

$$\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi^*(x,t)=iwk\xi^*(k,t)-\vartheta|k|^{\lambda}\xi^*(k,t) \tag{44}$$

where $\xi^*(k,t)$ is the Fourier transform of $\xi(x,t)$ with respect to variable x, now applying the Ψ-Formable transform on (44), we will get

$$\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^*(k,X,v)-\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)$$
$$=iwk\overline{\xi}^*(k,X,v)-\vartheta|k|^{\lambda}\overline{\xi}^*(k,X,v)$$

where $\overline{\xi}^*(k,X,v)$ is the Ψ-Formable integral transform of $\xi^*(k,t)$ with respect to variable t, therefore, we have

$$\overline{\xi}^*(k,X,v)\left[\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}+\vartheta|k|^{\lambda}-iwk\right]=\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)$$

$$\overline{\xi}^*(k,X,v)=\frac{\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\ \frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}<1$$

$$\overline{\xi}^*(k,X,v)=\frac{\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\left[1+\frac{\vartheta|k|^{\lambda}-iwk}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{-1}$$

$$\overline{\xi}^*(k,X,v)=\left(\frac{v}{X}\right)^{\nu(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}\left[\frac{-\vartheta|k|^{\lambda}+iwk}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{n}$$

$$\overline{\xi}^*(k,X,v)=\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})^{n}\left(\frac{v}{X}\right)^{\nu(1-\rho)+\rho+\rho n-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu-\gamma n-\gamma}g^*(k)$$

the solution of the problem was obtained by applying the inverse of both the Fourier and Ψ-Formable transforms

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(iwk-\vartheta|k|^{\lambda})^{n}t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\varpi(\Psi(t))^{\varrho})dk$$

□

Remark 1: If the values of w and ϑ in the equation (40) are set to 0 and $\frac{ih}{2m}$ respectively, the equation will be reduced to the one-dimensional space-time Schrodinger fractional equation, where the values of mass and plank constant are m and h respectively.

$$\xi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(-ikx)}g(k)\sum_{n=0}^{\infty}(-\frac{ih}{2m}|k|^{\lambda})t^{\nu(1-\rho)+n\rho+\rho-1}E^{\gamma(1+n)-\gamma\nu}_{\varrho,\nu(1-\rho)+\rho(n+1)}(\varpi(\Psi(t))^{\varrho})dk \quad (45)$$

Theorem 4.2. *Investigating the solution to a generalized Cauchy problem for the fractional heat equation*

$${}^{C}\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi(x,t)=N\frac{\partial^2}{\partial x^2}\xi(x,t) \quad (46)$$

subject to the initial condition

$$\xi(x,0)=g(x) \quad (47)$$
$$\lim_{x\to\infty}\xi(x,t)=0$$

with $\rho \in (0,1),\ \nu[0,1];\ \varpi, x \in \mathbb{R};\ N, \varrho > 0, \gamma \geq 0,$ *is given by*

$$\xi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) dk \sum_{n=0}^{\infty} t^{\rho n} E_{\varrho,\rho n+1}^{\gamma n}(\varpi(\Psi(t))^{\varrho})(-Nk^2)^n \tag{48}$$

Proof. Applying the Fourier and Ψ-Formable transform on equation (46) by using the equations (39), (47), first we will apply the Fourier transform

$${}^{C}\mathcal{D}_{\varrho,\varpi,0^+}^{\gamma,\rho,\nu,\Psi} \xi^*(k,t) = -Nk^2 \xi^*(k,t) \tag{49}$$

where $\xi^*(k,t)$ is the Fourier integral transform of $\xi(x,t)$ with respect to variable x, now applying the Ψ-Formable integral transform on (49)

$$\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} \overline{\xi}^*(k,X,v) - \left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} \xi^*(k,0) = -Nk^2 \overline{\xi}^*(k,X,v)$$

where $\overline{\xi}^*(k,X,v)$ is Ψ-the Formable integral transform of $\xi^*(k,t)$ with respect to variable t, therefore

