Special Associated Curves in Euclidean 3-Space

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Abstract

First, we study a new tip of unit speed associated curves in the E3 like a normal-direction curve and normal-donor curve. Then we achieve qualification for these curves. Moreover, we confer applications of normal-direction to some special curves such as helix, slant helix, plane curve or normal-direction (ND)-normal curves in E3. And, we show that slant helices and rectifying curves might be assemble by using normal-direction curves.

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MSC: 53A04.

Key words: Incorporated curve; normal-direction curve; normal-donor curve.

1. Introduction

In the curve theory of Euclidean space, the momentous question is achieve a characterization in order to a regular curve. The specification may be dedicated for a single curve or for a curve pair. Helix, slant helix, plane curve, spherical curve, etc. are well-known instance of single special curves [1,9,10,13,18] and these curves, exclusively the helices, are used in many applications [2,7,8,15]. Additionally, special curves can be defined by careful Frenet planes. Providing the position vector of a curve always lies on its rectifying, osculating or normal planes, then the curve is called rectifying curve, osculating curve or normal curve, seriatim [4]. Exclusively, therein obtain a basic correlation among rectifying curves and Darboux vectors, which trick some momentous parts in mechanics, kinematics as well as in differential geometry in describing the curves of constant motion [5,12].

Besides, special curve pairs are characterized by some relationships between their Frenet vectors or curvatures. Involute-evolute curves, Bertrand curves, Mannheim curves are admitted sample of curve pairs and studious by some mathematicians [3,11-13,16,17].

Hereabout, a new curve pair in the Euclidean 3-space E^3 has been defined by Choi and Kim [6]. They have considered an integral curve γ of a unit vector field X defined in the Frenet basis of a Frenet curve α and they have given the definitions and characterizations of principal-directional curve and principal-donor curve in E^3.They have also given some applications of these curves to some special curves.

In the current paper, we consider a new type of associated curve and define a new curve pair such as normal-direction curve and normal-donor curve in E^3. We obtain some characterizations for these curves and show that normal-direction curve is an evolute of normal-donor curve. Moreover, we give some applications of normal-direction curve to some special curves such as helix, slant helix or plane curve.

2. Preliminaries

This section includes a brief summary of space curves and definitions of general helix and slant helix in the Euclidean 3-space E^3.

A unit speed curve $\alpha : I \to E^3$ is called a general helix if there is a constant vector u, so that $\langle T, u\rangle = \cos\theta$ is constant along the curve, where $\theta \neq \pi/2$ and $T(s) = \alpha'(s)$ is unit tangent vector of α at s. The curvature (or first curvature) of α is defined by $\kappa(s) = \|\alpha''(s)\|$. Then, the curve α is called Frenet curve, if $\kappa(s) \neq 0$, and the unit principal normal vector $N(s)$ of the curve α at s is given by $\alpha''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of α at s. Then $\{T, N, B\}$ is called the Frenet frame of α. For the derivatives of the Frenet frame, the following Frenet-Serret formulae hold:

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (1)$$

where $\tau(s)$ is the torsion (or second curvature) of α at s. It is well-known that the curve α is a general helix if and only if $\frac{\tau}{\kappa}(s) = \text{constant}$ [17]. If both $\kappa(s) \neq 0$ and $\tau(s)$ are constants, we call α as a circular helix. A curve α with $\kappa(s) \neq 0$ is called a slant helix if the principal normal lines of α make a constant angle with a fixed direction. Also, a slant helix α in E^3 is characterized by the differential equation of its curvature κ and its torsion τ given by

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)' = \text{constant}.$$

(See [11]).

Now, we give the definitions of some associated curves defined by Choi and Kim [6]. Let $I \subset \mathbb{R}$ be an open interval. For a Frenet curve $\phi : I \to E^3$, consider a vector field X given by

$$\chi(s) = \vartheta(s)\mathbf{T(s)} + \upsilon(s)\mathbf{N(s)} + \omega(s)\mathbf{B(s)},$$

(2)

where ϑ , υ and ω are arbitrary differentiable functions of s which is the arc length parameter of ϕ. Let

$$\vartheta^2(s) + \upsilon^2(s) + \omega^2(s) = 1, \tag{3}$$

holds. Then the definitions of χ-direction curve and χ-donor curve in E^3 are given as follows.

