Study on resonance and bifurcation of fractional order nonlinear Duffing System

Xueting Bai[1], Jiaquan Xie[2], Qinle Yang[1], Xiaoqi Liu[3], and Jiale Zhang[3]

[1]Jinzhong College of Information
[2]Taiyuan University of Science and Technology
[3]Taiyuan Normal University

April 19, 2022

Abstract

In this paper, resonance and bifurcation of a nonlinear damped fractional-order Duffing system are studied. The amplitude and phase of the steady-state response of system are obtained by means of average method, and then the amplitude-frequency characteristic curves of the system under different parameters are drawn based on the implicit function equation of amplitude. Grunwald-Letnikov fractional derivative is used to discretize the system numerically, and the response curve and phase trajectory of the system under different parameters are obtained, and the dynamic behavior is analyzed. The forked bifurcation behavior and saddle bifurcation behavior of the system under different parameters are investigated by numerical simulation.

Study on resonance and bifurcation of fractional order

nonlinear Duffing System

Xueting Bai [1] | Jiaquan Xie[2] | Qinle Yang [1] | Jiale Zhang[2] | Xiaoqi Liu[2]

[1]JinZhong College of Information, Jinzhong, Shan Xi , China

[2]Department of Mathematics, Taiyuan Normal University, Jinzhong, Shanxi, China

Correspondence

Xueting Bai, JinZhong College of Information, Jinzhong, Shan Xi , China.

Email:bliuli@163.com

Funding information

This project is supported by National Natural Science Foundation of China (52005360), and Scientific and Technological innovation Programs of Higher Education Institution in Shanxi (2021L403), Major special projects of public bidding in Shanxi Province (20201101020).

Abstract: In this paper, resonance and bifurcation of a nonlinear damped fractional-order Duffing system are studied. The amplitude and phase of the steady-state response of system are obtained by means of average method, and then the amplitude-frequency characteristic curves of the system under different parameters are drawn based on the implicit function equation of amplitude. Grunwald-Letnikov fractional derivative is used to discretize the system numerically, and the response curve and phase trajectory of the system under different parameters are obtained, and the dynamic behavior is analyzed. The forked bifurcation behavior and saddle bifurcation behavior of the system under different parameters are investigated by numerical simulation.

Keywords : Duffing system; Fractional order system; Resonance; Bifurcation; Average method

1. Introduction

Fractional calculus was born in 1695 and appeared almost the same time as integer order calculus. However, because of no practical background for a long time, fractional calculus was not as widely used as integer order calculus. In recent years, with the successful applications of fractional calculus in various fields, people gradually realize that fractional calculus can appropriatly describe the viscoelastic materials and other materials with memory characteristics. Modeling methods by fractional calculus can reflect the dynamical characteristics of the object more accurately, and have unique advantages compared with integer order calculus. With the continuous exploration of natural science, fractional order system has been focused on by scholars over the word. Nowadays, with the rapid development computational technology, the problem of large amount of calculation of fractional calculus has been effectively solved. Fractional calculus has become a powerful mathematical tool for studying abnormal diffusion, soft matter, electrochemistry, bioengineering, complex quantum system, finance, porous medium mechanics, non-Newtonian hydrodynamics, viscoelasticity, vibration control, soft matter physics, signal analysis, image processing, automatic process control and other disciplines [1-12], with greatly encourages researchers to study the applications of fractional calculus.

Dynamic phenomena exist in problems such as mechanical system gap, polymer material, large deformation of structural material, soft material, nonlinear material constitutive relationship and so on. However, a fractional order model can be formed by a mathematical model with fewer parameters. The fractional derivative or integral does not reflect merely the nature or quantity of a local area or a single point, but a way of comprehensive consideration, which is more appropriate to describe these problems than an integer order model. Therefore, fractional calculus has been widely used in dynamic systems such as vibration control [13-16], viscoelastic material [17-19] modeling and so on. So far, the solution of fractional order dynamic system and the analysis of dynamic characteristics become very important. At present, the existing methods for solving fractional order dynamic systems include orthogonal function methods [20-21], perturbation methods [22-23], ADM decomposition method [24-25], etc. Fractional order dynamical systems mainly include linear and nonlinear systems. Different scholars have carried out in-depth research on different systems. K. A. propose a nonlinear classical and fractional order dynamic system for COVID-19 [26]. Khan applied the homotopy perturbation aided optimization process to oscillatory fractional order nonlinear dynamical systems [27]. X.Cai used multiple fractional differential equations to simulate fractional order control systems [28]. G.D. Zhao proposed a new hierarchical fast terminal sliding mode control strategy for a class of uncertain dynamical systems [29]. Y. Li analyzed the Mittag Leffler stability automation of fractional order nonlinear dynamical systems [30].Y. Qin studied an effective analysis method for solving some fractional order dynamic systems [31].

In this paper, the response of a class of fractional order nonlinear damped Duffing systems is solved by analytical and numerical methods, and the amplitude frequency characteristics are analyzed. It is proved that the fractional differential term shows damping characteristics in the whole dynamic system. With the increase of fractional differential order or fractional differential term coefficient, the vibration amplitude of the system is suppressed. This paper also explores the saddle node bifurcation caused by the variation of fractional differential order and external excitation amplitude through another form of numerical simulation.

2. Fractional nonlinear Duffing system

In the current study, the following fractional-order Duffing system with nonlinear damping is studied

whereis the system mass, linear stiffness, nonlinear damping, nonlinear stiffness coefficient, excitation amplitude and external excitation frequency respectively. andare fractional differential term coefficient and fractional order respectively. There are many ways to define fractional differential. Caputo type fractional differential definition is adopted here

whereis Gamma function, satisfying.

Perform the following coordinate transformation on the system:

Eq. (1) becomes

Consider the case of principal resonance. is introduced, whereis the tuning parameter of excitation frequency, then Eq. (1) is

Let the form of the steady-state response solution of Eq. (4) be

Among them.According to the average method, one obtain

where,

.