$$\begin{aligned}
&\overline{\xi}^*(k,X,v) \left[\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} + Nk^2 \right] = \left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} g^*(k), \\
&\overline{\xi}^*(k,X,v) = \frac{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} g^*(k)}{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} \left[1 + \frac{Nk^2}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]},\ if \frac{Nk^2}{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}} < 1 \\
&\overline{\xi}^*(k,X,v) = \frac{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} g^*(k)}{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}} \left[1 + \frac{Nk^2}{\left(\frac{v}{X}\right)^{-\rho} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{-1}, \\
&\overline{\xi}^*(k,X,v) = g^*(k) \sum_{n=0}^{\infty} (-Nk^2)^n \left(\frac{v}{X}\right)^{\rho n} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma n} g^*(k), \\
&\overline{\xi}^*(k,X,v) = \sum_{n=0}^{\infty} (-Nk^2)^n \left(\frac{v}{X}\right)^{\rho n} \left(1 - \varpi \left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma n} g^*(k),
\end{aligned}$$

the solution of the problem was obtained by applying the inverse of both the Fourier and Ψ-Formable transforms

$$\xi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) dk \sum_{n=0}^{\infty} t^{\rho n} E_{\varrho,\rho n+1}^{\gamma n}(\varpi(\Psi(t))^{\varrho})(-Nk^2)^n$$

□

Theorem 4.3. *Investigating the solution to a generalized Cauchy problem for the fractional heat equation*

$$\mathcal{D}_{\varrho,\varpi,0^+}^{\gamma,\rho,\nu,\Psi} \xi(x,t) = M \frac{\partial^2}{\partial x^2} \xi(x,t), \tag{50}$$

$$\mathcal{I}_{\varrho,(1-\nu)(1-\rho),\varpi,0^+}^{-\gamma(1-\nu),\Psi} \xi(x,t)|_{t=0} = g(x), \tag{51}$$

$$\lim_{x \to \infty} \xi(x,t) = 0,$$

with $\rho \in (0,1),\ \nu[0,1];\ \varpi, x \in \mathbb{R};\ M, \varrho > 0, \gamma \geq 0,$ *is given by*

$$\xi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) \sum_{n=0}^{\infty} (-Mk^2)^n t^{\rho(n+1)-\nu(\rho-1)-1} E_{\varrho,\rho(n+1)+\nu(1-\rho)}^{\gamma(n+1-\nu)}(\varpi(\Psi(t))^{\varrho}) dk \tag{52}$$

Proof. Applying the Fourier and Ψ-Formable transform on equation (50) by using equations (38), (51), first we will apply the Fourier transform

$$\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,\varpi,0^+}\xi^*(x,t) = -Mk^2\xi^*(k,t) \tag{53}$$

where $\xi^*(k,t)$ is the Fourier transform of $\xi(x,t)$ with respect to variable x, now applying the Ψ-Formable integral transform on equation (53)

$$\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^*(k,X,v) - \left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}\mathcal{I}^{-\gamma(1-\nu),\Psi}_{\varrho,(1-\nu)(1-\rho),\varpi,0^+}f^*(x,0) = -Mk^2\overline{\xi}^*(k,X,v)$$

where $\overline{\xi}^*(k,X,v)$ is the Ψ-Formable integral transform of $\xi^*(k,t)$ with respect to variable t, therefore we have

$$\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\overline{\xi}^*(k,X,v) - \left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k) = -Mk^2\overline{\xi}^*(k,X,v)$$

$$\overline{\xi}^*(k,X,v)\left[\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} + Mk^2\right] = \left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)$$

$$\overline{\xi}^*(k,X,v) = \frac{\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma} + Mk^2}$$

$$\overline{\xi}^*(k,X,v) = \frac{\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{Mk^2}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\ \left(\frac{Mk^2}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right) < 1$$

$$\overline{\xi}^*(k,X,v) = \frac{\left(\frac{v}{X}\right)^{\nu(1-\rho)-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu}g^*(k)}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\left[1+\frac{Mk^2}{\left(\frac{v}{X}\right)^{-\rho}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{-1}$$

$$\overline{\xi}^*(k,X,v) = \left(\frac{v}{X}\right)^{\nu(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu-\gamma}g^*(k)\sum_{n=0}^{\infty}(-Mk^2)^n\left(\frac{v}{X}\right)^{\rho n}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma n}$$

$$\overline{\xi}^*(k,X,v) = g^*(k)\sum_{n=0}^{\infty}(-Mk^2)^n\left(\frac{v}{X}\right)^{\rho n+\nu(1-\rho)+\rho-1}\left(1-\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma\nu-\gamma n-\gamma},$$

the solution of the Cauchy type fractional differential equation was obtained by applying the inverse of both the Fourier and Ψ-Formable transforms.

