Preliminaries for muon tracking in GEANT4 simulations

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Abstract

The preliminary information about the muon tracking by means of the GEANT4 simulations is summarized in accordance with the physics reference manual of GEANT4 10.7.

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Preliminaries for muon tracking in GEANT4 simulations

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1 Introduction

In this chapter, the preliminary information about the muon tracking by means of the GEANT4 simulations [1] is summarized in accordance with the physics reference manual of GEANT4 10.7 [2] along with the implementation steps in [3], all the equations as well as all the expressions of which are narrated by respecting the existing notations therewithin. In addition to the formal references present in [2,3], a number of auxillary references are introduced when/if appropriate. For the sake of consistency with the physics reference manual for GEANT4 10.7, the total energy is denoted by E, whereas the kinetic energy is indicated by T. To briefly outline, section 2 includes the basic tracking notions such as transportation and mean free path, while section 3 is composed of muon ionization together with the muon-induced radiative processes. Finally, the practical implementation of muon interactions is concisely reported in section 4.

2 Basic tracking notions in GEANT4

2.1 Transportation

Particle transport in the GEANT4 simulations [1] is the consequence of the joined actions of the GEANT4 kernel's Stepping Manager class and the involved physical processes. The process called Transportation classifies the previous geometrical volume as well as the next geometrical volume in the course of the tracking progress. The expected length at which an interaction is supposed to take place is governed by counting all processes feasible at each step. As a result, the life cycle of the corresponding particle in a particular volume is determined before traversing another volume for the other potential interactions.

2.2 True step length

The GEANT4 simulation for the passage of particles through matter is realized step by step [1]. A true step length for the next particle interaction is arbitrarily sampled by utilizing the mean free path of the interaction or by various step limitations established through different GEANT4 components. The new true step length is identified by the smallest step.

2.2.1 Mean free path

The mean free path of a particle in a medium is computed by utilizing the cross section of a specific process and the density of atoms in the GEANT4 simulations. The number of atoms per volume in a simple material is determined as shown in

$$n = \frac{\rho N_A}{A} \tag{1}$$

where N_A is the Avogadro's number, ρ is the density of medium, and A is the molar mass. Furthermore, the number of atoms per volume of the i^{th} element in a compound material is obtained by the following expression:

$$n_i = \frac{w_i \rho N_A}{A_i} \tag{2}$$

where w_i is the proportion by mass of the i^{th} element, and A_i is the molar mass of the i^{th} element. At a given energy, the mean free path of a process denoted by λ that is also called the interaction length can be expressed in terms of the total cross section as written in

$$\lambda(E) = \left(\sum_i \sigma(Z_i, E) n_i \right)^{-1} \tag{3}$$

where $\sigma(Z_i, E)$ is the total cross section per atom of the associated process, and $\sum_i$ sums over all elements constituting the compound material. $\sum_i \sigma(Z_i, E) n_i$ also refers to the macroscopic cross section, and the mean free path is defined as the inverse of the macroscopic cross section.

2.2.2 Interaction point

The mean free path of a particle for a particular process is contingent on the medium and cannot be exactly employed in order to sample the probability of a process in a heterogeneous medium. The number of mean free paths crossed by a particle between two positions denoted by x_1 and x_2 is described as expressed in

$$n_\lambda = \int_{x_1}^{x_2} \frac{dx}{\lambda(x)} \tag{4}$$

that is independent of the crossed medium. By assuming that n_r is an arbitrary parameter indicating the number of mean free paths between an initial point and the location of interaction, it is possible to reveal that n_r has the following distribution function:

$$P(n_r < n_\lambda) = 1 - e^{-n_\lambda} \tag{5}$$

The total number of mean free paths that a particle traverses before arriving the interaction point is sampled at the beginning of the trajectory as follows:

$$n_\lambda = -\log \eta \tag{6}$$

where η is a random number lying on the interval between 0 and 1, and n_λ is updated in accordance with the following formula after each step denoted by Δx:

$$n_\lambda^{'} = n_\lambda - \frac{\Delta x}{\lambda(x)} \tag{7}$$

until the step generated from $s(x) = \lambda(x) n_\lambda$ becomes the shortest step where the specific process is activated.