According to the average method, the integral average of Eq. (6) is carried out on the interval, one can get

When the integral is averaged, taking(ifis periodic) or (ifis aperiodic). Integrate the first part of Eq. (7), one have

Integrate the second part of Eq. (7), and

Two basic formulas are introduced:

Therefore

Using the coordinate transformation,, we get

Eq. (15) is divided into two parts, the first part is defined as, and the second part is defined as, which can be obtained by

Analysis of,we can get

Using a similar method, when, we found

Then

By analyzing,in a similar way ,we can get

Based on the above calculation, it can be obtained by

By substituting the initial parameters into Eqs. (23) and (24), we can get

Making, it can be obtained by

Then

where

„„, °

3 Numerical simulation

3.1 Amplitude-frequency characteristic

Based on the above analytic method discussion, this paper draws the amplitude-frequency characteristic curves of the system under different parameters according to the implicit function equation of Eq. (27). FIG.1 shows the amplitude-frequency characteristic curves of the system under different fractional differential orders. It can be seen from FIG.1 that with the increase of fractional differential order, the amplitude of system response decreases. It shows that the larger the value of fractional differential order is, the greater the hindrance effect of fractional differential term on the system is. FIG.2 shows the amplitude-frequency characteristic curves of the system with different values of nonlinear stiffness. It can be seen from FIG.2 that with the increase of nonlinear stiffness coefficient, the peak value of system response amplitude moves to the right and drops slightly, indicating that the resonance frequency is increasing. FIG.3 shows the amplitude-frequency characteristic curve of the system when the coefficient of fractional differential term is different. It can be clearly seen from FIG.3 that with the increase of the coefficient of the fractional differential term, the amplitude of the system response will decrease significantly, indicating that the fractional differential term presents damping characteristics in the system. FIG. 4 shows the amplitude frequency characteristic curve of the system when the external excitation amplitude takes different values. As can be seen from FIG. 4, with the increase of external excitation amplitude, the system response amplitude is gradually increasing, the span of the curve is becoming larger and larger, and the resonance frequency is obviously increasing.

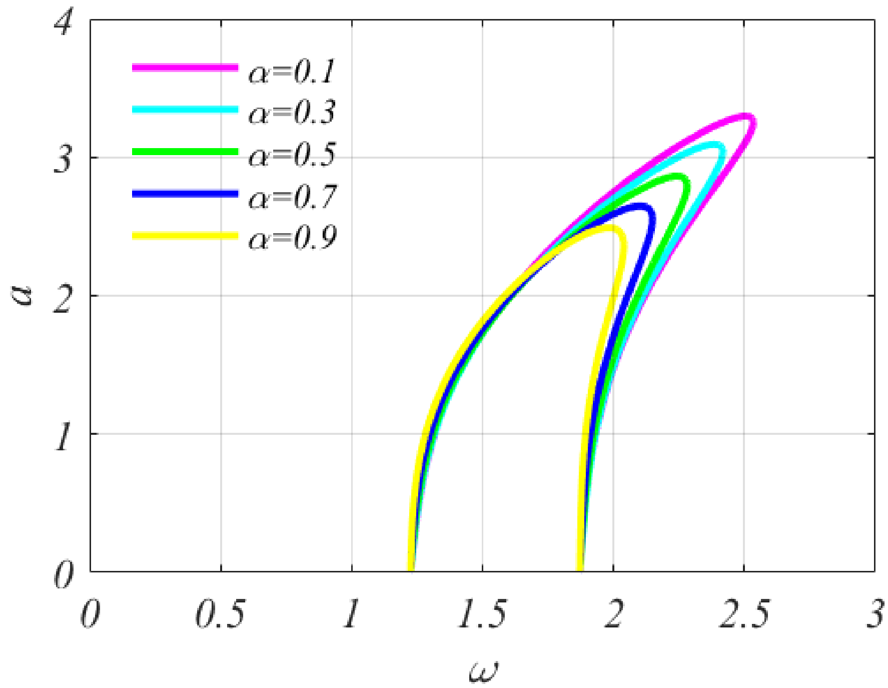

FIG. 1 Amplitude-frequency characteristic curve of the system with different values of fractional differential order, and other calculated parameters are

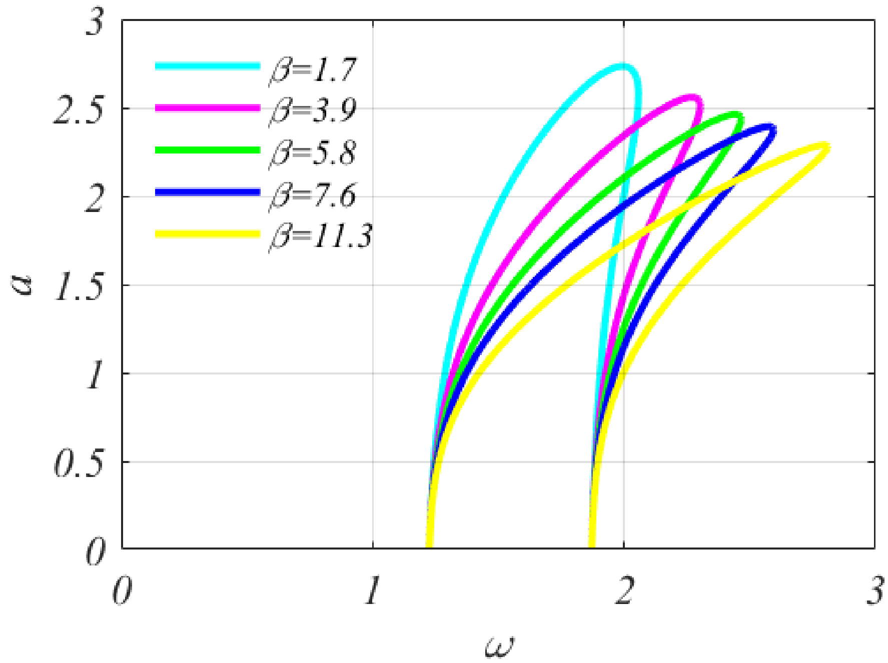

FIG. 2 Amplitude-frequency characteristic curves of the system with different values of nonlinear stiffness, and other calculated parameters are