Definition 2.1. ([6]) Let ϕ be a Frenet curve in Euclidean 3-space E^3 and χ be a unit vector field satisfying the equations (2) and (3). The integral curve $\delta : I \to E^3$ of χ is called an χ-direction curve of ϕ. The curve ϕ whose χ-direction curve is δ is called the ϕ-donor curve of δ in E^3.

Definition 2.2. ([6]) An integral curve of principal normal vector $\mathbf{N(s)}$ (resp. binormal vector $\mathbf{B(s)}$) of ϕ in (2) is called the principal-direction curve (resp. binormal-direction curve) of ϕ in E^3.

Remark 2.1. ([6]) A principal-direction (resp. the binormal-direction) curve is an integral curve of $\phi(s)$ with $\vartheta(s) = \omega(s) = 0,\ \ \upsilon(s) = 1$ (resp. $\vartheta(s) = \upsilon(s) = 0,\ \ \omega(s) = 1$) for all s in (2).

3. Normal-direction curve and normal-donor curve in E^3

In this section, we will give definitions of normal-direction curve and normal donor curve in E^3. We obtain some theorems and results characterizing these curves. First, we give the following definition.

Definition 3.1. Let α be a Frenet curve in E^3 and X be a unit vector field lying on the normal plane of α and defined by

$$X(s) = v(s)N(s) + w(s)B(s), v(s) \neq 0,\ \ w(s) \neq 0, \tag{4}$$

and satisfying that the vectors $X'(s)$ and $T(s)$ are linearly dependent. The integral curve $\gamma : I \to E^3$ of $X(s)$ is called a normal-direction curve of α. The curve α whose normal -direction curve is γ is called the normal-donor curve in E^3.

The Frenet frame is a rotation-minimizing with respect to the principal normal N[9]. If we consider a new frame given by $\{T, X, M\}$ where $M = T \times X$, we have that this new frame is rotation-minimizing with respect to T, i.e., the unit vector X belongs to a rotation-minimizing frame.

Since, $X(s)$ is a unit vector and $\gamma : I \to E^3$ is an integral curve of $X(s)$, without loss of generality we can take s as the arc length parameter of γ and we can give the following characterizations in the view of these information.

Theorem 3.1. *Let $\alpha : I \to E^3$ be a Frenet curve and an integral curve of $X(s) = v(s)N(s) + w(s)B(s)$ be the curve $\gamma : I \to E^3$. Then, γ is a normal-direction curve of α if and only if the following equalities hold,*

$$v(s) = \sin\left(\int \tau ds\right) \neq 0, \quad w(s) = \cos\left(\int \tau ds\right) \neq 0.$$

(5)

Proof: Since γ is a normal-direction curve of α, from Definition 3.1, we have

$$X(s) = v(s)N(s) + w(s)B(s),$$

(6)

and

$$v^2(s) + w^2(s) = 1.$$

(7)

Differentiating (6) with respect to s and by using the Frenet formulas, it follows

$$X'(s) = -v\kappa T + (v' - w\tau)N + (w' + v\tau)B.$$

(8)

Since we have that X' and T are linearly dependent. Then from (8) we can write

$$\begin{cases} -v\kappa \neq 0, \\ v' - w\tau = 0, \\ w' + v\tau = 0. \end{cases}$$

(9)

The solutions of second and third differential equations are

$$v(s) = \sin\left(\int \tau ds\right) \neq 0, \quad w(s) = \cos\left(\int \tau ds\right) \neq 0,$$

respectively, which completes the proof.

Theorem 3.2. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal-direction curve of α, then γ is a space evolute of α.*

Proof: Since γ is an integral curve of X, we have $\gamma' = X$. Denote the Frenet frame of γ by $\{\bar{T}, \bar{N}, \bar{B}\}$. Differentiating $\gamma' = X$ with respect to s and by using Frenet formulas we get

$$X' = \bar{T}' = \bar{\kappa}\bar{N}.$$

(10)

Furthermore, we know that X' and T are linearly dependent. Then from (10) we get $\bar{N}$ and T are linearly dependent, i.e, γ is a space evolute of α.

