# In-situ estimation of erosion model parameters using an advection-diffusion model and Bayesian inversion

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### Abstract

We describe a framework for the simultaneous estimation of model parameters in a partial differential equation using sparse observations. Monte Carlo Markov Chain (MCMC) sampling is used in a Bayesian framework to estimate posterior probability distributions for each parameter. We describe the necessary components of this approach and its broad potential for application in models of unsteady processes. The framework is applied to three case studies, of increasing complexity, from the field of cohesive sediment transport. We demonstrate that the framework can be used to recover posterior distributions for all parameters of interest and the results agree well with independent estimates (where available). We also demonstrate how the framework can be used to compare different model parameterizations and provide information on the covariance between model parameters.

## In-situ estimation of erosion model parameters using an advection-diffusion model and Bayesian inversion

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### 6 Key Points:

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7	•	Probabilistic framework to estimate unobserved erosion model parameters using
8		sparse measurements collected above the seabed.
9	•	General approach can be updated with any model parameterization and quanti-
10		tatively compared.
11	•	The framework is applicable to many similar data sets with both unsteady or quasi-

steady forcing and response.

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#### 13 Abstract

We describe a framework for the simultaneous estimation of model parameters in a par-14 tial differential equation using sparse observations. Monte Carlo Markov Chain (MCMC) 15 sampling is used in a Bayesian framework to estimate posterior probability distributions 16 for each parameter. We describe the necessary components of this approach and its broad 17 potential for application in models of unsteady processes. The framework is applied to 18 three case studies, of increasing complexity, from the field of cohesive sediment trans-19 port. We demonstrate that the framework can be used to recover posterior distributions 20 for all parameters of interest and the results agree well with independent estimates (where 21 available). We also demonstrate how the framework can be used to compare different 22 model parameterizations and provide information on the covariance between model pa-23 rameters.

### <sup>25</sup> Plain Language Summary

We describe a framework for the simultaneous estimation of multiple unobserved 26 parameters by combining observations of a tracer with a numerical model. This frame-27 work uses Bayesian inference techniques established in statistical literature to estimate 28 the unobserved parameters of interest used in the model with uncertainty quantification. 29 We explain the key components of this framework in simple terms to encourage its use 30 for analysing other unsteady processes and performing quantitative inference on param-31 eters that are difficult or impossible to measure directly. We then demonstrate the frame-32 work's efficacy by applying it to three case studies from the field of cohesive sediment 33 transport that all use the transport equation (advection-diffusion). Inferred parameter 34 values show good agreement with independent estimates, where available. 35

### 36 1 Introduction

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Numerical solutions to partial differential equations (PDE) are a key tool in mod-37 elling complex processes. There is, however, often a natural variation of the key parameters required for accurate modelling which presents a challenge in many fields of research 39 (e.g. boundary layer physics (Souza et al., 2020), pulmonary circulation (Păun et al., 2018), 40 groundwater flow (Ghouili et al., 2017), population spread (Soubeyrand & Roques, 2014), 41 and sediment transport (Manning & Schoellhamer, 2013; Valipour et al., 2017). Mea-42 surements can provide insight into these complex processes, but in many cases the pa-43 rameter of interest is difficult to observe directly (e.g. Liñán Baena et al., 2009; Egan 44 et al., 2021; Maa & Kwon, 2007; Mattsson et al., 2016; Smyth et al., 2021). Furthermore, 45 the unsteady time- and space-dependent nature of some processes can mean that meth-46 ods developed under controlled settings (e.g. steady forcing in a laboratory) are not read-47 ily implementable in real world examples. 48

To overcome this challenge, we describe a framework to estimate PDE parameters by combining time- and space-dependent observations with a numerical model of a known PDE to solve the so-called inverse problem. The typical approach when solving the inverse problem is to determine the set of model parameters,  $\theta$ , that minimises the residual,  $\epsilon$ , between the observations and model outcome. For a time-dependent, 1-dimensional spatial model this is then written

$$C_{obs}(z,t) = f(z,t;\theta) + \epsilon, \tag{1}$$

which states that the observations,  $C_{obs}(z,t)$  (herein C), can be represented by a chosen model,  $f(z,t;\theta)$ , plus the residual,  $\epsilon$ . We have used z to represent the spatial dimension and t to represent time. Here we employ a probabilistic framework based on Bayesian inference to solve the inverse problem (described in Section 2). This allows for the simultaneous estimation of probability distributions for all model parameters and the residual  $\epsilon$ , thus providing full uncertainty quantification and parameter covariance information.

When modelling environmental processes the required parameters may take a range 63 of probable values, i.e., a probabilistic description of the parameters is often most suit-64 able. Quantifying this range provides more meaningful insights than maximum likelihood 65 estimation alone, especially when the parameter probability distribution is non-Gaussian 66 or multi-modal. Bayesian frameworks for solving inverse problems with uncertainty quan-67 tification have been described extensively in the statistical literature, but uptake by researchers in many fields has been slow. For this reason, we take a pedagogical approach when describing the method used in this study. The Bayesian inference framework can 70 be employed for parameter estimation in a wide range of numerical model applications 71 that use a known (but potentially complex) state equation that is computationally in-72 expensive to solve. 73

To demonstrate this method we examine a numerical model of the one-dimensional 74 unsteady advection-diffusion equation (ADE), widely used to model tracer evolution in 75 fluid flows. This type of process has intrinsic variability and measurements of tracer con-76 centration can be noisy. In addition, typical numerical models of the ADE may be ide-77 alised, as in this work, and boundary or initial conditions may also contain errors or re-78 quire estimation with sub-models. These sub-models may also be idealised or mis-specified, 79 and can include additional unknown parameters that contribute uncertainty. This makes 80 a numerical model of the ADE a good example for simultaneous parameter estimation 81 with uncertainty quantification. 82

We test this methodology on three different case studies of increasing complexity, 83 all from the field of cohesive sediment transport. This field is a suitable test for the method-84 ology used herein because there are a number of parameters that cannot be directly observed. Recent research in this field continues to identify large differences in all key pa-86 rameter estimates depending on site, time, and/or method (see Maa and Kwon (2007) 87 for an example of particle settling velocity, Valipour et al. (2017) for critical bed strength, 88 and Egan et al. (2021) for the erosion rate parameter). Similar physics-driven approaches 89 to parameter estimation using observations have been performed before in cohesive sed-90 iment environments (e.g. Brand et al., 2015; Zhang et al., 2021), but not using a Bayesian 91 framework. Schmelter et al. (2011) assessed bed load transport sediment transport in 92 a Bayesian framework and provide an excellent summary of the key fundamentals for Bayesian 93 modelling. Before analysing the case studies we first describe the Bayesian framework 94 used for parameter estimation in more detail. Then we describe the numerical model used 95 for the case studies, along with two boundary flux (sub-)models implemented within the 96 numerical model. Following that we describe the observations, model specifics, and re-97 sults of the inference for each case study. 98