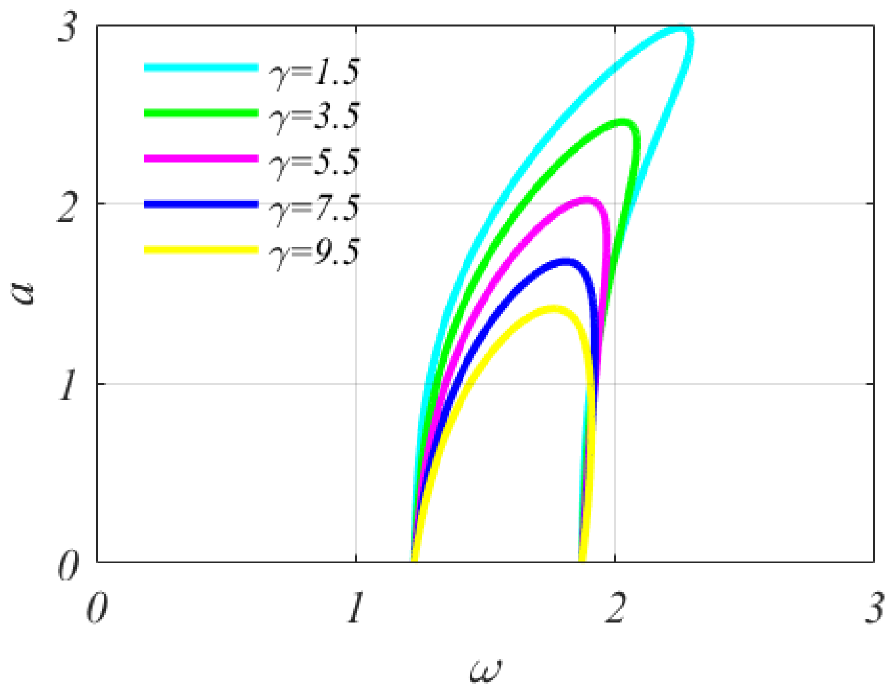

FIG. 3 Amplitude-frequency characteristic curve of the system when the coefficient of the fractional differential term is different, and other calculated parameters are

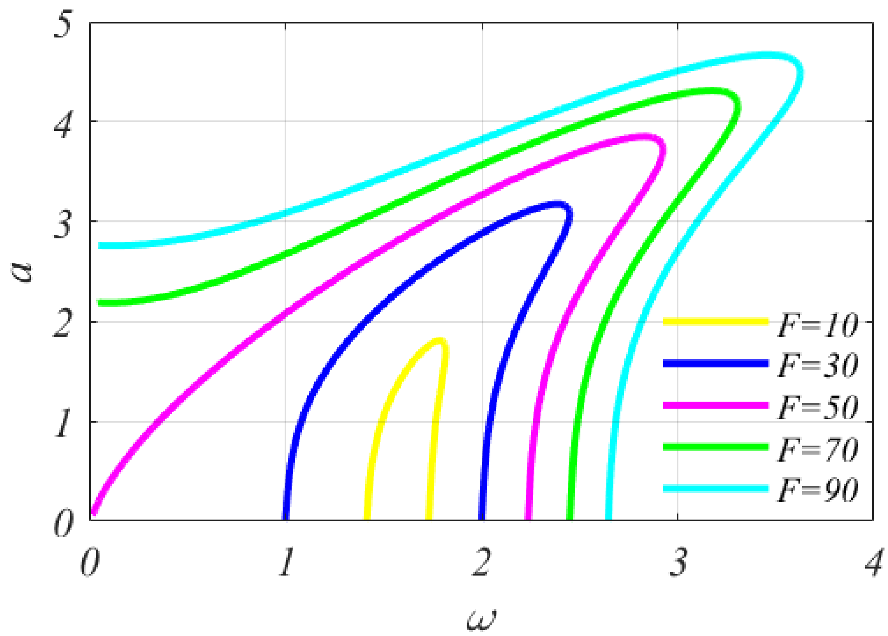

FIG. 4 Amplitude-frequency characteristic curves of the system with different external excitation amplitudes, and other calculated parameters are

3.2 Response and phase trajectory

In this paper, the Grünwald-Letnikov fractional derivative is used to simulate the system response. The Grunwald-Letnikov fractional derivative of a given function is defined as follows:

In Eq. (28), the binomial coefficient is

Then reduce the order of the original system, then Eq. (1) becomes

FIG.5 shows the response curves of the system under different fractional differential orders, and FIG.6 shows the phase trajectory curve of the system under the above parameters. FIG.7 shows the response curves of the system under different fractional differential orders, and FIG.8 shows the phase trajectory curves of the system under the above parameters. It can be seen from FIG.5 and 7 that the order of fractional differential will affect the response curve of the system. And the smaller the fractional differential order value is, the larger the peak value of the system response curve is. This is consistent with the analytical results. The two figures show the response curves of the system under different external excitation frequencies. When the external excitation frequency is greater than the natural frequency of the system, the response curve of the system will appear smooth and stable. When the external excitation frequency is less than the natural frequency, the response curve of the system oscillates violently. It shows that when the high frequency signal is applied to the system, the high frequency signal will change the oscillation law of the original system and make it become stable and orderly. When the low frequency signal is applied to the system, it is equivalent to adding a noise signal to the system, which makes the system appear chaotic phenomenon. It can also be seen from FIG. 8 that the system is moving around two equilibrium points.

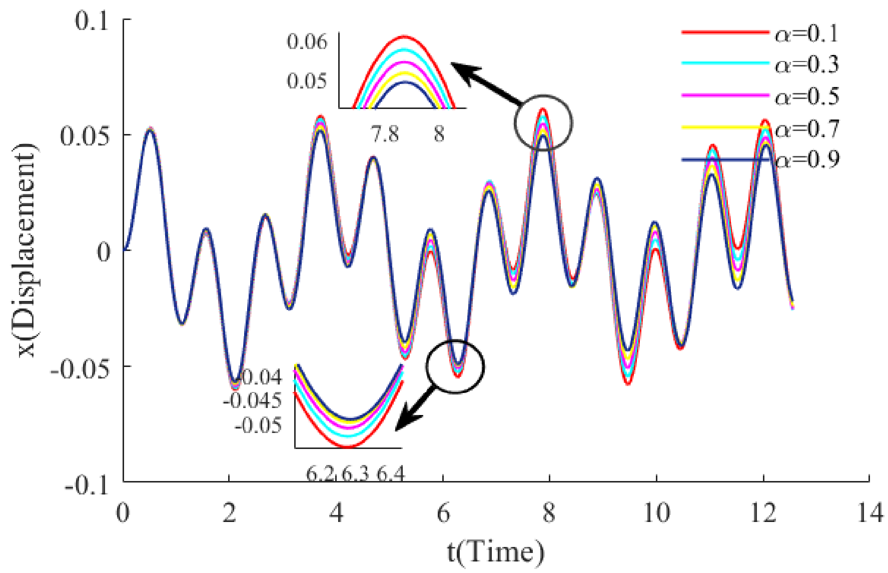

FIG. 5 Response analysis of the system when the fractional differential order takes different values, and other calculation parameters are