Theorem 3.3. *Let $\alpha : I \to E^3$ be a Frenet curve. If γ is the normal direction curve of α, then the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ of γ are given as follows,*

$$\bar{\kappa} = \kappa \left| \sin\left(\int \tau ds\right) \right|, \bar{\tau} = \kappa \cos\left(\int \tau ds\right).$$

(11)

Proof: From (8), (9) and (10), we have

$$\bar{\kappa}\bar{N} = -v\kappa T.$$

(12)

By considering (12) and (5) we obtain

$$\bar{\kappa}\bar{N} = -\kappa \sin\left(\int \tau ds\right) T,$$

(13)

which gives us

$$\bar{\kappa} = \kappa \left| \sin\left(\int \tau ds\right) \right|.$$

(14)

Moreover, from (13) and (14), we can write

$$\bar{N} = T.$$

(15)

Then, we have

$$\bar{B} = \bar{T} \times \bar{N} = \cos\left(\int \tau ds\right) N - \sin\left(\int \tau ds\right) B.$$

(16)

Differentiating (16) with respect to s gives

$$\bar{B}' = -\kappa \cos\left(\int \tau ds\right) T.$$

(17)

Since $\bar{\tau} = -\left\langle \bar{B}', \bar{N} \right\rangle = -\left\langle \bar{B}', T \right\rangle$, from (17) it follows

$$\bar{\tau} = \kappa \cos\left(\int \tau ds\right),$$

(18)

that finishes the proof.

Corollary 3.1. *Letγ be a normal-direction curve of the curve α. Then the relationships between the Frenet frames of curves are given as follows,*

$$X = \bar{T} = \sin\left(\int \tau ds\right) N + \cos\left(\int \tau ds\right) B, \bar{N} = T, \bar{B} = \cos\left(\int \tau ds\right) N - \sin\left(\int \tau ds\right) B.$$

Proof: The proof is clear from Theorem 3.3.

Theorem 3.4. *Let γ be a normal-direction curve of α with curvature $\bar{\kappa}$ and torsion $\bar{\tau}$. Then curvature κ and torsion τ of α are given by*

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}, \tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}}\right)'.$$

Proof: From (14) and (18), we easily get

$$\kappa = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}. \tag{19}$$

Substituting (19) into (14) and (18), it follows

$$\left| \sin\left(\int \tau ds \right) \right| = \frac{\bar{\kappa}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \tag{20}$$

$$\cos\left(\int \tau ds \right) = \frac{\bar{\tau}}{\sqrt{\bar{\kappa}^2 + \bar{\tau}^2}}, \tag{21}$$

respectively. Differentiating (20) with respect to s, we have

$$\tau \cos\left(\int \tau ds \right) = \frac{\bar{\tau}(\bar{\kappa}'\bar{\tau} - \bar{\kappa}\,\bar{\tau}')}{(\bar{\kappa}^2 + \bar{\tau}^2)^{3/2}}. \tag{22}$$

From (21) and (22), it follows

$$\tau = \frac{\bar{\kappa}'\,\bar{\tau} - \bar{\kappa}\,\bar{\tau}'}{\bar{\kappa}^2 + \bar{\tau}^2},$$

or equivalently,

$$\tau = \frac{\bar{\tau}^2}{\bar{\kappa}^2 + \bar{\tau}^2} \left(\frac{\bar{\kappa}}{\bar{\tau}} \right)'. \tag{23}$$

Theorem 3.4 leads us to give the following corollary whose proof is clear.

Corollary 3.2. *Let γ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$ be a normal-direction curve of α. Then*

$$\frac{\tau}{\kappa} = -\frac{\bar{\kappa}^2}{\left(\bar{\kappa}^2 + \bar{\tau}^2\right)^{3/2}} \left(\frac{\bar{\tau}}{\bar{\kappa}}\right)',$$

(24)

is satisfied, where κ and τ are curvature and torsion of α, respectively.