The three case studies are similar in nature so they use the same numerical model, 99 which limits the repetition of model specifics that are incidental to the general method-100 ology. Case Study 1 is from an annular flume erosion experiment conducted in Baltimore 101 Harbour by Maa et al. (1998) (digitised from Sanford and Maa (2001), Figure 1b). Case 102 Study 2 is a synthetic data set generated from the numerical model itself, with the in-103 puts designed to mimic typical unsteady tidal forcing. Case Study 3 is an application 104 of the method to a set of complex (unsteady and noisy) real-world observations from Edge et al. (2021). Our goal is to use these case studies of sediment transport to gradually in-106 troduce complexity and demonstrate the efficacy of the method for estimating un-observable, 107 yet important, parameters used in the prediction of the response of a geophysical sys-108 tem. 109

### <sup>110</sup> 2 Parameter inference methods

### 111 2.1 General approach

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<sup>112</sup> We first describe the general framework used in this study to solve the inverse prob-<sup>113</sup> lem posed in Equation 1. Bayesian statistical and computational techniques are used to <sup>114</sup> simultaneously estimate the posterior probability of the model parameters ( $\theta$ ), given some <sup>115</sup> observations C, stated  $p(\theta|C)$ . The approach is derived from Bayes' theorem,

$$p(\theta|C) \propto p(C|\theta)p(C).$$

(2)

The posterior probability density (herein the posterior) is proportional to the product 117 of the likelihood,  $p(C|\theta)$ , and the prior probability density, p(C). The prior and poste-118 rior are m-dimensional distributions, where m is the number of parameters to sample. 110 The likelihood is a cost function that evaluates the chosen model and penalises model outcomes that are further from the observations. Only the likelihood function is max-121 imised in a frequentist approach; there is no incorporation of prior beliefs in that frame-122 work. One advantage of the Bayesian approach is that the prior on each parameter can 123 represent information that is already known or expected, such as a plausible range of re-124 ported values from literature. 125

Note that in Equation 2 the parameter vector  $\theta$  includes  $\sigma$ , which represents the 126 standard deviation of  $\epsilon$ , the residual from Equation 1. For this work we have assumed 127 that the distribution of values in  $\epsilon$  is homoskedastic (invariant with respect to predic-128 tor variables) and Gaussian, i.e.  $\epsilon = \mathcal{N}(0, \sigma)$ . Differences between the model outcome 129 and the observations may be the result of natural variation, measurement error, or model 130 mis-specification, all of which are encapsulated in  $\sigma$ . By estimating the magnitude of  $\sigma$ , 131 Bayesian inference is able to partition these sources of uncertainty from the uncertainty in the parameter estimates. This allows for better parameter estimates while still accu-133 rately predicting the plausible range C may take. 134

The use of Markov Chain Monte Carlo (MCMC) sampling techniques allow the posterior to be approximated in the proportional form in Equation 2. For each step in a Markov Chain a new  $\theta$  is proposed and Equation 2 is evaluated. By doing this, MCMC sampling is able to explore the posterior space and move towards regions of proportionally higher probability until a stationary distribution is found. The specifics of defining priors, computing the likelihood function, and sampling the posterior using MCMC sampling are described in more detail below.

### 2.2 Likelihood

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The likelihood function quantifies the probability that  $\theta$  was used to generate a set of observations, C. This allows us to compare the probability of different proposals for  $\theta$ . At this point we separate  $\theta$  into the model parameters,  $\theta_{mod}$  and  $\sigma$  for clarity. The likelihood is assumed to be a Gaussian distribution for a sample of observations,  $\mathbf{C} =$  $(C_i, ..., C_n)$ . For computational reasons the likelihood function is evaluated in log-space and can be written as

$$\mathcal{L}(y|\theta_{mod},\sigma) = -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma}\sum_{i=1}^{n} (C_i - y_i)^2.$$
 (3)

Here  $y = f(\theta_{mod})$  is the output from our chosen model and n is the number of observations. Note that each element in  $\theta_{mod}$  and  $\sigma$  are free parameters estimated during MCMC sampling.

### 153 2.3 Priors

The prior distributions on each parameter form the second component to the righthand-side of Equation 2. They represent the probability that we assign to each parameter before we have performed the inference. Priors can be informative, weakly informative, or non-informative (diffuse). If we already have detailed information on a parameter, say from well-established physical theory, literature, or independent measurements, we can use an informative prior to limit the space the posterior can move to. Conversely, for parameters that potentially could occupy a large range, we can set priors to be diffuse so that MCMC sampling can explore this range without being penalised.

Our aim when defining the prior distributions for this research was to limit the posterior sample space to a reasonable range of values that were supported by physical theory and literature. This increases the efficiency of MCMC sampling and reduces the time required for individual chains to converge on a stationary distribution (Tarantola, 2004). When the chosen model is a numerical model, priors can also be used to restrict the parameter space to regions of model stability. For example, an explicit advection-diffusion numerical model becomes unstable when the Courant number exceeds one, so any parameter that may cause this to occur must be restricted.

The specific priors used for each case study are described before the results. All the priors used in this study were either half-normal  $(\mathcal{H})$  or log-normal  $(\ln \mathcal{N})$  distributions. These were used because all of the parameters in our chosen model could not be negative, including  $\sigma$ . Parameters that could be zero were assigned half-normal priors, while parameters that could not be zero were assigned log-normal distributions. This study was focused on the development and testing of the described methodology. Priors were set to be weakly informative, based on a range of possible values determined from literature.

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### 2.4 MCMC sampling and evaluation

With the likelihood and prior distribution functions defined we can now use Markov
Chain Monte Carlo (MCMC) sampling to investigate the posterior. The aim of MCMC
sampling is to explore the posterior space until the chain (or chains) converge on a stationary target distribution. For a thorough explanation of MCMC sampling methods the
reader is referred to Gelman et al. (2013, Part III).