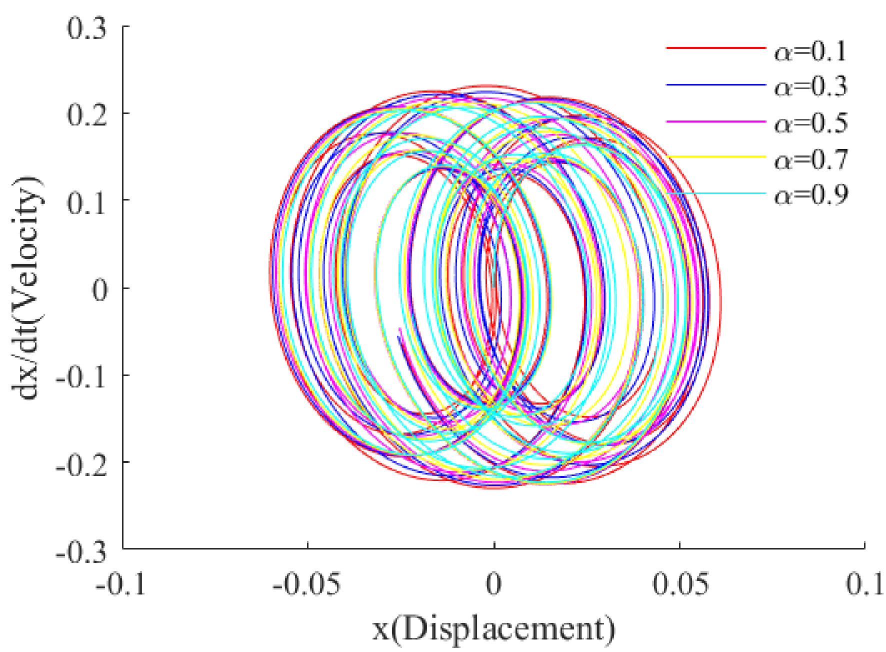

FIG. 6 Phase trajectory analysis of the system with different values of fractional differential order, and other calculated parameters are

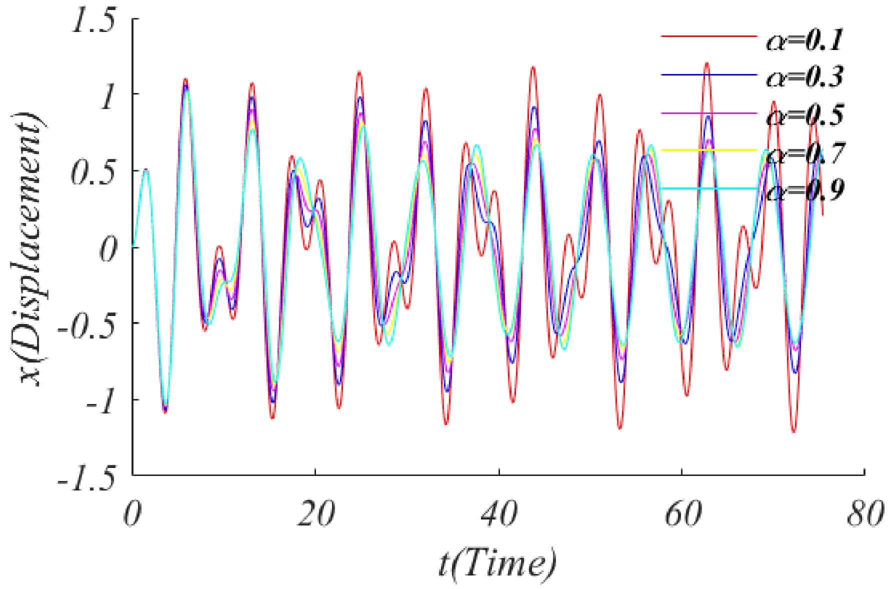

FIG. 7 Response analysis of the system with different values of fractional differential order, and other calculated parameters are

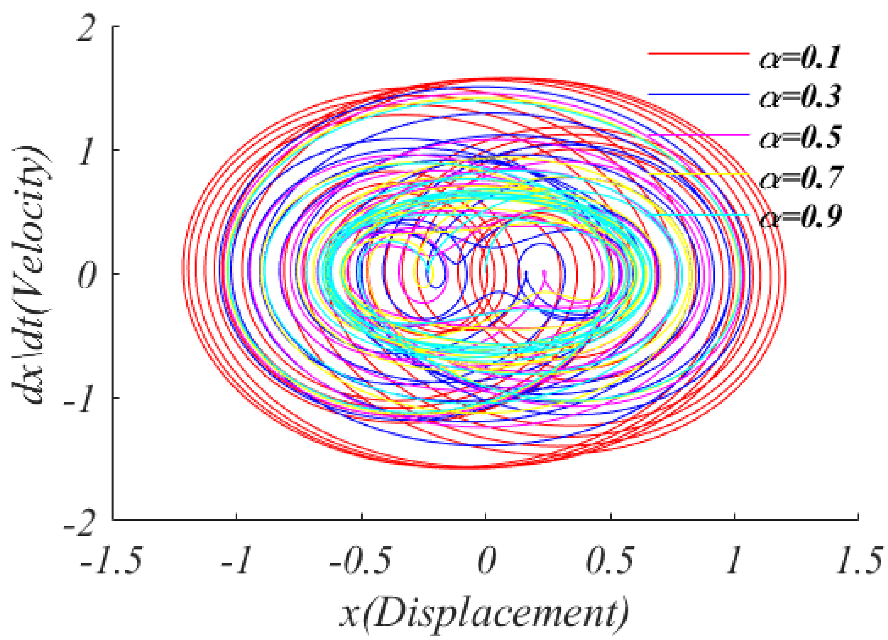

FIG. 8 Phase trajectory analysis of the system with different values of fractional differential order, and other calculated parameters are

3.3 System forked bifurcation

In this paper, the forked bifurcation of the system is simulated numerically and can be determined by calculating the Fourier series of time series. According to the analytic method, the equilibrium point of the system is the constant part of the Fourier series. The Fourier series ofexpanded inis where the coefficients of the sine and cosine Fourier components are