$$\xi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}g(k)\sum_{n=0}^{\infty}(-Mk^2)^n t^{\rho(n+1)-\nu(\rho-1)-1}E^{\gamma(n+1-\nu)}_{\varrho,\rho(n+1)+\nu(1-\rho)}(\varpi(\Psi(t))^{\varrho})dk$$

□

Theorem 4.4. *Investigating the solution to a Cauchy problem for a fractional differential equation*

$$^{C}\mathcal{D}^{\gamma,\rho,\nu,\Psi}_{\varrho,-\varpi,0^+}\xi(x,t) = -\lambda(1-x)\xi(x,t),\ \ for\ \ |x| \leq 1 \tag{54}$$

$$\xi(x,0) = 1 \tag{55}$$

with the constraints that
$t > 0,\ \lambda > 0,\ \gamma \geq 0,\ 0 < \varrho \leq 1, 0 < \rho \leq 1,$ *is obtained*

$$\xi(x,t) = \sum_{n=0}^{\infty}(\lambda x-\lambda)^n t^{\rho n}E^{\gamma n}_{\varrho,\rho n+1}(-\varpi(\Psi(t))^{\varrho}) \tag{56}$$

Proof. Let $\overline{\xi}(x,X,v)$ be the Ψ-Formable integral transform of $\xi(x,t)$ with respect to variable t. Now applying the Formable transform on equation (54) by using (39), (55), then we get

$$\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\overline{\xi}(x,X,v)-\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\overline{\xi}(x,0)$$
$$=-\lambda(1-x)\overline{\xi}(x,X,v)$$

$$\overline{\xi}(x,X,v)\left[\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}+\lambda(1-x)\right]=\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}$$

$$\overline{\xi}(x,X,v)=\frac{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}\left[1+\frac{\lambda(1-x)}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]},\ if\frac{\lambda(1-x)}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}<1$$

$$\overline{\xi}(x,X,v)=\frac{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\left[1+\frac{\lambda(1-x)}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{-1}$$

$$\overline{\xi}(x,X,v)=\sum_{n=0}^{\infty}\left[\frac{-\lambda(1-x)}{\left(\frac{v}{X}\right)^{-\rho}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{\gamma}}\right]^{n}$$

$$\overline{\xi}(x,X,v)=\sum_{n=0}^{\infty}(-\lambda)^{n}(1-x)^{n}\left(\frac{v}{X}\right)^{\rho n}\left(1+\varpi\left(\frac{v}{X}\right)^{\varrho}\right)^{-\gamma n},$$

the solution of the Cauchy type fractional differential equation was obtained by applying the inverse of the Ψ-Formable integral transform on both sides of the above equation.

$$\xi(x,t)=\sum_{n=0}^{\infty}(\lambda x-\lambda)^{n}t^{\rho n}E^{\gamma n}_{\varrho,\rho n+1}(-\varpi(\Psi(t))^{\varrho})$$

□

5 Conclusion

The generalised Foramable transform called as Ψ-Formable transform was used on both the Ψ-Hilfer-Prabhakar fractional derivative and its regularized version. The applications of this transform were then demonstrated by solving Cauchy type fractional differential equations using the Ψ-Hilfer-Prabhakar fractional derivative and the three parameter Mittage-Leffler function. The results indicate that the Ψ-Formable transform is a useful tool for solving fractional differential equations.

References

[1] Magar, Sachın, et al., Generalized Shehu Transform to Ψ-Hilfer-Prabhakar Fractional Derivative and its Regularized Version., Advances in the Theory of Nonlinear Analysis and its Application 6.3 (2022), 364-379.

[2] Hilfer, R., Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore-New jersey-Hong Kong (2000), 87-130.