For this work we have used Differential Evolution MCMC sampling (DE-MC) (Ter 184 Braak, 2006), though we note that the method is independent of the specific MCMC sam-100 pling method. DE-MC is a variation of the Metropolis algorithm and uses information from other chains to improve the sampling efficiency for all chains. The advantage of DE-187 MC is it's speed of convergence, particularly in high dimensional problems. A practical 188 advantage of the DE-MC sampler is that it does not require the computation of gradi-189 ents of the likelihood function with respect to the parameters. Note that the framework 190 was implemented with a different MCMC method (Dynesty, described in Speagle (2020)) 191 with equivalent results. 192

Evaluation of convergence between chains was performed using common tests described in Gelman et al. (2013). Specifically, we determined convergence was met when the  $\hat{R}$  statistic was less than 1.05 for each parameter and the auto-correlation for all chains had dropped to near zero. If satisfactory convergence of the chains was achieved the results were deemed suitable for analysis and prediction.

### <sup>198</sup> 3 The numerical model

Here we describe the specific numerical model used in this study, f, noting that in theory any model may take its place. The numerical model is a finite difference discretisation of the 1-dimensional unsteady advection-diffusion equation (ADE). The ADE, described next, can be used to model the transport of any scalar, C, in fluid flows (among other applications). The model is applied to three case studies from the field of cohesive sediment transport. In this field the fluxes of C at the boundary (sediment-water interface) of the ADE model are key drivers of changes in C and much research has been undertaken to determine the best parameterizations for this process. In this section, we describe the chosen sub-models, herein boundary flux models, that we have used and define the parameters of interest.

#### 3.1 The advection-diffusion equation

The full three dimensional time-space dependent form of the ADE can be written as

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$$\frac{\partial C}{\partial t} + \nabla .(\boldsymbol{u})C = \nabla .(\gamma \nabla C) + S, \tag{4}$$

where C is the tracer concentration,  $\boldsymbol{u}$  is the three-dimensional velocity vector,  $\gamma$  is the diffusivity, and S is a source or sink term.

In order to simplify this equation to one dimension we have made several assump-215 tions. The primary assumption is that the observed process can be adequately represented 216 in a single (vertical) dimension. First, we have assumed that vertical turbulent diffusion, 217 denoted  $\gamma_z$ , is the only important component in the diffusivity term. Second, we have 218 assumed that the horizontal flux divergence components (in  $\nabla . \boldsymbol{u} C$ ) are small and can be ignored. Last, we have assumed that the only source and sink terms come from the 220 model bottom boundary. The two boundary flux models are: a source of sediment to the 221 water column by erosion, E, and a loss of sediment from the water column by deposi-222 tion, D. Both terms are only applied to the lowest grid cell. The one-dimensional-vertical 223 (1Dv) equation used in this study is thus 224

$$\frac{\partial C}{\partial t} + \frac{\partial (w - w_s)C}{\partial z} = \frac{\partial}{\partial z} \left( \gamma_z \frac{\partial C}{\partial z} \right) + E - D.$$
(5)

Note that the tracer used for this study is sediment, which is negatively buoyant, and the tracer settling velocity,  $w_s$ , is a parameter of interest and is applied to the vertical direction only (herein defined as positive downward). For this study  $w_s$  is assumed to be independent of time, space, and C.

In order to utilise the ADE we discretised each of the terms using an implicit finite difference method (2nd-order centred-time centred-space). The model was forced with observations of vertical current velocity (w) and bottom friction velocity  $(u_*, used$ to estimate the time-dependent bed shear stress and profile of the vertical turbulent diffusivity  $\gamma_z$ ). The output from the model can then be compared to observations of C. The parameters of interest that we want to estimate are  $w_s$  and any parameters required for the chosen boundary flux model (described next).

### 3.2 Boundary flux models

In sediment transport studies, the upward fluxes into the water column from erosion and the downward fluxes out of the water column to the bed from deposition are key processes that control the magnitude of C. Erosion and deposition have been the subject of considerable investigation over the past six decades and the optimal models to represent these processes are still the subject of ongoing research.

To demonstrate the method described in this study we compared two erosion models. Both erosion models translate a known bed stress,  $\tau_{bed}$ , into a flux C that is added to the lowest grid cell as a source term. The deposition flux was simply calculated as the product of  $w_s$  and the concentration in the lowest grid cell,  $C_0$ . We note that the method described in this research could be used for comparison and parameter estimation with any erosion or deposition boundary flux models, or combinations thereof. The first erosion model tested was a variation of the popular Ariathurai-Partheniadesequation, given by

$$E = M(\tau_{bed} - \tau_{cr})^b \quad \text{where} \quad \tau_{cr} = \tau_{cr0} \tag{6}$$

(Ariathurai, 1974), shown here without the denominator. Here  $\tau_{bed}$  is the bed shear stress, which is known, and  $\tau_{cr0}$ , M, and b are the erosion model parameters to infer. Herein this boundary flux model is referred to as E1.

The second erosion model tested was another variation of the Ariathurai-Partheniades equation where the term  $\tau_{cr}$  was allowed to vary in proportion to the total C in the numerical model,  $m_c$ ,

$$E = M(\tau_{bed} - \tau_{cr}) \quad \text{where} \quad \tau_{cr} = \tau_{cr0} + \tau_m m_c^{\tau_b}. \tag{7}$$

For this erosion model M,  $\tau_{cr0}$ ,  $\tau_m$ , and  $\tau_b$  are the parameters to infer. This erosion model is an approximation of what is commonly referred to as depth-limited erosion. Accurate modelling of depth-limited erosion requires a multi-layer bed model (e.g. Sanford, 2008) which was beyond the scope of this study. We used available literature to set the prior distributions for the free parameters in each boundary flux model, discussed below, before presenting the case study results.

### 3.3 Forcing the numerical model

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There are several quantities necessary to force the numerical model which need to be either measured or estimated before the model can be run. The main observation required is a time series of the bottom friction velocity,  $u_*$ , or the bed stress,  $\tau_{bed}$ , which are related to each other by  $\tau_{bed} = \rho u_*^2$ , where  $\rho$  is a measured or assumed water density. For all examples  $\tau_{bed}$  was the input into the erosion models. In addition, if fluid vertical velocities are important at the site of interest, they can be included in the model (as they have been in Case Study 3).