The constant obtained from Eq. (32) is that

In order to obtain more accurate calculation results offrom time series, a longer time series can be calculated, then

where,is a sufficiently large integer. According to Eq. (34), it can be seen that the constant in the time series israther than, so the equation for calculating the equilibrium point through the time response series is

The discrete form of Eq. (35) is

FIG.9 shows the forked bifurcation of the system caused by the change of the amplitude of the external excitation. It can be seen from the figure that when the amplitude of external excitation is zero, the equilibrium point of the system is at zero scale. With the increase of the amplitude of external excitation, the equilibrium point of the system is gradually divided into two branches and diverges to the distance. The larger the fractional differential order value is, the smaller the equilibrium point is. FIG.10 shows the forked bifurcation of the system caused by nonlinear damping parameter variation. As can be seen from the figure, with the increase of nonlinear damping parameters, the equilibrium point of the system gradually converging from the upper and lower branches at zero scale to zero scale. It shows that supercritical bifurcation occurs in the system, and with the increase of fractional differential order, the value of nonlinear damping parameter of supercritical bifurcation occurs in the system gradually increases. FIG.11 shows fork bifurcation of the system caused by coefficient variation of fractional differential term. When the coefficient of fractional differential term is negative, the equilibrium point of the system will fluctuate. As the coefficient of fractional differential term increases, the equilibrium point of the system gradually returns to the zero scale, and supercritical bifurcation occurs at a certain position.

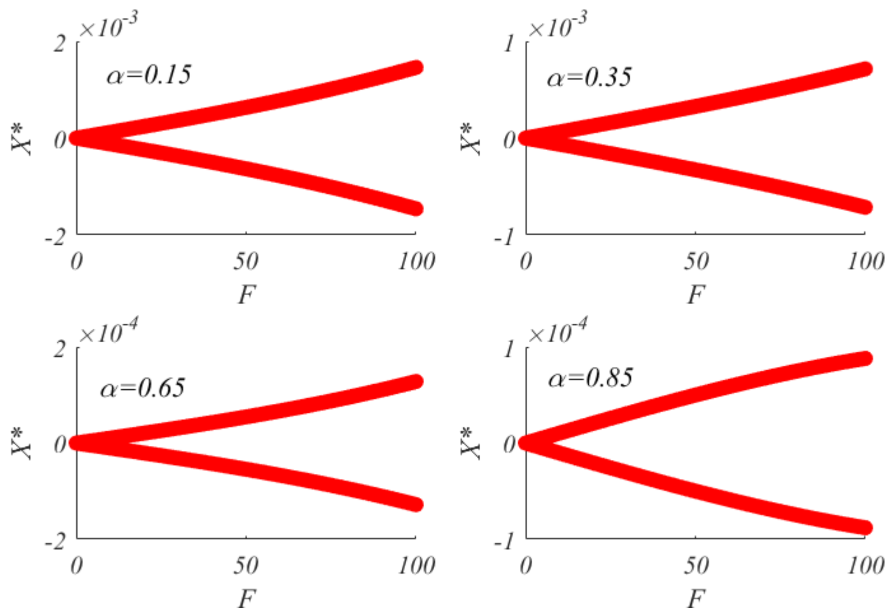

FIG. 9 Forked bifurcation of the system caused by external excitation amplitude, and other calculated parameters are

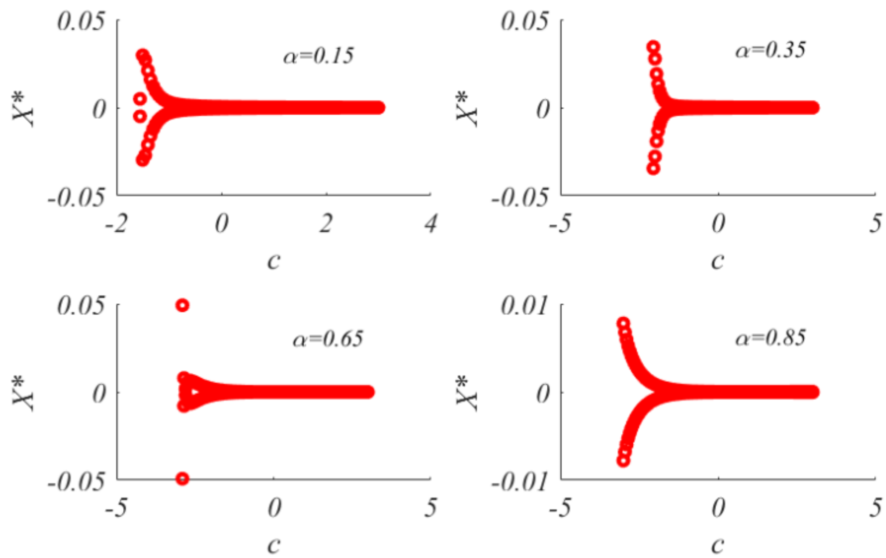

FIG. 10 Forked bifurcation of the system caused by nonlinear damping coefficient, and other calculated parameters are

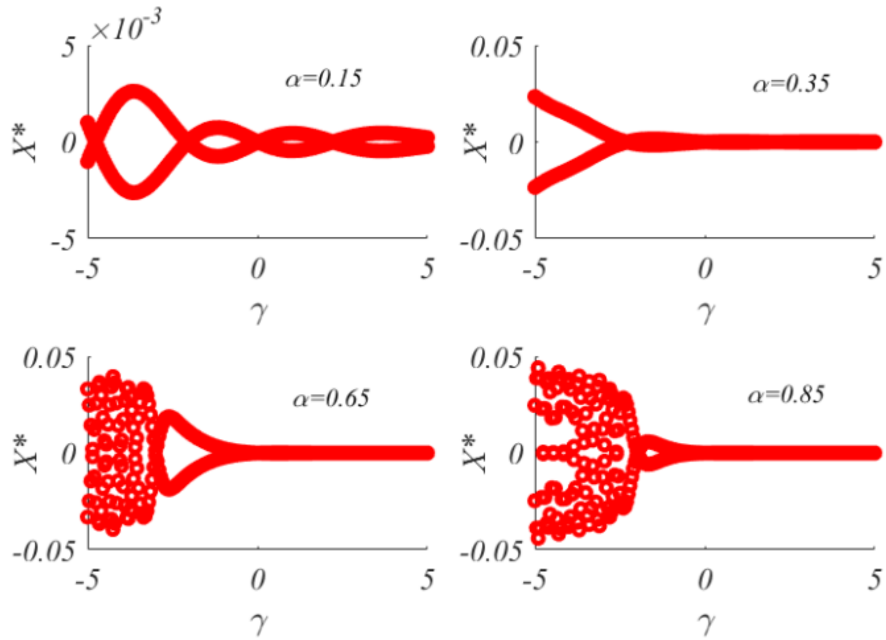

FIG. 11 Fork bifurcation of the system caused by coefficient of fractional differential term, and other calculation parameters are