3 Basic muon-induced processes in GEANT4

3.1 Muon ionization

Fairly relativistic charged particles apart from electrons deposit energy in a medium predominantly via ionization and atomic excitation. The mean rate of energy loss (or stopping power) is given by the **Bethe-Bloch equation** as written in [4]

$$< -\frac{dE}{dx} >= Kz^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\frac{1}{2}\ln\left(\frac{2m_e c^2\beta^2\gamma^2 T_{\max}}{I^2}\right) - \beta^2 - \frac{\delta}{2}\right] \tag{8}$$

where E is the total energy in MeV, $K = 4\pi N_A r_e^2 m_e c^2$ in MeV mol^{-1} cm^2, Z is the atomic number, A is the atomic weight in g/mol, $\beta = v/c$, $\gamma = 1/(1-\beta^2)^{\frac{1}{2}}$, z is the particle charge, m_e is the electron mass, $T_{\max}$ is the maximum kinetic energy which can be imparted to a free electron in a single collision, I is the mean excitation energy in eV, and δ is the density effect correction to the ionization energy loss. It should be carefully noted that if x is defined as the thickness of the absorber in cm, then the units of $-dE/dx$ is in MeV/cm; on the other hand, if x is defined as the mass thickness, i.e. the product of density and thickness, then the units of $-dE/dx$ is in MeV g^{-1} cm^2. According to [4], Eq. (8) is defined in MeV g^{-1} cm^2 based on the listed variables.

The continuous energy loss due to ionization is determined by the aid of G4MuIonisation in GEANT4 [1], and the "discrete" part of the ionization, i.e. the delta rays produced by muons, is simulated. The corresponding models utilized in this class depending on the kinetic energy denoted by T are listed as indicated in [2]:

- G4BetheBlochModel is valid for muons with 0.2 MeV $< T <$ 1 GeV;
- G4MuBetheBlochModel is valid for muons with $T >$ 1 GeV.

For $T >$ 1 GeV, the G4MuBetheBlochModel covers the corrections [5] for bremsstrahlung on the atomic electrons. A practical analytical expression for the cross section is used, and the calculation results are significantly different from the usual elastic $\mu - e$ scattering in the region of high energy transfers and yield a non-negligible correction to the total average energy loss of high energy muons. The total cross section is written as shown in

$$\sigma(E,\epsilon) = \sigma_{\rm BB}(E,T)\left[1 + \frac{\alpha}{2\pi}\ln\left(1+\frac{2\epsilon}{m_e}\right)\ln\left(\frac{4m_e E(E-\epsilon)}{m_\mu^2(2\epsilon+m_e)}\right)\right] \tag{9}$$

where $\sigma_{\rm BB}(E,T)$ is the Bethe-Bloch cross section that is defined as

$$\sigma_{\rm BB}(E,T) = 2\pi r_e^2 mc^2 Z\frac{z_p^2}{\beta^2}\frac{1}{T^2}\left[1 - \beta^2\frac{T}{T_{\max}} + s\frac{T^2}{2E^2}\right] \tag{10}$$

in which $s = 0$ for the spinless particles and $s = 1$ for the other particles. In Eq. (9), m_μ is the muon mass, E is the muon energy, α is a constant, and $\epsilon = \omega + T_e$ is the energy transfer where T_e is the electron kinetic energy, and ω is the energy of radiative photons.

3.2 Elastic scattering

3.2.1 Coulomb scattering of muons

Single elastic scattering process is considered an alternative to the multiple scattering (MSC) process. The asset behind the single scattering process is hinged on the utilization of the

theory-based cross sections in contrast with the GEANT4 MSC model [6] that uses a number of phenomenological approximations along with the Lewis theory [7]. The process called G4CoulombScattering was implemented to simulate the single scattering of muons, but it is also applicable to electrons and ions with some physical limitations. Since each of elastic collisions is simulated, the number of steps for the charged particles is significantly increasing in comparison with the MSC approach, and its CPU performance is correspondingly weak. Howbeit, in the low-density media (e.g. vacuum or low-density gas), the MSC approach may lead to the incorrect results, and the single scattering processes are assumed to be more appropriate. The single scattering model of Wentzel [8] is employed in a number of the MSC models, e.g. the Penelope code [9]. The **Wentzel model** for describing the elastic scattering of particles with charge ze (z = -1 for electron) by the atomic nucleus with the atomic number denoted by Z is based on the simplified scattering potential as described in