The eddy diffusivity profile,  $\gamma_z$ , was generated using a parabolic model for unstratified flow (Dyer & Soulsby, 1988),

$$\gamma_z = \kappa u_* z \left( 1 - \frac{z}{B_h} \right). \tag{8}$$

In this model  $\kappa$  is the von Karman constant, z is the height above the bed, and  $B_h$  is 276 the height of the diffusivity profile. Above the boundary layer the diffusivity was set to a constant value of  $10^{-4}$  m<sup>2</sup> s<sup>-1</sup>. A single point at the top edge of the boundary layer 278 was smoothed (quadratic interpolation) to remove the discontinuous transition between 279 the parabolic model and the area of constant diffusivity. The height  $B_h$  was either con-280 stant or time-dependent, depending on the case study. Note that any diffusivity profile 281 could be used with this method. To demonstrate our method we have thus chosen a sim-282 ple profile, but the implementation of more sophisticated diffusivity models is a possi-283 ble focus for future research. Estimation of the diffusivity profiles is only required one 284 time before inference is performed, so this would not significantly increase the compu-285 tation time. 286

### <sup>287</sup> 4 Case studies

### 4.1 Setting priors

All three case studies chosen for this work come from the field of cohesive sediment transport and as such, they all exhibit similar behaviour. For this work we used weakly informative priors determined from the literature and used any independent results for comparison. Because we used weakly informative priors we were able to specify the same priors for all three case studies [Table 1].

Parameter	Prior	95% CI	References
$w_s \times 10^{-3} \mathrm{~m~s^{-1}}$	$ln\mathcal{N}(0.5, 0.7)$	(0.50, 5.34)	Neville (1986), Cartwright et al. (2013), Voulgaris and Meyers (2004), Maa and Kwon (2007), Manning and Schoellhamer (2013)
$M \times 10^{-4}$ kg m <sup>-2</sup> s <sup>-1</sup> (Pa <sup>-b</sup> )*	$\mathcal{H}(20)$	(44.83)	Winterwerp and van Kesteren (2004), Maa et al. (1998), Egan et al. (2021)
$\tau_{cr0}\times 10^{-2}$ Pa	$\mathcal{H}(3)$	(6.72)	Egan et al. (2021), Maa et al. (1998), Edge et al. (2021)
$b \ / \  au_b$	$ln\mathcal{N}(0.25, 0.5)$	(0.48,  3.42)	Egan et al. (2021), Brand et al. (2015)
$\tau_m$ Pa $m_c^{-\tau_b}$	$ln\mathcal{N}(0.5, 1.5)$	(0.09, 31.05)	Sanford and Maa (2001), Brand et al. (2015)
$\sigma \times 10^{-3} \rm ~kg~m^{-3}$	$\mathcal{H}(3)$	(6.72)	-

**Table 1.** Specified prior distributions for all case studies, including the 95% credible intervals(CI) and relevant references.

\**M* units include the term  $Pa^{-b}$  for the boundary flux model *E*1 but not for *E*2.

All the priors were defined using distributions that could not be negative based on physical theory and literature. Each parameter was scaled to be (close to) order one for MCMC sampling. For parameters where the literature indicated that the underlying value may span many orders of magnitude, such as M and  $\tau_m$ , we used initial (user specified) testing to narrow down the likely range, and then specified priors to cover a few orders of magnitude. Our initial uncertainty in these parameters was reflected by the large 95% credible intervals (CI) of the priors [Table 1]. We compare and discuss the priors in light of the inferred posteriors in the case study results below.

### 4.2 Case Study 1: annular flume data

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### 4.2.1 Observations

This case study is based on an annular flume experiment conducted in Baltimore 304 Harbour (described in Maa et al., 1998). Note the data was digitised from the corrected 305 results of total mass eroded  $(m_c)$  and bed shear stress  $(\tau_{bed})$  shown in Sanford and Maa 306 (2001), Figure 1b, with  $m_c$  related to the depth-averaged tracer concentration  $(\overline{C})$  by 307 the height of the flume. This case study is typical of controlled erosion experiments per-308 formed over naturally deposited cohesive sediment beds. It involved a series of stepped 309 increases to the applied bed stress and regular measurement of  $\overline{C}$  over time [Figure 1]. 310 In such tests the next step is typically initiated once  $\overline{C}$  stabilises ( $\partial \overline{C}/\partial t$  reaches zero). 311 The relationship between bed strength  $(\tau_c)$  and mass eroded  $(m_c)$  can then be examined 312 by using the steady state values at the end of each step. 313

Here, the inference on the model parameters is performed using the entire time series of data, not just the steady points at the end of each bed stress step. Note that because this case study only uses the depth-averaged tracer concentration,  $\overline{C}$ , the advectiondiffusion model is superfluous and the problem may be simplified to a boundary condition (as demonstrated by Zhang et al., 2021). Later case studies use spatially sparse observations of C. For brevity we have used the same method for all three case studies.



Figure 1. Observations from the Maa et al. (1998) (digitised from Sanford and Maa (2001), Figure 1b) annular flume experiment for site White Rocks showing measured bed stress and  $\overline{C}$  over a period of 250 minutes. The steady state values ( $\hat{\tau}_b$  and  $\hat{C}$ ) used to estimate the bed strength profile are shown as the larger scatter points.  $\overline{C}$  has been converted to a mass concentration (from mass per unit of bed area) for comparison with the model output.

### 320 4.2.2 Model specification

We digitised the time series of bed stress and  $\overline{C}$  from Sanford and Maa (2001) and 321 used these to estimate the numerical model parameters. The digitised data was (near-322 est neighbour) interpolated onto a time grid with resolution,  $\Delta_z$ , of 60 seconds. The ver-323 tical grid was divided into four equal cells and the initial  $\overline{C}$  was set to the observed con-324 centration at t = 0. Deposition was set to zero for the entire simulation, a common as-325 sumption for cohesive sediment erosion events, thus  $\partial \overline{C} / \partial t = E$ . Therefore, in this case study we only estimated the parameters in the boundary flux (erosion) model, E, and 327 not  $w_s$ . We still present  $w_s$  in the results to show that the posterior of  $w_s$  did not sig-328 nificantly change from its prior, because any value proposed for  $w_s$  gave the same model 320 outcome). We implemented both erosion models, E1 (Equation 6) and E2 (Equation 330 7). The priors used for each scenario were unchanged from those specified previously. 331 The likelihood (Equation 3) was modified to use  $\overline{C}$  by taking the depth-average of the 332 model output for each time step before evaluation. 333

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### 4.2.3 Results and discussion

Scenarios E1 and E2 were both sampled for 25,000 iterations using 12 chains. The first 5,000 iterations of each chain were discarded as burn-in samples. Chain auto-correlation and  $\hat{R}$  convergence analysis indicated the chains were suitable for inference (not shown).