3.4 System saddle bifurcation

A numerical method is presented below to simulate the change of equilibrium point. The specific steps are as follows: all points within an interval are selected as the initial conditions for calculating system response. Here, the point withinis selected as, whereandare the starting point and end point of the interval respectively, andis the interval step size. The computing time for selected,, the positions of all paths determined under different initial conditions at this time are recorded to study the bifurcation behavior of the system. FIG. 12 and 13 respectively show the system equilibrium point changes caused by fractional differential order changes under different nonlinear damping parameters and stiffness parameters. As the value range of fractional differential order given in this paper is, when the fractional differential order changes in this range, the bifurcation behavior of the equilibrium point of the system is not seen. The reasons are analyzed as follows: on the one hand, it may be related to the selection range of original parameters or initial points of the system. On the other hand, it may be related to the narrow range of fractional differential order. If the variation range of fractional differential order is, it may be possible to clearly observe the saddle node bifurcation caused by the variation of fractional differential order.

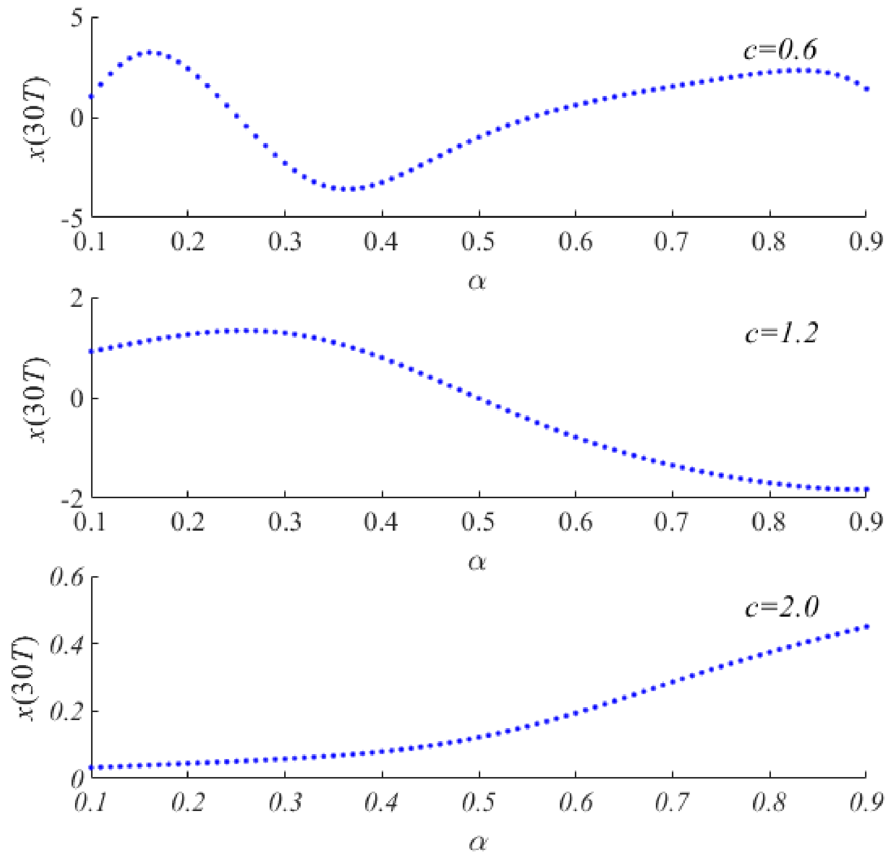

FIG. 12 Change of system equilibrium point caused by fractional differential orders with different values of nonlinear damping and other parameters

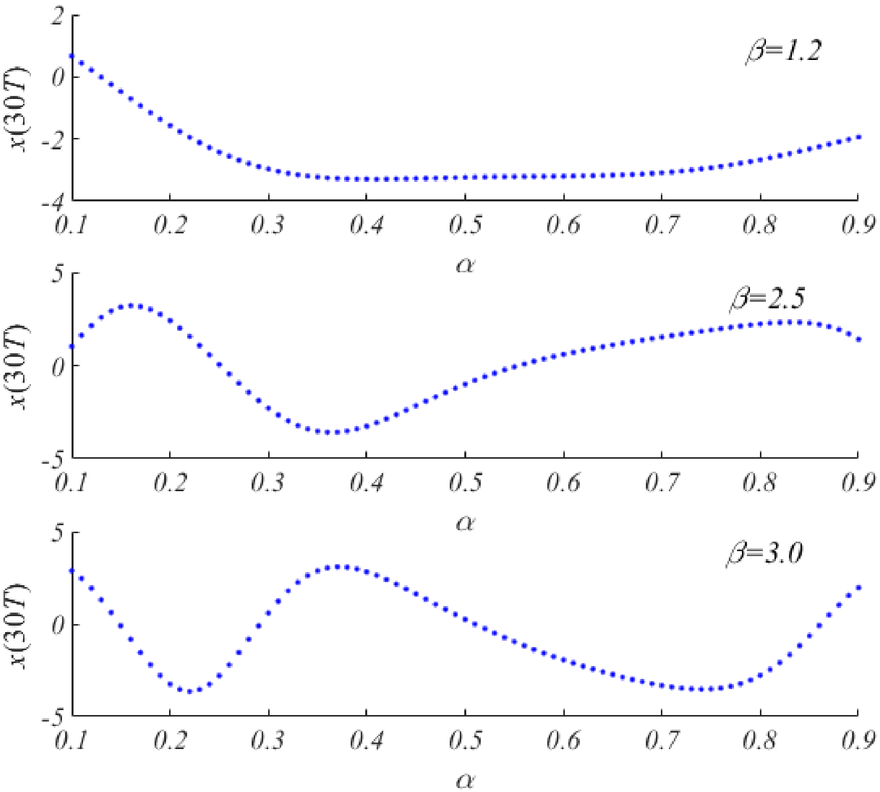

FIG. 13 Changes of system equilibrium point caused by fractional differential orders with different values of nonlinear stiffness and other parameters

FIG. 14 and FIG. 15 show the curves of the system response with nonlinear damping parameter or fractional differential term coefficient when the amplitude of external excitation is given different values. It can be seen from these two figures that when the nonlinear damping parameter or fractional differential term coefficient is given a small value, the equilibrium point of the system is divergent. With the increase of these two parameters, the equilibrium point of the system gradually returns to a stable value, indicating that the phase trajectory is a limit cycle moving around a stable equilibrium point, so the response will not diverge.