$$V(r) = \frac{zZe^2}{r} e^{-\frac{r}{R}} \tag{11}$$

where the exponential factor attempts to reproduce the effect of screening. The parameter R is a screening radius that is expressed in

$$R = 0.885 Z^{-\frac{1}{3}} r_B \tag{12}$$

where r_B is the Bohr radius. In **the first Born approximation** [10], the elastic scattering cross section indicated by σ^W can be obtained as

$$\frac{d\sigma^W(\theta)}{d\Omega} = \frac{(ze^2)^2}{(p\beta c)^2} \frac{Z(Z+1)}{(2S+1-cos\theta)^2} \tag{13}$$

where β is the velocity ratio of the projectile particle, and p is the momentum. The screening parameter symbolized by S is defined according to Moliere and Bethe [11] as written in

$$S = \left(\frac{\hbar}{2pR}\right)^2 \left(1.13 + 3.76 \left(\frac{\alpha Z}{\beta}\right)^2\right) \tag{14}$$

where α is the fine structure constant, and the factor in brackets is introduced to take into account the second order corrections to the first Born approximation. The total elastic cross section indicated by σ can be expressed by means of the Wentzel cross section as shown in

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{\sigma^W(\theta)}{d\Omega} \left(\frac{Z}{\left(1 + \frac{q^2 R_N^2}{12}\right)^2 + 1}\right) + \frac{1}{Z+1} \tag{15}$$

where R_N is the nuclear radius, and q is the momentum transfer to the nucleus. This term takes the nuclear size effect [12] into consideration, while the second term takes into account scattering off electrons.

3.2.2 Multiple scattering of muons

Elastic scattering of muons is a pivotal component of any transport code that is associated with the passage of the muons through matter. The elastic cross section is high when the particle energy diminishes, so the MSC approach should be implemented in order to have acceptable CPU performance in the GEANT4 simulations. A universal interface called G4VMultipleScattering is employed by the muon-induced GEANT4 MSC process, i.e. G4MuMultipleScattering [13, 14]. For the concrete simulation, the G4VMscModel interface, which is an extension of the base G4VEmModel interface, is utilized. The following models are available for muons:

- G4**Urban**MscModel is applicable to all types of particles and is the default model for electrons and positrons below 100 MeV [15];

- G4**Wentzel**VIModelis the default model for all changed particles including electrons and positrons above 100 MeV, and it is included in the physics list together with the G4CoulombScattering process that is is responsible for large angle scattering [14–16].

The MSC simulation algorithms can be categorized as either detailed or condensed. In the detailed algorithms, all the collisions/interactions undergone by the associated particle are simulated. This type of simulation can be regarded as exact, and it provides the same outcomes as the solution of the transport equation. However, it can be only utilized when the number of collisions is not drastically large, which is a condition satisfied merely for the special geometries, e.g. thin foils or low density gas. In the solid or liquid media, the average number of collisions is so high, hence the detailed simulation becomes very inefficient. The high energy simulation codes employ the condensed simulation algorithms where the global effects of the collisions are simulated at the end of a track segment. The net energy loss, displacement, and change of direction of the charged particle are routinely calculated in these codes. The last two quantities are calculated from the MSC theories employed in the codes, and the accuracy of the condensed simulations is restricted by the accuracy of the MSC approximation.

The MSC theories of Molière [17], Goudsmit and Saunderson [18], and Lewis [7] are utilized in most of the particle physics simulation codes. The Lewis theory also determines the moments of the spatial distribution, whereas the theories of Molière and Goudsmit-Saunderson deliver only the angular distribution after a step. The probability distribution of the spatial displacement is not provided by any of these MSC theories. Each of the MSC simulation codes incorporates its own algorithm to determine the angular deflection, the true path length correction, and the spatial displacement of the charged particle after a given step. These algorithms responsible for most of the uncertainties of the transport codes since they are not exact. Furthermore, due to inaccuracy of MSC, the simulation results might depend on the value of the step length, and generally user has to select the value of the step length carefully. A new class of MSC simulation, which also refers to the mixed simulation algorithms (e.g. see [19]), started to be present in the literature. The mixed algorithm simulates the hard collisions one by one, and a MSC theory is used in order to incorporate the effects of the soft collisions at the end of a given step. Such algorithms might avert the number of steps from becoming too large and might also decrease the dependence on the step length. G4WentzelVIModel [14] includes the GEANT4 original implementation of a similar methodology. The Urban MSC models existing in the GEANT4 simulations appertain to the category of the condensed simulations, and these models employ the model functions to compute the spatial as well as angular distributions after a step. These model functions have been selected in such a way as to yield the same moments of the spatial and angular distributions as described by the Lewis theory [7].