The posterior distribution for each parameter was approximated as a kernel density estimate histogram of all the accepted samples in the 12 chains (240,000 samples) [Figures 2 and 3]. The median and 95% CI of the posterior were calculated for each pa-



Figure 2. Posterior distributions (blue) for Case Study 1 - scenario *E1*. Prior distributions (gray) are shown over the range of the posterior, except for  $\sigma$ , where the range was extended to show both complete distributions. The median and lower and upper 95% CI (brackets) are shown in the axis titles and white dots on the posteriors. Note the prior distribution density may be difficult to see (close to zero) when the posterior is much more narrow than the prior.



**Figure 3.** Posterior distributions (blue) for Case Study 1 - scenario *E2*. Prior distributions (gray) are shown over the range of the posterior. The median and lower and upper 95% CI (brackets) are shown in the axis titles and white dots on the posteriors.

rameter. Each prior distribution was sampled the same number of times for comparison. For both scenarios (E1 & E2) the posterior distributions were found to be well within the limits of the prior distributions, except for  $\sigma$  in E1, which was much higher than the prior value. This implied that the erosion model in E1 was a poor choice for modelling the observed data in this annular flume data.

We randomly selected 10,000 samples of  $\theta$  from the posterior and re-ran the numerical model for each  $\theta$  to generate the posterior predictive distribution. From this we estimated the 50%, 80%, and 95% CI of the predictions, and compared them to the observations for *E1* [Figure 4 left] and *E2* [Figure 5 left].

In scenario E1 there was significant model mis-specification. The chosen erosion model was not able to replicate the observed erosion rate decay that occurred rapidly after each increase in bed stress. Instead the results generally fit "through" the stepped observations of  $\overline{C}$  [Figure 4]. This meant that the residuals were larger than expected. For scenario E2, however, the posterior predictions fit the observations well and the estimated  $\sigma$  was lower, indicating a reduction in model mis-specification (i.e. E2 is a better model than E1).



Figure 4. Left: observations of  $\overline{C}$  (yellow/red) and the posterior predictive distribution of  $\overline{C}$  based on 10,000 samples of the posterior for scenario E1. The 50%, 80%, and 95% CI of the posterior predictive distribution are shown as graduated gray shading. Right: Plots of the posterior samples from scenario E1 for correlated parameters only ( $|r^2|>0.3$ ). Black lines are the 50%, 80%, and 95% density contours.

Investigation of the posterior samples in scenario E1 indicated that all the parameters used in the erosion model (Equation 6) were correlated ( $|r^2|>0.3$ ) [Figure 4 right]. This resulted in a wide range of parameter choices that would produce a similar erosion flux. For example, a reduction in the erosion rate M could be compensated for by either an increase in  $\tau_{cr0}$  or a reduction in b, or some combination of both. The posteriors of  $w_s$  and  $\sigma$  were not correlated with any other parameters (not shown).

The posterior samples in scenario E2 indicated that some parameter pairs were correlated [Figure 5 right]. While the observed parameter covariance can be sampled by this method, high correlations can result in decreased efficiency of MCMC sampling. For the later case studies we still test both erosion models E1 and E2 because they are commonly used in sediment transport modelling. We note, however, that future work could investigate erosion models more suited to inverse method sampling techniques.

For scenario E2 we used the results to show the estimated relationship between the mass eroded,  $m_c$ , and the bed strength [Figure 6]. Note that the relationship is not just a fit of the data points (shown in blue) but of the full time series of steady and unsteady data. The bed strength estimate agrees well with the data points originally used to estimate the parameter values by Sanford and Maa (2001). This demonstrates that the method used here gave equivalent results for the estimation of the bed strength parameters, while also simultaneously estimating the erosion rate parameter, M, and  $\sigma$ .

This example demonstrates that the method described here can be successfully employed to estimate multiple model parameters simultaneously, using unsteady data in a controlled setting. It is important to note that the full time series of observations from the experiment has contributed to the inference process, not just the steady-state points at the end of each step. In the following case studies, the same process will be performed on data sets where neither the forcing nor the concentration C reach a steady-state.



Figure 5. Left: observations of  $\overline{C}$  (yellow/red) and the posterior predictive distribution of  $\overline{C}$  based on 10,000 samples of the posterior for scenario E2. The 50%, 80%, and 95% CI of the posterior predictive distribution are shown as graduated gray shading. Right: Plots of the posterior samples from scenario E2 for correlated parameters only ( $|r^2|>0.3$ ). Black lines are the 50%, 80%, and 95% density contours.



Figure 6. Total mass eroded,  $m_c$ , shown with bed strength,  $\tau_{bed}$  (shown as per Sanford and Maa (2001) Figure 2a, site White Rocks). The blue dots are the applied stress and measured  $\overline{C}$  values at the end of each step (as shown by the blue and red markers in Figure 1). The gray shading is the 50%, 80%, and 95% CI of the posterior predictive distribution for 10,000 samples from the bed strength parameters in erosion model E2 (Equation 7,  $\tau_{cr0}$ ,  $\tau_m$ , and  $\tau_b$ ). Note  $m_c$  is related to  $\overline{C}$  using the height of the flume.



Figure 7. Top: synthetic tidal forcing data for Case Study 2, including the friction velocity,  $u_*$  (black, shown multiplied by 10), and the bed stress,  $\tau_b$  (gray). Middle: filled color plot of synthetic C using erosion model E1 (Equation 6) for the entire model grid with added Gaussian noise. Bottom: synthetic observations of C for selected heights above the bottom boundary.

### 4.3 Case Study 2: synthetic tidally-driven erosion

### 383 4.3.1 Observations

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For this case study we generated synthetic observations with the numerical model 384 in order to recover specified model parameters. Forcing variables were chosen to describe idealised spring tide conditions. Guided by the field observations in Edge et al. (2021), 386 the friction velocity,  $u_*$ , was modelled as sinusoidal with a 12-hour period, thus approx-387 imating the semi-diurnal tide. The range of  $u_*$  was from 1 to 16 mm s<sup>-1</sup> [Figure 7] (note 388 the flow never comes entirely to rest in the field observations). Using the quadratic stress 389 equation,  $\tau_b = \rho u_*^2$ , where  $\rho$  was 1025 kg m<sup>-3</sup>, the bed stress thus varied between 0.001 390 and 0.262 Pa. The erosion model E1 from Equation 6 was used to calculate the erosion 391 flux. The underlying parameter values used to generate the synthetic observations were; 392  $w_s = 1 \times 10^{-3} \text{ m s}^{-1}, M = 3 \times 10^{-4} \text{ kg m}^{-2} \text{ s}^{-1} \text{ Pa}^{-b}, \tau_{cr0} = 7 \times 10^{-2} \text{ Pa}, \text{ and}$ b = 1.1.394