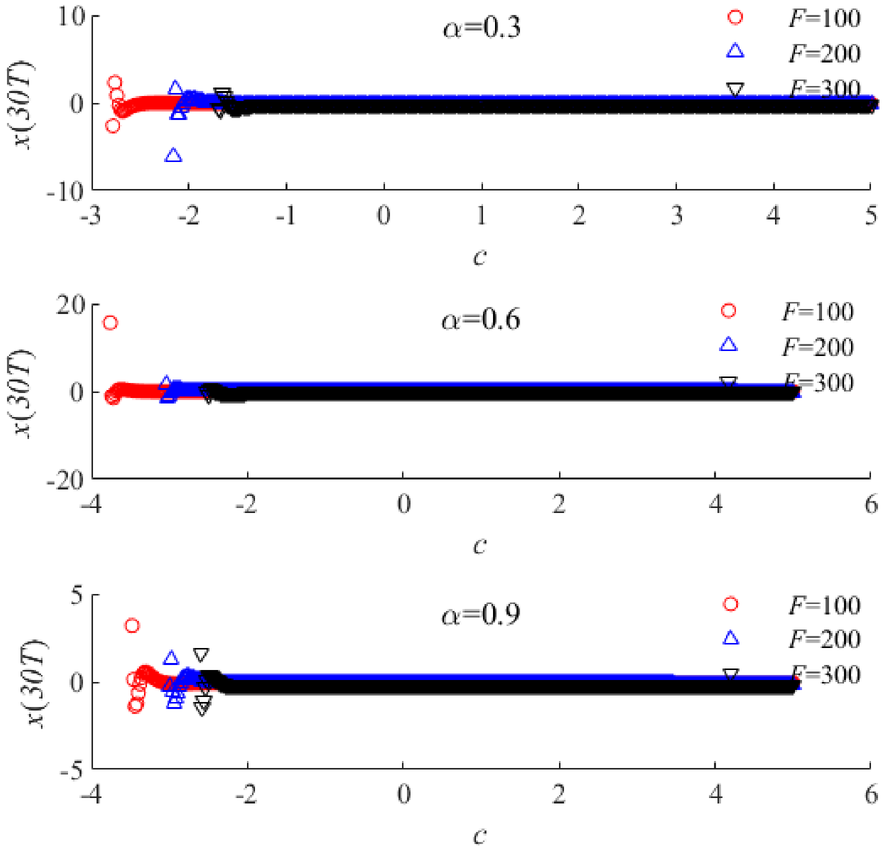

FIG. 14 Saddle node bifurcation caused by nonlinear damping when the external excitation amplitude takes different values, and other calculation parameters are

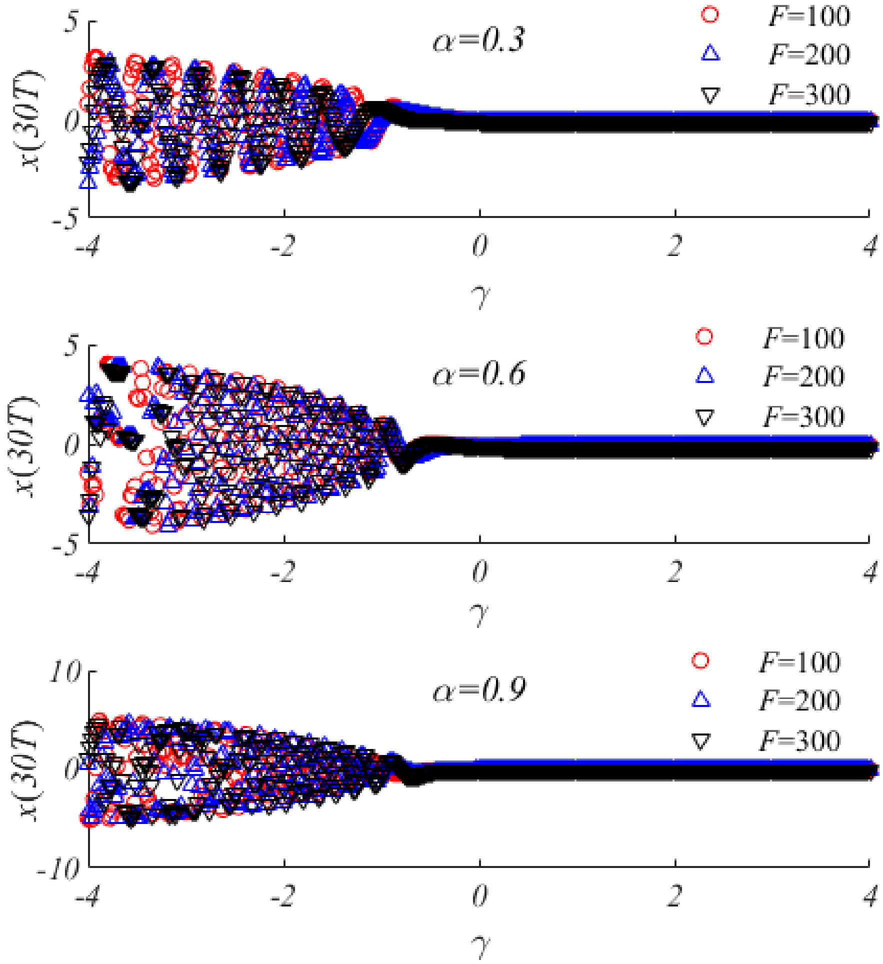

FIG. 15 Saddle node bifurcation of the system caused by fractional differential term coefficient when the external excitation amplitude takes different values, and other calculation parameters are

4 Conclusion

In this paper, the response of a class of fractional order nonlinear damped Duffing systems is solved by analytical and numerical methods, and the amplitude frequency characteristics are analyzed. By analyzing the amplitude-frequency characteristics of the system, it is shown that the fractional differential order, nonlinear damping parameters, nonlinear stiffness parameters, external excitation amplitude and fractional differential term coefficients all affect the vibration amplitude of the system. It has been proved that the fractional differential term presents damping characteristics in the whole dynamic system. With the increase of the fractional differential order or the coefficient of the fractional differential term, the vibration amplitude of the system is suppressed. According to analytic theory, the equilibrium point of the system is the constant part of the Fourier series expansion. Based on this idea, the bifurcation of the equilibrium point of the system caused by the variation of external excitation amplitude, nonlinear damping parameters and the coefficient of the fractional differential term in different fractional differential orders is investigated by numerical simulation. The saddle-node bifurcation caused by fractional differential order and external excitation amplitude variation is also investigated by another form of numerical simulation.