[3] Kilbas, A.A., Srivastava, H.M. and Trujillo, J. Theory and Applications of Fractional Differential Equations, Elsevier Science, USA (2006).

[4] Prabhakar, T.R., *et al.*, A singular integral equation with a generalized Mittag Leffler function in the kernel, Yokohama Math. J., 19 (1971), 7-15.

[5] Jafari, H.: A new general integral transform for solving integral equations. J. Adv. Res. (2020). https://doi.org/10.1016/j.jare.2020.08.016

[6] Meddahi, M., Jafari, H. and Ncube, M.N., New general integral transform via Atangana–Baleanu derivatives., Advances in Difference Equations, 2021(1), 1-14.

[7] Garra, Roberto, et al., Hilfer–Prabhakar derivatives and some applications., Applied mathematics and computation 242 (2014), 576-589.

[8] S.K. Panchal, P.V. Dole, A.D. Khandagale., k-Hilfer-Prabhakar Fractional Derivatives and Applications., Indian journal of mathematics, 59, (2017), 367-383

[9] Panchal SK, Khandagale AD, Dole PV., Sumudu transform of Hilfer-Prabhakar fractional derivatives and applications., arXiv preprint arXiv:1608.08017. (2016).

[10] Singh, Yudhveer, et al., On the Elzaki transform and its applications in fractional free electron laser equation., Acta Universitatis Sapientiae, Mathematica 11(2) (2019), 419-429.

[11] Belgacem, Rachid, Ahmed Bokhari, and Boualem Sadaoui., Shehu transform of Hilfer-Prabhakar fractional derivatives and applications on some Cauchy type problems., Advances in the Theory of Nonlinear Analysis and its Application 5(2) (2021), 203-214.

[12] Agrawal, Om P., Fractional optimal control of a distributed system using eigenfunctions., Journal of Computational and Nonlinear Dynamics 3(2) (2008).

[13] Liu, Fawang, Pinghui Zhuang, and Kevin Burrage., Numerical methods and analysis for a class of fractional advection–dispersion models., Computers and Mathematics with Applications 64(10) (2012), 2990-3007.

[14] Mainardi, Francesco., An introduction to mathematical models, Fractional calculus and waves in linear viscoelasticity., Imperial College Press, London (2010).

[15] Miller, K. S., and B. Ross., An Introduction to the Fractional Integrals and Derivatives-Theory and Applications., John Willey and Sons, Inc., New York (1993).

[16] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic, New York, (1974).

[17] Samko, G., A. Kilbas, and O. Marichev., Fractional Integrals and Derivatives: theory and applications., Amsterdam: Gordon and Breach, (1993).

[18] Saadeh, Rania Zohair, and Bayan fu'ad Ghazal., A new approach on transforms: Formable integral transform and its applications., Axioms 10(4) (2021) 332.

[19] Belgacem, Rachid, Dumitru Baleanu, and Ahmed Bokhari., Shehu transform and applications to Caputo-fractional differential equations., (2019).

[20] Miller, K., The weyl fractional calculus, Springer, Berlin Heidelberg (1975), 80–89

[21] Saxena, R.K., Mathain, A.M. and Haubold, H.J., Reaction diffusion systems and nonlinear waves., Astrophys. Space Sci., 305(3) (2006), 297–303

[22] Metzler, R. and Klafter, J., The random walk's guide to anomalous diffusion: A fractional dynamics approach., Phys. Rep., 339(1) (2000), 1–77.

[23] Debnath L. and Bhatta D., Integral transforms and their applications, Chapman and Hall/CRC, Taylor and Francis Group, New York (2007).

[24] Singh, Gyanvendra Pratap, and Sudhanshu Aggarwal, Sawi transform for population growth and decay problems, International Journal of Latest Technology in Engineering, Management and Applied Science 8(8) (2019), 157-162.

[25] Higazy, M., and Sudhanshu Aggarwal, Sawi transformation for system of ordinary differential equations with application, Ain Shams Engineering Journal 12(3) (2021), 3173-3182.

[26] Aggarwal, Sudhanshu, and Anjana Rani Gupta, Dualities between some useful integral transforms and Sawi transform, International Journal of Recent Technology and Engineering 8(3) (2019), 5978-5982.