The model was run for one tidal cycle as a spin up, with the profile of C on the fi-395 nal time step used as the initial quasi-steady condition for a second run. This second run 396 was used to generate the synthetic observations of C. We then added noise to the entire time-height grid of C in the form of random samples drawn from a Gaussian distribution with a standard deviation of  $2 \times 10^{-3}$  kg m<sup>-3</sup> [Figure 7b]. Discrete observations 399 near the bottom boundary (0.5 m ASB) and near the top of the model domain (8.5 m s)400 ASB) were extracted as the forcing observations [Figure 7 bottom]. The exact under-401 lying standard deviation of the noise was calculated as the standard deviation of the resid-402 uals between the noisy and noise-free model observations of C ( $\sigma = 0.199 \times 10^{-3}$  kg 403  $m^{-3}$ ). 404

### 405 4.3.2 Model specification

The model grid was created with a vertical resolution of 1 m and a time step of 300 s. The grid resolution was chosen to be as coarse as possible while still adequately re-



**Figure 8.** Posterior distributions (blue) for Case Study 2, synthetic tidally-driven erosion - scenario *E1*. Prior distributions (gray) are shown over the range of the posterior. The median and lower and upper 95% CI (brackets) are shown in the axis titles and white dots on the posteriors. The teal vertical line in each subplot was the underlying value used to generate the observations.

solving the vertical profile of C and ensuring the majority of the prior parameter space
was in a region of model stability (e.g., the Courant number was generally less than one
during sampling). The implicit model form provided more flexibility in this regard. The
total model height was 16 m and the boundary layer height, required to calculate the
eddy diffusivity profile using Equation 8, was set to a constant value of 15 m ASB. Vertical fluid velocities were set to zero for this case study.

We ran one scenario which used the erosion model E1 in Equation 6. There was 414 no model mis-specification in this example (the model determined the underlying obser-415 vations perfectly), thereby a comparison of multiple numerical model scenarios (such as 416 using both erosion models) was not warranted (note Case Study 2 was tested using ero-417 sion model E2 with similar results, but was not presented for brevity). The priors used 418 for each scenario were unchanged from those specified previously. For this example, the 419 observations and modelled C at 0.5 and 8.5 m ASB were supplied to the likelihood func-420 tion (Equation 3) as a 2D array. 421

The numerical model was specified with a constant deposition rate for this case. There is ongoing research into the apparent disconnect between deposition behaviour in laboratory experiments and common numerical modelling practices (see Letter and Mehta (2011) for a detailed explanation). For this work we simply note that numerical modelling of sediment transport over several tidal cycles is generally performed with a constant deposition flux. This method could be used to compare other more complex deposition models, such as Krone's probabilistic model (Winterwerp and van Kesteren (2004), Equation 5.21), a potential topic for future research.

### 430 4.3.3 Results and discussion

The scenario E1 was sampled for 25,000 iterations using 12 chains. The first 5,000 samples were discarded as burn-in samples. Chain auto-correlation and  $\hat{R}$  convergence analysis indicated the chains were suitable for inference (not shown). The priors and posteriors were compared as per Case Study 1 [Figure 8]. The approximate form and 95% CI of the posterior was calculated for each parameter [see Figure 8 axis titles].

The inference method was able to accurately capture the underlying parameter values, with the mode and median of the posterior close to the true value [Figure 8]. The posterior distribution of  $w_s$  was narrow and the posterior samples were not correlated



Figure 9. Top: observations of C (lines) for Case Study 2 from the two selected heights above the sea bed (ASB) and posterior predictive distributions of C at each height based on 10,000 samples of the parameter posteriors for E1, showing 50%, 80%, and 95% CI as graduated gray and blue shading. Bottom: Plots of posterior distribution samples of E1 for correlated parameters only ( $|r^2|>0.3$ ). Maroon dots are the specified (true) parameter values. Black lines are the 50%, 80%, and 95% density contours.

with any other parameter. The posterior distributions for all of the erosion model parameters  $(M, \tau_{cr0}, b)$  were wide in comparison to their median value.

We randomly selected 10,000 samples of  $\theta$  from the posterior and re-ran the numerical model for each  $\theta$  to generate the posterior predictive distribution. From this we estimated the 50%, 80%, and 95% CI of the predictions at the two observation heights and compared them to the noisy synthetic observations [Figure 9]. All of the erosion model parameters were correlated [Figure 9]. When estimating the posterior predictive distribution of C, however, the majority of the distribution width came from  $\sigma$ . This suggested that the resulting erosion flux was similar for all samples from the posterior, despite the wide posteriors.

This case study demonstrated that the method can recover the underlying spec-449 ified values of the numerical model parameters even when using noisy observations. In 450 general, the underlying parameter value was at or close to the posterior distribution me-451 dian, and the erosion model posterior distributions were wide and correlated. The width 452 of the erosion model posteriors was surprising, given the lack of model mis-specification. 453 Testing indicated that these posteriors were more narrow and less correlated when the 454 exponent b (Equation 6) was further from one and the erosion flux response was less lin-455 ear (not shown). In cases where b is known to be approximately one this parameter could 456 be removed from the model, thus increasing the inference efficiency. In addition, the ero-457 sion model itself could be re-parameterised to reduce parameter covariance. 458

### 4.4 Case Study 3: Continental shelf bottom boundary observations

### 4.4.1 Observations

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In this case study we attempt to estimate unobserved parameters using 2017 field 461 observations from the Northwest Shelf of Australia (Zulberti et al., 2022; Edge et al., 2021). 462 These studies presented data over 15 days from 2 to 16 April 2017. As it was too computationally expensive to include the entire period in the numerical model using this method, 464 we selected a period of 3 consecutive tidal cycles (about 37.6 hours) [Figure 10]. This 465 period was chosen because it captured a range of forcing conditions and sediment response, 466 and because no large amplitude nonlinear internal waves (NLIW) propagated past the 467 site at this time; NLIW introduce strong vertical fluid velocities and are likely to cre-468 ate horizontal gradients of C, hence creating horizontal advective effects that violate the 469 assumptions of our model. To evaluate the method, we generated predictions using in-470 ferred posterior distributions of model parameters and compared these with independent observations not used in the parameter estimation (noting that the primary purpose of 472 the method is parameter estimation and not prediction of C). 473