4. Applications of normal-direction curves

In this section, we focus on relations between normal-direction curves and some special curves such as general helix, slant helix, plane curve or rectifying curve in E^3 .

4.1. General helices, slant helices and plane curves

Considering Corollary 3.2, we have the following theorems which gives a way to construct the examples of slant helices by using general helices.

Theorem 4.1. *Let $\alpha : I \to E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α. Then the followings are equivalent,*

i) A Frenet curve α is a general helix in E^3.

ii) α is a normal-donor curve of a slant helix.

iii) A normal-direction curve of α is a slant helix.

Theorem 4.2. *Let $\alpha : I \to E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α. Then the followings are equivalent,*

i) A Frenet curve α is a plane curve in E^3.

ii) α is a normal-donor curve of a general helix.

iii) A normal-direction curve of α is a general helix.

Example 4.1. Let consider the general helix given by the parametrization $\alpha(s) = \left(\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right)$ in E^3 (Fig. 1). The Frenet vectors and curvatures of α are obtained as follows,

$$T(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),$$

$$N(s) = \left(-\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ 0\right),$$

$$B(s)=\left(\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),$$

$$\kappa=\tau=\frac{1}{2}.$$

Then we have $X(s)=(x_1(s),\ x_2(s),\ x_3(s))$ where

$$\begin{array}{l} x_1(s)=-\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}, \\ x_2(s)=\sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}, \\ x_3(s)=\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right). \end{array}$$

and c is integration constant. Now, we can construct a slant helix γ which is also a normal-direction curve of α (Fig. 2):

$$\gamma=\int_0^s\gamma'(s)ds=\int_0^s X(s)ds=(\gamma_1(s),\ \gamma_2(s),\ \gamma_3(s)),$$

where

$$\begin{array}{l} \gamma_1(s)=\int_0^s\left[-\sin\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}\right]ds, \\ \gamma_2(s)=\int_0^s\left[\sin\left(\frac{s}{2}+c\right)\sin\frac{s}{\sqrt{2}}-\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\cos\frac{s}{\sqrt{2}}\right]ds, \\ \gamma_3(s)=\int_0^s\frac{1}{\sqrt{2}}\cos\left(\frac{s}{2}+c\right)\ ds. \end{array}$$

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Fig. 1. General helix α. **Fig. 2.** Slant helix γ constructed by α.

4.2. ND-normal Curves

In this subsection we define normal-direction (ND)-normal curves in E^3 and give the relationships between normal-direction curves and ND -normal curves.

A space curve whose position vector always lies in its normal plane is called normal curve [5]. Moreover, if the Frenet frame and curvatures of a space curve are given by $\{T,N,B\}$ and $\kappa,\ \tau$, respectively, then the vector $\tilde{D}(s)=\frac{\tau}{\kappa}(s)T(s)+B(s)$ is called modified Darboux vector of the curve [11].

Let now α be a Frenet curve with Frenet frame $\{T, N, B\}$ and γ a normal-direction curve of α. The curve γ is called normal-direction normal curve (or ND -normal curve) of α, if the position vector of γ always lies on the normal plane of its normal-donor curve α.

The definition of ND-normal curve allows us to write the following equality,

$$\gamma(s) = m(s)N(s) + n(s)B(s),$$

(25)

where $m(s),\ n(s)$ are non-zero differentiable functions of s. Since γ is normal-direction curve of α, from Corollary 3.1, we have

$$\left\{\begin{array}{l} N = \sin\left(\int \tau ds\right)\bar{T} + \cos\left(\int \tau ds\right)\bar{B}, \\ B = \cos\left(\int \tau ds\right)\bar{T} - \sin\left(\int \tau ds\right)\bar{B}. \end{array}\right.$$

(26)

Substituting (26) in (25) gives

$$\gamma(s) = \left[m\sin\left(\int \tau ds\right) + n\cos\left(\int \tau ds\right)\right]\bar{T} + \left[m\cos\left(\int \tau ds\right) - n\sin\left(\int \tau ds\right)\right]\bar{B}.$$

(27)