[27] Patil, D. P, Application of Sawi transform in Bessel functions, Aayushi International Interdisciplinary Research Journal (AIIRJ) (2021).

[28] Higazy, M., Sudhanshu Aggarwal, and Taher A. Nofal, Sawi decomposition method for Volterra integral equation with application., Journal of Mathematics 2020 (2020).

[29] Bhuvaneswari, R., and K. Bhuvaneswari., Application of Sawi transform in Cryptography., International Journal of Modern Agriculture 10(1) (2021), 185-189.

[30] Sahoo, Mrutyunjaya, and Snehashish Chakraverty., Sawi Transform Based Homotopy Perturbation Method for Solving Shallow Water Wave Equations in Fuzzy Environment., Mathematics 10(16) (2022), 2900.

[31] Agarwal, Ritu, Sonal Jain, and R. P. Agarwal., Analytic solution of generalized space time advection-dispersion equation with fractional Laplace operator., J Nonlinear Sci (2016), 3545-54.

[32] Mainardi, Francesco. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. World Scientific, (2022).

[33] Magin, Richard L., Fractional calculus models of complex dynamics in biological tissues., Computers & Mathematics with Applications 59(5) (2010), 1586-1593.

[34] Ionescu, C., et al., The role of fractional calculus in modeling biological phenomena: A review., Communications in Nonlinear Science and Numerical Simulation 51 (2017), 141-159.

[35] Bagley, Ronald L., and Peter J. Torvik., On the fractional calculus model of viscoelastic behavior., Journal of Rheology 30(1) (1986), 133-155.

[36] Carpinteri, Alberto, and Francesco Mainardi, eds. Fractals and fractional calculus in continuum mechanics., Springer 378 (2014).

[37] Hamoud, Ahmed A., Abdulrahman A. Sharif, and Kirtiwant P. Ghadle., Existence, Uniqueness and Stability Results of Fractional Volterra-Fredholm Integro Differential Equations of ψ-Hilfer Type., Discontinuity, Nonlinearity, and Complexity 10(03) (2021), 535-545.

[38] Magar, Sachin K., Pravinkumar V. Dole, and Kirtiwant P. Ghadle., Prabhakar and Hilfer-Prabhakar fractional derivatives in the setting of ψ-fractional calculus and its applications., Krak. J. Math 2024(48) (2024), 515-533.

[39] Sousa, J. Vanterler da C., and E. Capelas De Oliveira., On the ψ-Hilfer fractional derivative., Communications in Nonlinear Science and Numerical Simulation 60 (2018), 72-91.

[40] Ghadle, Kirtiwant P., Sachin K. Magar, and Pravinkumar V. Dole., A new Sumudu type integral transform and its applications., Proc. Fract. Differ. Appl. 7(3) (2021), 145-152.

[41] Almeida, Ricardo., A Caputo fractional derivative of a function with respect to another function., Communications in Nonlinear Science and Numerical Simulation 44 (2017), 460-481.

[42] Hamoud, Ahmed., Existence and uniqueness of solutions for fractional neutral volterra-fredholm integro differential equations., Advances in the Theory of Nonlinear Analysis and its Application 4(4) (2020), 321-331.

[43] Hamoud, Ahmed, Nedal Mohammed, and Kirtiwant Ghadle., Existence and Uniqueness Results for Volterra-Fredholm Integro Differential Equations., Advances in the Theory of Nonlinear Analysis and its Application 4(4) (2020), 361-372.

[44] A. Khalouta and A. Kadem., A New combination method for solving nonlinear Liouville-Caputo and Caputo-Fabrizio time-fractional Reaction-Diffusion-Convection equations., Malaysian Journal of Mathematical Sciences, 15(2) (2021), 199-215.

[45] Jarad, Fahd, and Thabet Abdeljawad., A modifi ed Laplace transform for certain generalized fractional operators., Results in Nonlinear Analysis 1(2) (2018), 88-98.

[46] A.A. Hamoud, N.M. Mohammed and K.P. Ghadle, Existence and uniqueness results for Volterra-Fredholm integro differential equations, Advances in the Theory of Nonlinear Analysis and its Application, 4(4) (2020) 361-372. https://doi.org/10.31197/atnaa.703984