The friction velocity and bed stress were derived from high frequency current mea-474 surements collected at 0.49 m ASB. Zulberti et al. (2018) showed that use of the quadratic drag law with  $C_D = 0.00185$  was an accurate proxy for measured Reynolds stress ( $u_{*Re} =$ 476  $0.99u_{*Cd} + 0.00, r^2 = 0.92$ ) within the log-layer (typically 1 m thick) at this site and 477 provided a more complete record. Eddy diffusivity profiles,  $\gamma_z$ , were calculated as per 478 Equation 8 with the time-varying bottom boundary layer height (BBL) (typically 10 m 479 thick) estimated using backscatter from an acoustic Doppler current profiler (ADCP), 480 as per Zulberti et al. (2022) [Figure 11]. Direct estimates of eddy diffusivity at 1.4 m ASB 481 were typically three times greater than modelled estimates using Equation 8 (the model 482 is highly idealised). Zulberti et al. (2022) analysed backscatter, current profiles and temperature profiles, and determined that backscatter could be used as a suitable proxy to 484 estimate the BBL height. Fluid vertical velocities were supplied by the ADCP. Points 485 close to the sea bed where the ADCP returned poor vertical velocity data were linearly 486 extrapolated to zero at the bed. Both the BBL height and vertical velocities were low-487 pass filtered. 488

The observations of C, taken from Edge et al. (2021), were chosen at two discrete heights, 1.4 m and 9.5 m ASB [Figure 10, bottom]. In addition, we performed the inference with a single point measurement (at 1.4 m ASB) in order to test the method sensitivity and applicability to cases where only a single observation height was available.

Closer examination of the forcing data and observations highlights the complex be-493 haviour of the continental shelf boundary layer during this period with both barotropic 494 and (out-of-phase) baroclinic tides influencing the currents and hence the bottom stress 495 [Figure 11]. The bed stress is not a simple sinusoid: on odd-numbered half-tides the stress 496 was intensified by internal tides that generate an additional near-bed current. For even-497 numbered tides the opposite was true, with internal tides generating an adverse near-498 bed current, reducing the estimated bed stress and  $u_*$ . The near-bed C exhibited varied responses to similar magnitudes of bed stress and there was a time-lag in the response at different observation heights. Note that the periods where C was greater at the higher 501 observation point were likely the result of calibration uncertainty and not a negative sed-502 iment gradient [Figure 11, bottom] (calibration uncertainty was not included in this work, 503 but is a topic of future research). 504

### 505 4.4.2 Model specification

The model grid was similar to Case Study 2, with a vertical resolution of 1 m and a time step of 300 s. The total model height was 24 m with the maximum BBL height remaining under 20 m. We ran two different numerical model scenarios for this case study,



Figure 10. 15 days of oceanographic data from Zulberti et al. (2022) and Edge et al. (2021). The top panel shows the estimated bed shear stress,  $\tau_{bed}$ , and friction velocity,  $u_*$ , from high frequency current measurements at 0.49 m ASB. The middle panel shows a compilation of C observations, primarily from the ADCP over the bottom 23 m ASB. The bottom panel shows the two observed time series of C used for Case Study 3. The gray shaded area shows the three tidal cycle periods used in Case Study 3.



Figure 11. Forcing data and observations of C for Case Study 3. The top panel shows the measured bed stress and friction velocity at the bed. The middle panel shows the eddy diffusivity and the low-pass filtered BBL height (white line). The bottom panel shows the observations of C used for this case study.



Figure 12. Posterior distributions (blue) for Case Study 3 - scenario *E1*. Prior distributions (gray) are shown over the range of the posterior. The median and lower and upper 95% CI (brackets) are shown in the axis titles and white dots on the posteriors.



Figure 13. Posterior distribution samples for Case Study 3 - scenario E1 for the correlated parameters only ( $|r^2|>0.3$ ). Black lines are the 50%, 80%, and 95% density contours.

E1 using the erosion model in Equation 6, and E2 using the erosion model in Equation 7. The priors used for each scenario were unchanged from those specified previously. In addition, a modified scenario was also assessed based on scenario E2 that only used observations from a single height at 1.4 m ASB, rather than two heights (denoted E2-10bs). Note the numerical model output was interpolated (quadratic) to estimate C at 1.4 m, as the grid points were spaced at 1 m intervals starting at 0.5 m.

515

### 4.5 Results and discussion

All three scenarios (E1, E2, and E2-10bs) were sampled for 25,000 iterations us-516 ing 12 chains. The first 5,000 iterations of each chain were discarded as burn-in samples. 517 Chain auto-correlation and R convergence analysis indicated the chains were suitable for 518 inference (not shown). Kernel density estimates were used to visually compare the prior 519 and posterior distributions for scenario E1 [Figure 12]. Parameters M,  $\tau_{cr0}$ , and b were correlated  $(|r^2|>0.3)$  in scenario E1 [Figure 13], similar to the previous case studies. Pa-521 rameter estimates from the E2 and E2-10bs scenario variations were compared visually 522 using kernel density estimates [Figure 14] and several of the parameters were positively 523 correlated  $(r^2 > 0.3)$  [Figure 15]. 524

Scenarios E1 and E2 provided similar estimates for comparable parameters ( $w_s$ ,  $M, \tau_{cr0}$ , and  $\sigma$ ). The settling velocity,  $w_s$ , was estimated to be around  $0.16 \times 10^{-4}$  m  $s^{-1}$  slower in E2 compared to E1. Mean particle size and effective density measurements from Edge et al. (2021) gave an estimated mean  $w_s$  of  $0.6 \times 10^{-4}$  m s<sup>-1</sup> using Stokes'



Figure 14. Posterior distributions for Case Study 3 - scenario E2 (blue) and scenario E2-10bs (orange). Prior distributions (gray) are shown over the range of the posteriors. The median and lower and upper 95% CI (brackets) are shown for only scenario E2 in the axis titles and white dots on the posteriors.



Figure 15. Posterior distribution samples for Case Study 3 - scenario E2 (blue) and scenario E2-10bs (orange) for the correlated parameters only ( $|r^2|>0.3$ , shown without contours for clarity).

law. Posterior distributions of the initial bed strength (critical shear stress),  $\tau_{cr0}$ , could be approximated by a half-normal distribution shape in both scenarios. The observations do not contain significant time periods where  $\tau_{bed}$  was in the range of  $\tau_{cr0}$  and so the likelihood function was insensitive to changes in  $\tau_{cr0}$  at small values. The posterior predictions were narrower for scenario E2, compared to E1 [Figure 16], indicating a better fit using the E2 erosion model.

The two variations of scenario E2 (E2 and E2-10bs) provided interesting insight into how much data was needed to infer the parameters and how changing the input observations affected  $\sigma$  [Figure 14]. Notably, the inferred parameters were both in a similar range, which suggested the method was (relatively) insensitive to the chosen height of measurements of C. Using observations from only a single height in scenario E2-10bs also shifted the posterior distribution for some parameters [Figure 14].