Before expounding the MSC models, it might be necessary to define the acting terms. In the GEANT4 simulations, a particle is transported by steps through the detector geometry. The shortest distance between the endpoints of a step is called the geometrical path length indicated by l_{geo}. In the absence of a magnetic field, this is a straight line. For the non-zero fields, l_{geo} is the length along a curved trajectory. Constraints on l_{geo} are imposed when particle tracks cross the volume boundaries. The path length of an actual particle, however, is usually longer than l_{geo} due to the MSC. This distance is called the true path length denoted by l_{true}. Constraints on l_{true} are imposed by the physical processes acting on the particle.

The properties of the MSC process are determined by the transport mean free paths symbolized by λ_k that are the functions of the energy in a given material. The k^{th} transport mean

free path is defined as

$$\frac{1}{\lambda_k} = 2\pi n_a \int_{-1}^{1} [1 - P_k(\cos\chi)] \frac{d\sigma(\chi)}{d\Omega} d\cos\chi \tag{16}$$

where $d\sigma(\chi)/d\Omega$ is the differential cross section of the scattering, $P_k(\cos\chi)$ is the k^{th} order Legendre polynomial, and n_a is the number of atoms per volume.

Most of the mean properties of MSC computed in the simulation codes depend only on the first and second transport mean free paths. The mean value of the l_{geo} (first moment) corresponding to a given l_{true} is determined by the following expression:

$$< l_{\text{geo}} >= \lambda_1 \left[1 - e^{-\frac{l_{\text{true}}}{\lambda_1}}\right] \tag{17}$$

Eq. (17) is an exact result for the mean value of l_{geo} if the differential cross section has axial symmetry, and the energy loss can be neglected. The transformation between l_{true} and l_{geo} is called the path length correction. This formula and the other expressions for the first moments of the spatial distribution were taken from either [19] or [20], but they were originally calculated by Goudsmit and Saunderson [18] and Lewis [7]. At the end of l_{true}, the scattering angle is θ. The mean value of $\cos\theta$ is

$$< \cos\theta >= e^{-\frac{l_{\text{true}}}{\lambda_1}} \tag{18}$$

The variance of $\cos\theta$ can be written as

$$\delta^2 =< \cos^2\theta > - < \cos\theta >^2= \frac{1 + 2e^{-2\kappa\tau}}{3} - e^{-2\tau} \tag{19}$$

where $\tau = l_{\text{geo}}/\lambda_1$ and $\kappa = \lambda_1/\lambda_2$. The mean lateral displacement is given by a more complicated formula [19], but this quantity can also be calculated relatively easily and accurately. The square of the mean lateral displacement is

$$< x^2 + z^2 >= \frac{4\lambda_1^2}{3}\left[\tau - \frac{\kappa+1}{\kappa} + \frac{\kappa}{\kappa-1}e^{-\tau} - \frac{1}{\kappa(\kappa-1)}e^{-\kappa\tau}\right] \tag{20}$$

Here, it is assumed that the initial particle direction is parallel to the the y-axis.

As mentioned earlier, the path length correction refers to the transformation $l_{\text{true}} \rightarrow g$ and its inverse. The $l_{\text{true}} \rightarrow g$ transformation is given by Eq. (17) if the step is small and the energy loss can be neglected. If the step is not small, the energy dependence makes the transformation more complicated. For this case, Eqs. (17) and (18) should be modified as

$$< \cos\theta >= e^{-\int_0^{l_{\text{true}}} \frac{du}{\lambda_1(u)}} \tag{21}$$

and

$$< l_{\text{geo}} >= \int_0^{l_{\text{true}}} < \cos\theta >_u \, du \tag{22}$$

The quantity $u = \cos\theta$ is sampled according to a model function $g(u)$. The shape of this function has been chosen such that Eqs. (17) and (18) are satisfied. The functional form of $g(u)$ is