Writing

$$\left\{\begin{array}{l} \rho(s) = m\sin\left(\int \tau ds\right) + n\cos\left(\int \tau ds\right), \\ \sigma(s) = m\cos\left(\int \tau ds\right) - n\sin\left(\int \tau ds\right), \end{array}\right.$$

(28)

in (27) and differentiating the obtained equality we obtain

$$\bar{T} = \rho'\bar{T} + (\rho\bar{\kappa} - \sigma\bar{\tau})\bar{N} + \sigma'\bar{B}.$$

(29)

Then we have

$$\sigma = a = \text{constant},\ \rho = s + b = \frac{\bar{\tau}}{\bar{\kappa}} a,$$

(30)

where $a,\ b$ are non-zero integration constants. From (30), it follows that

$$\gamma(s) = a\left(\frac{\bar{\tau}}{\bar{\kappa}}\bar{T} + \bar{B}\right)(s) = a\tilde{\bar{D}}(s),$$

(31)

where $\tilde{\bar{D}}$ is the modified Darboux vector of γ.

Now we can give the followings which characterize ND-normal curves.

Theorem 4.3. *Let $\alpha : I \to E^3$ be a Frenet curve in E^3 and γ be a normal-direction curve of α. If γ is a ND-normal curve in E^3, then we have the followings,*

i) γ is a rectifying curve in E^3 whose curvatures satisfy $\frac{\bar{\tau}}{\bar{\kappa}} = \frac{s+b}{a}$ where $a,\ b$ are non-zero constants .

ii) The position vector and modified Darboux vector $\tilde{\bar{D}}$ of γ are linearly dependent.

Theorem 4.3 gives a way to construct a rectifying curve by using normal-donor curve as follows:

Corollary 4.1. *Let $\alpha : I \to E^3$ be a Frenet curve in E^3 and γ a ND-normal curve of α in E^3. Then the position vector of γ is obtained as follows,*

$$\gamma(s) = \left[(s+b)\sin\left(\int \tau ds\right) + a\cos\left(\int \tau ds\right)\right] N(s) + \left[(s+b)\cos\left(\int \tau ds\right) - a\sin\left(\int \tau ds\right)\right] B(s)$$

(32)

where $a,\ b$ are non-zero integration constants.

Proof. The proof is clear from (25), (28) and (30).

Example 4.2. Let consider the general helix given by the parametrization

$$\alpha(s) = \left(\sqrt{1+s^2}, s, \ln(s + \sqrt{1+s^2})\right),$$

and drawn in Fig. 3. Frenet vectors and curvatures of the curve are

$$T(s)=\frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(s,\sqrt{1+s^2},1\right),$$

$$N(s)=\frac{1}{\sqrt{1+s^2}}\left(1,0,-s\right),$$

$$B(s)=\frac{1}{\sqrt{2}\sqrt{1+s^2}}\left(-s,\sqrt{1+s^2},-1\right),$$

$$\kappa=\tau=\frac{1+s^2}{2},$$

respectively. Then from Corollary 4.1, a ND-normal curve γ is obtained as follows,

$$\begin{aligned}\gamma(s)=\Big(&\tfrac{1}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)+a\cos\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)\right]\\&-\tfrac{s}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)-a\sin\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)\right],\\&-\tfrac{1}{\sqrt{2}}\left[(s+b)\cos\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)-a\sin\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)\right],\\&-\tfrac{s}{\sqrt{1+s^2}}\left[(s+b)\sin\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)+a\cos\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)\right]\\&-\tfrac{1}{\sqrt{2(1+s^2)}}\left[(s+b)\cos\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)-a\sin\left(\tfrac{s}{2}+\tfrac{s^3}{6}+c\right)\right]\Big)\end{aligned}$$

which is also a rectifying curve in the view of Theroem 4.3 and drawn in Figures 4,5,6 by choosing $a=b=1,\ \ c=0$.

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Fig. 3. General helix α. **Fig 4.** ND-normal curve γ for $-\pi\leq s\leq\pi$.

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Fig 5. ND-normal curve γ for $\frac{-3\pi}{2}\leq s\leq\frac{3\pi}{2}$. **Fig 6.** ND-normal curve γ for $-2\pi\leq s\leq 2\pi$.

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