We did not expect the results for scenarios E2 and E2-1obs to be similar as the 541 use of observations at a single point could in theory be matched with many values of  $w_s$ 542 (and hence profiles of C). It appeared, however, that the information contained within 543 the time series of C at a single point was sufficient to infer the model parameters, not-544 ing that the prior distributions help inform the estimate. In addition, using less obser-545 vations reduced  $\sigma$ , indicating that the erosion model in scenario E2 - 2obs could bet-546 ter represent the time series of C close to the bottom boundary (compared to C further 547 from the bed). Note that the distributions of  $\sigma$  for scenarios E2 and E2-10bs were not directly comparable as the observation data sets were different (Scenario E2-1 obs used less data points). 550

We compared the observations to posterior predictive distributions of C using 10,000 randomly selected samples from  $\theta$ . The 50%, 80%, and 95% CI of the predictions were calculated at the two observation heights above the sea bed (1.4 m and 9.5 m). The results were similar for all three scenarios, so only predictions for scenarios E1 and E2 are presented [Figure 16 top and bottom, respectively].

To demonstrate the ability of the model to make predictions beyond the chosen inference period, we extended the model run. We supplied the forcing data; bed stress, bottom friction velocity, vertical velocity, and BBL height, to the model for the extended period. We then generated posterior predictive samples for this period using the scenario E2-10bs and calculated the 50%, 80%, and 95% CI at 1.4 m and 9.5 m ASB [Figure 17]. While this scenario did not use any observations of C at 9.5 m for the inference, it still made good predictions at this height for the original model period. Beyond the original model period the predictions and observations diverged at both heights.

It is clear from the results with the Case Study 3 field observations, that in addi-564 tion to natural variation and measurement errors, in both erosion model scenarios there 565 is missing physics, as reflected in  $\sigma$ . The primary sources of model mis-specification are 566 likely to come from simplification of the ADE and the boundary flux models used (i.e. 567 the erosion and deposition models). Simplification of the ADE for this model may have 568 discounted potentially important horizontal advection processes, such as horizontal sediment gradients induced by the internal tide. Improvement of the boundary flux mod-570 els is a common aim for sediment transport studies and this method shows promise in 571 quantifying relative model performance. Future work could also include the implemen-572 tation of a multi-layer bed model with consolidation and multiple sediment size classes. 573

### 574 5 Conclusions

We have demonstrated that MCMC sampling can be used in conjunction with a simple 1D advection-diffusion numerical model (including its boundary model) to estimate parameters of interest with uncertainty quantification. The numerical model itself used a well known discretisation method and has the potential for inclusion of any bound-



Figure 16. Top plots: Case Study 3 observations of C (solid lines) and the 50%, 80%, and 95% CI of the posterior predictive distributions of C from Scenario E1 (graduated shading) at 1.4 m above the sea bed (ASB) (gray) and 9.5 m ASB (blue). Bottom plots: as per the top plots for Scenario E2.



Figure 17. Top: Case Study 3 - scenario E2-10bs extended observations of C (solid line) at 1.4 m above the sea bed (ASB) and 50%, 80%, and 95% CI of the posterior predictive distributions of C (graduated gray shading). Bottom: as per top panel but for C at 9.5 m ASB (blue line and shading). The original model period is shown as light gray shading in both plots.

ary model (in this case erosion or deposition). This allows for a quantitative comparison of the performance of any model selected from literature as applied to a given data set of observations.

We have applied this method to three case studies from the field of cohesive sed-582 iment transport. Case Study 1 was an example of a common erosion test conducted us-583 ing an in-situ annular flume. In this scenario, the depth-averaged C was supplied which 584 simplified the analysis. We demonstrated that the entire time series of bed stress and 585 C could be used to infer model parameters, where previously only steady state periods have been used. This indicates that the method should be suitable to apply to similar investigations with unsteady forcing, such as channel or estuary flows. In-situ param-588 eter estimation in cohesive sediment environments such as these is a difficult task, with 589 conflicting results between methods (e.g. Maa & Kwon, 2007), and state-of-the-art mea-590 surement techniques giving a wide range of (potentially noisy) estimates (e.g. Egan et 591 al., 2021). 592

We then applied the method to Case studies 2 and 3, synthetic and real examples 593 of complex BBL forcing and response on the continental shelf. These observations mo-594 tivated this work due to the difficulty of measuring the inferred parameters directly in 595 deep environments (the development and testing of new in-situ measurement techniques 596 is better suited to shallow environments such as channels and estuaries). Using the method 597 described in this paper we were able to infer quantitative information on difficult to observe parameters using measurements that are simple to collect (current velocity and suspended sediment concentration from acoustic backscatter). We note that this simple coarse 600 resolution model still required computation time of around 2 days (Intel(R) Core(TM) 601 i7-7700 CPU @ 3.60GHz), so the method may be too computationally costly to apply 602 to 2- and 3-dimensional numerical models (note there is much literature available on other 603 methods for larger models (e.g. Bui-Thanh et al., 2012)). 604

The key output from this method was the posterior distributions of each parameter, with predictions being a secondary outcome. Nevertheless, posterior predictions of C were in agreement with observations (as they must be to achieve good results) but extended predictions diverged from observations outside the original inference period. This
was likely due to the simplified representation of baroclinic tides in the model, which,
in reality, were not phase-locked with the barotropic tide. The divergence in predictions
may also be due to changes in the key parameters with time or some other aspect of model
mis-specification.

In summary, the method described here was able to infer key model parameters with uncertainty. The method inherently partitions uncertainty from the parameters and the model mis-specification and the measurement error term ( $\sigma$ ). A more sophisticated model (e.g., a discretized bed model or a more realistic diffusivity model) could readily be implemented to potentially improve the results. Furthermore, we expect this method would perform even better if applied to a less complex system, such as a shallow tidal channel.

### 620 6 Open Research

Forcing and fitting data used for Case Study 3 are archived on two UWA library research data repositories (Zulberti et al., 2020; Edge, 2021). No proprietary software has been used for this research. All analysis was conducted using Python 3 with the exception of exporting raw instrument data using manufacturer's software. Specific packages utilised frequently for this work (although not specifically required) include *Pymc3* (Salvatier et al., 2016), *Seaborn* (Waskom, 2021), *Xarray* (Hoyer & Hamman, 2017), and any packages they depend on. Jupyter Notebooks can be supplied upon request.

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