$$g(u) = q[pg_1(u) + (1-p)g_2(u)] + (1-q)g_3(u) \tag{23}$$

where $0 \leq p,\, q \leq 1$, and g_i are in the form of simple functions of $u = \cos\theta$ normalized over the range $u \in [-1, 1]$. $g_i(u)$ have been chosen as

$$g_i(u) = \begin{cases} g_1(u) = C_1 e^{-a(1-u)} \text{ if } -1 \leq u_0 \leq u \leq 1 \\ g_2(u) = C_2 \frac{1}{(b-u)^d} \text{ if } -1 \leq u \leq u_0 \leq 1 \\ g_3(u) = C_3 \text{ if } -1 \leq u \leq 1 \end{cases} \tag{24}$$

where $a > 0$, $b > 0$, $d > 0$, and u_0 are the model parameters, and C_i are the normalization constants. It is worth noting that, for small scattering angles, $g_1(u)$ is nearly Gaussian if $\theta_0^2 \approx 1/a$, while $g_2(u)$ has a Rutherford-like tail for large scattering angle if $b \approx 1$ and d is not far from 2. Regarding the model parameters, a, b, d, u_0, p, and q are not independent. The requirement that the angular distribution function $g(u)$ and its first derivative be continuous at $u = u_0$ imposes two constraints on the parameters as shown in

$$pg_1(u_0) = (1-p)g_2(u_0) \tag{25}$$

and

$$pag_1(u_0) = (1-p)\frac{d}{b-u_0}g_2(u_0) \tag{26}$$

A third constraint, which implies that $g(u)$ must give the same mean value for u, comes from Eq. 21 as written in

$$q\{p < u >_1 + (1-p) < u >_2\} = [1 - \alpha l_{\text{true}}]^{\frac{1}{\alpha\lambda_{10}}} \tag{27}$$

where α can be expressed by using λ_{10} and λ_{11} in which λ_{11} is the value of the transport mean free path at the end of the step as described in

$$\alpha = \frac{\lambda_{10} - \lambda_{11}}{l_{\text{true}}\lambda_{10}} \tag{28}$$

In Eq. (27), $< u >_i$ denotes the mean value of u computed from the distribution $g_i(u)$. The parameter indicated by a was chosen according to a modified Highland-Lynch-Dahl formula for the width of the angular distribution [21, 22] as expressed in

$$a = \frac{0.5}{1 - \cos(\theta_0)} \tag{29}$$

where θ_0 that is the width of the approximate Gaussian projected angle distribution is defined as written in

$$\theta_0 = \frac{13.6 \text{ MeV}}{\beta cp} z \sqrt{\frac{l_{\text{true}}}{X_0}} \left[1 + 0.038 \ln \frac{l_{\text{true}}}{X_0}\right] \tag{30}$$

in which p is the momentum, βc is the velocity, z is the charge, and l_{true}/X_0 is the true path length in radiation length unit.

3.3 Radiative processes

3.3.1 Muon bremsstrahlung

Muon bremsstrahlung [23–25] is the electromagnetic radiation produced by the acceleration or especially the deceleration of a muon after passing through the electric and magnetic fields of a nucleus. Bremsstrahlung dominates the other muon interaction processes in the region of catastrophic collisions ($\nu \geq 0.1$), i.e. at the "moderate" muon energies above the kinematic

limit for the knock–on electron production. At the high energies ($T \geq 1$ TeV), this process contributes about 40% of the average muon energy loss.

The differential cross section for the muon bremsstrahlung in the units of cm^2/(g GeV) can be written as

$$\frac{d\sigma(E,\epsilon,Z,A)}{d\epsilon} = \begin{cases} \frac{16}{3}\alpha N_A \left(\frac{m_e}{m_\mu} r_e\right)^2 \frac{1}{\epsilon A} Z(Z\Phi_n + \Phi_e)(1 - \nu + \frac{3}{4}\nu^2) \\ \\ 0 \quad \text{if} \quad \epsilon \geq \epsilon_{\max} = E - m_\mu \end{cases} \tag{31}$$

where m_μ and m_e are the muon and electron masses, r_e is the electron radius, Z and A are the atomic number and atomic weight of the material, and N_A is the Avogadro's number. By considering that E and T are the initial total and kinetic energy of the muon, and ϵ is the emitted photon energy, then $\epsilon = T - T'$ and the relative energy transfer $\nu = \epsilon/E$. Φ_n and Φ_e are the contributions from the nucleus and the electrons, respectively.