Reference

1. Xu H , Jiang X . Creep constitutive models for viscoelastic materials based on fractional derivatives,

Comput Math Appl, 2017, 73(6):1377-1384.
2. Atanacković T M, Konjik S, Pilipović S, et al. Complex order fractional derivatives in viscoelasticity, Mech Time-Depend Mat, 2016, 20(2): 175-195.
3. Mainardi F. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models, World Scientific, 2010.
4. Gao H B, Fu L, Niu J N. Mechanical properties research on fractional derivative standard linear solid viscoelastic material, J Henan Univ (Natural Science), 2018, 48(1): 86-91.
5. Caputo M, Carcione J M. Hysteresis cycles and fatigue criteria using anelastic models based on fractional derivatives, Rheol Acta, 2011, 50(2): 107-115.
6. Magin R L. Fractional calculus models of complex dynamics in biological tissues, Comput Math Appl, 2010, 59(5): 1586-1593.
7. Tenreiro Machado J, Kiryakova V, Mainardi F. A poster about the old history of fractional calculus, Fract Calc Appl Anal, 2010, 13(4): 447-454.
8. Munkhammar J. Fractional calculus and the Taylor-Riemann series, Rose-Hulman Undergraduate Mathematics Journal, 2005, 6(1): 6..
9. Bagley R L, Torvik P J. Fractional calculus-a different approach to the analysis of viscoelastically damped structures, AIAA J, 1983, 21(5): 741-748.
10. Nandal A , Gamboa-Rosales H , Dhaka A , et al. Image Edge Detection Using Fractional Calculus with Feature and Contrast Enhancement[J]. Circuits Systems & Signal Processing, 2018.
11. Srivastava H M, Owa S. Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, Nagoya Math J, 1987, 106: 1-28.
12. Luo D , Wang J R , Fekan M . Applying Fractional Calculus to Analyze Economic Growth Modelling[J]. Journal of Applied Mathematics Statistics and Informatics, 2018, 14(1):25-36.
13. Ali M, Khan M W, Abid M, et al. Variation of fraction in fopid controller for vibration control of Euler–Bernoulli beam, Sn Appl Sci, 2020, 2(12): 1-9.
14. Khan K N, Lamb W, McBride A C. Fractional calculus of periodic distributions, Fract Calc Appl Anal, 2011, 14(2): 260-283.
15. Singhal R, Padhee S, Kaur G. Design of fractional order PID controller for speed control of DC motor, International Journal of Scientific and Research Publications, 2012, 2(6): 1-8.
16. Al-Dhaifallah M, Nassef A M, Rezk H, et al. Optimal parameter design of fractional order control based INC-MPPT for PV system, Sol Energy, 2018, 159: 650-664.
17. Ezzat M A . Bio-thermo-mechanics behavior in living viscoelastic tissue under the fractional dual-phase-lag theory[J]. Archive of Applied Mechanics, 2021, 91(1–2).
18. Yang Y G , Sun Y H , Xu W . Bifurcation Analysis of an Energy Harvesting System with Fractional Order Damping Driven by Colored Noise[J]. International Journal of Bifurcation and Chaos, 2021, 31(15).
19. Dabiri A, Butcher E A, Nazari M. Coefficient of restitution in fractional viscoelastic compliant impacts using fractional Chebyshev collocation, J Sound Vib, 2017, 388: 230-244.
20. Kazem S, Abbasbandy S, Kumar S. Fractional-order Legendre functions for solving fractional-order differential equations, Appl Math Model, 2013, 37(7): 5498-5510.
21. Sabermahani S, Ordokhani Y, Yousefi S A. Fractional-order Fibonacci-hybrid functions approach for solving fractional delay differential equations, Eng Comput-Germany, 2020, 36(2): 795-806.
22. Khan N A, Hameed T, Ahmed S. Homotopy perturbation aided optimization procedure with applications to oscillatory fractional order nonlinear dynamical systems, Int J Model Simul Sc, 2019, 10(04): 1950026.
23. Johnston S J, Jafari H, Moshokoa S P, et al. Laplace homotopy perturbation method for Burgers equation with space-and time-fractional order, Open Phys, 2016, 14(1): 247-252.
24. Xu Y, Sun K, He S, et al. Dynamics of a fractional-order simplified unified system based on the Adomian decomposition method, The European Physical Journal Plus, 2016, 131(6): 1-12.
25. Momani S, Odibat Z. Numerical comparison of methods for solving linear differential equations of

fractional order, Chaos Soliton Fract, 2007, 31(5): 1248-1255.

26. Shah K, Din R U, Deebani W, et al. On nonlinear classical and fractional order dynamical system addressing COVID-19, Results Phys, 2021, 24: 104069.
27. Khan N A, Hameed T, Ahmed S. Homotopy perturbation aided optimization procedure with applications to oscillatory fractional order nonlinear dynamical systems, Int J Model Simul Sc, 2019, 10(04): 1950026.
28. Cai X, Liu F. Numerical simulation of the fractional-order control system, J Appl Math Comput, 2007, 23(1): 229-241.
29. Zhao G. Fractional-order fast terminal sliding mode control for a class of dynamical systems, Math Probl Eng, 2013, 2013.
30. Li Y, Chen Y Q, Podlubny I. Mittag–Leffler stability of fractional order nonlinear dynamic systems, Automatica, 2009, 45(8): 1965-1969.
31. Qin Y, Khan A, Ali I, et al. An efficient analytical approach for the solution of certain fractional-order dynamical systems, Energies, 2020, 13(11): 2725.