3.3.2 Positron - electron pair production by muons

Positron - electron pair production by muons [23] is defined as indicated in the following process:

$$\mu^{+,-} + \text{nucleus} \rightarrow \mu^{+,-} + e^+ + e^- + \text{nucleus} \tag{32}$$

The direct electron pair production is one of the most important muon interaction processes. At the TeV muon energies, the pair production cross section exceeds those of the other muon interaction processes over a range of energy transfers between 100 MeV and $0.1E_\mu$. The average energy loss for the pair production increases linearly with the muon energy, and this process contributes more than half of the total energy loss rate in the TeV region.

To adequately describe the number of pairs produced, the average energy loss and the stochastic energy loss distribution, the differential cross section behavior over an energy transfer range of 5 MeV $\leq \epsilon \leq 0.1E_\mu$ must be accurately reproduced. This is because the main contribution to the total cross section is given by transferred energies 5 MeV $\leq \epsilon \leq 0.01E_\mu$ and because the contribution to the average muon energy loss is determined mostly in the region $0.001E_\mu \leq \epsilon \leq 0.1E_\mu$. For a theoretical description of the cross section, the formulation in [26] is employed on top of a correction for the finite nuclear size [27]. To take the electron pair production into consideration in the field of atomic electrons, the inelastic atomic form factor contribution in [28] is also implemented.

The differential cross section denoted by $\sigma(E,\epsilon,Z,A)$ for the pair production by muons can be written as:

$$\sigma(E,\epsilon,Z,A) = \frac{4}{3\pi}\frac{Z(Z+\zeta)}{A} N_A(\alpha r_e)^2 \frac{1-\nu}{\epsilon} \int_0^{\rho_{\max}} G(E,\nu,\rho,Z)d\rho \tag{33}$$

where

$$G(E,\nu,\rho,Z) = \Phi_e + (m_e/m_\mu)^2\Phi_\mu \tag{34}$$

In Eqs. (33) and (34), ρ is an auxiliary variable, and ζ takes the process on atomic electrons (inelastic atomic form factor contribution) into consideration in addition to the above-listed parameters that are already defined in Eq. (31).

3.3.3 Muon photonuclear reactions

The inelastic interaction of muons with nuclei is important at the high muon energies ($E \geq 10$ GeV) and at the relatively high energy transfers $\nu(\nu/E \geq 10^{-2})$. It is especially crucial for the light materials as well as for the study of detector response to the high energy muons, the muon propagation, and the muon-induced hadronic background. The average energy loss for this process increases almost linearly with energy, and it constitutes about 10% of the energy loss rate at the TeV muon energies.

3.4 Muon decay

The muon decay [29] rarely occurs in the forms of

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \tag{35}$$

and

$$\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \tag{36}$$

G4MuonDecayChannel governs the muon decay according to the **vector minus axial vector (V-A) theory** [30]. The electron/positron energy is sampled from the following distribution:

$$d\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} 2\epsilon^2(3 - 2\epsilon) \tag{37}$$

where Γ is the decay rate, G_F is the Fermi coupling constant, m_μ is the muon mass, and $\epsilon = E_e/E_{\max}$ in which E_e is the electron energy and $E_{\max}$ is $m_\mu/2$.

4 Implementation of muon interactions in GEANT4

Implementation of the above-mentioned interactions in GEANT4 is summarized as documented in [3]. The energy loss mechanisms in GEANT4 have continuous as well as discrete parts [1, 31,32]. The high energy transfers are expressed as a real discrete act of an interaction, whereas the low energy transfers are treated as a continuous process of the energy loss. The average value of this restricted energy loss is acquired via the integration of the following equation:

$$-\frac{dE}{dx} = \int_0^{\epsilon_m} \epsilon\sigma(\epsilon, E)d\epsilon = E\int_0^{\epsilon_m} \epsilon\nu\sigma(\epsilon, E)d\ln\nu \tag{38}$$

The restricted cross section of each process is determined and listed in the separate lambda tables as expressed in

$$\lambda(E) = \int_{\epsilon_m}^{E} \sigma(\epsilon, E)d\epsilon \tag{39}$$

where $\lambda(E)$ is the inverse interaction length, and ϵ_m is the same threshold value as used in Eq. (38).

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