

New constitutive equations for the teardrop and parabolic lens yield curves in viscous-plastic sea ice models

Damien Ringeisen¹, Martin Losch², and Bruno Tremblay¹

¹McGill University

²Alfred Wegener Institute for Polar and Marine Research

November 22, 2022

Abstract

Most viscous-plastic sea ice models use the elliptical yield curve. This yield curve has the fundamental flaw that it excludes acute angles between deformation features at high resolution. Conceptually, the teardrop and parabolic lens yield curves offer an attractive alternative. These yield curves feature a non-symmetrical shape, a Coulombic behavior for the low-medium compressive stress, and a continuous transition to the ridging-dominant mode. We show that the current formulation of the teardrop and parabolic lens viscous-plastic yield curves with normal flow rules results in negative or zero bulk and shear viscosities and consequently poor numerical convergence and representation of stress states on or within the yield curve. These issues are mainly linked to the assumption that the constitutive equation applicable for the elliptical yield curve also applies to non-symmetrical yield curves and yield curves with tensile strength. We present a new constitutive relation for the teardrop and parabolic lens yield curves that solves the numerical convergence issues naturally. Results from simple uni-axial loading experiments show that we can reduce the residual norm of the numerical solver with a smaller number of total solver iterations, resulting in significant improvements in numerical efficiency and representation of the stress and deformation field.

New constitutive equations for the teardrop and parabolic lens yield curves in viscous-plastic sea ice models

Damien Ringeisen^{1,2,3}, Martin Losch¹, L.Bruno Tremblay³

¹Alfred-Wegener-Institut, Helmholtz-Zentrum für Polar- und Meeresforschung, Bremerhaven, Germany

²MARUM – Center for Marine Environmental Sciences, Leobener Str. 8, 28359, Bremen, Germany

³Department of Atmospheric and Oceanic Sciences, McGill University, Montreal, QC, Canada

Key Points:

- The teardrop and parabolic lens yield curves are alternatives to solve the shortcomings of the elliptical yield curve to model sea ice.
- Using the constitutive equation of the elliptical yield curve does not apply to yield curves with other shapes, such as the teardrop.
- We propose new constitutive equations that solves this problem and show improved numerical convergence and stress fields.

Abstract

Most viscous-plastic sea ice models use the elliptical yield curve. This yield curve has the fundamental flaw that it excludes acute angles between deformation feature at high resolution. Conceptually, the teardrop and parabolic lens yield curves offer an attractive alternative. These yield curves feature a non-symmetrical shape, a Coulombic behavior for the low-medium compressive stress, and a continuous transition to the ridging-dominant mode. We show that the current formulation of the teardrop and parabolic lens viscous-plastic yield curves with normal flow rules results in negative or zero bulk and shear viscosities and consequently poor numerical convergence and representation of stress states on or within the yield curve. These issues are mainly linked to the assumption that the constitutive equation applicable for the elliptical yield curve also applies to non-symmetrical yield curves and yield curves with tensile strength. We present a new constitutive relation for the teardrop and parabolic lens yield curves that solves the numerical convergence issues naturally. Results from simple uni-axial loading experiments show that we can reduce the residual norm of the numerical solver with a smaller number of total solver iterations, resulting in significant improvements in numerical efficiency and representation of the stress and deformation field.

Plain Language Summary

Most sea ice models simulate sea ice as a viscous-plastic material. The stress parametrization in viscous-plastic models commonly uses an elliptical yield curve that delimits between the fast plastic deformations and the slow viscous creep. To overcome some shortcomings of the elliptical yield curve, other shapes of yield curves can be used, like the teardrop and parabolic lens yield curves. In this work, we analyze the current formulation of these two yield curves and show that three problems in these formulations lead to bad numerical convergences and nonphysical behaviors. We propose solutions to each of these problems and we use a simple experiment to show that our proposed formulation leads to significant improvements in the computing time and the representation of stresses and deformation.

1 Introduction

Sea ice dynamical models are an integral part of the CMIP6 models (Notz & Community, 2020). Following the increase of computation power, the models' resolution was

increased and can now represent Linear Kinematic Features or LKFs in the sea ice deformation (N. Hutter et al., 2019; Hutchings et al., 2005). LKFs are narrow bands where most of the sea ice deformation takes place (Kwok, 2001). Many of the LKFs are leads with open water or thin ice where the bulk of the heat and matter transfer between atmosphere and ocean takes place (Maykut, 1978), so that it is important to represent LKFs adequately in sea ice models. The capacity of a sea ice dynamical model to represent LKFs explicitly depends mainly on its resolution and its rheological model (Bouchat et al., 2021; N. C. Hutter et al., 2021).

Sea ice deformation is simulated using a rheological model that parameterizes sea ice physical properties and relates stresses and strain rates. Sea ice rheological models can use different material properties to represent sea ice: Elastic-Plastic (EP, Coon et al., 1974), Viscous-Plastic (VP, Hibler, 1977), Elastic-Anisotropic-Plastic (EAP, Tsamados et al., 2013), or Maxwell-Elasto-Brittle (MEB, Dansereau et al., 2016). Still today, sea ice models most commonly use the VP model because in spite of critique they perform well compared to observations and other rheologies, especially at high resolution (N. C. Hutter et al., 2021).

The VP rheological model requires the definition of a yield curve and a flow rule. First, the yield curve sets the stress limit at which sea ice deforms plastically. Several yield curve shapes have been used: elliptical (Hibler, 1979), triangular or Mohr–Coulomb (Ip et al., 1991; Hibler & Schulson, 2000), and teardrop or parabolic lens (Zhang & Rothrock, 2005). Second, the flow rule sets the relative amount of shear and divergence or convergence of the ice for a given stress state. The flow rule can be normal (Hibler, 1979; Zhang & Rothrock, 2005) or non-normal (Ip et al., 1991; Ringeisen et al., 2021) to the yield curve. In this terminology, a rheology is defined by a specific yield curve shape and a flow rule.

Rothrock (1975) proposed two yield curve shapes: the teardrop and parabolic lens yield curves. These two yield curves satisfy Drucker’s convexity postulate for stability (Palmer et al., 1967; Drucker, 1950) and represent both divergence and convergence, in contrast to a Mohr–Coulomb yield curve with a normal flow rule. These two yield curve shapes have not been used until Zhang and Rothrock (2005) proposed VP constitutive equations of these yield curves with normal flow rules. These sea ice VP rheologies are implemented in the PIOMAS model (Zhang, 2020) but, to our knowledge, have not been used much elsewhere.

Sea ice models require expensive solvers because of the non-linear equations of the rheology (Lemieux & Tremblay, 2009; Losch & Danilov, 2012; Koldunov et al., 2019). For useful climate modeling, sea ice models need to be stable and efficient, while giving an accurate prediction of sea ice motion. The stability of the sea ice model can be expressed by system energy considerations (Dukowicz, 1997; Schulkes, 1996; Pritchard, 2005). For example, a negative bulk viscosity would make the rheology act as a spurious energy source that leads to model instabilities. Instabilities lead to poor numerical convergence and inefficiency as the numerical convergence is more difficult to obtain.

In this paper, we document three issues in the formulation of the Teardrop (TD) and Parabolic Lens (PL) yield curves with normal flow rules, as they are described in Zhang and Rothrock (2005). First and second, regions of the yield curve where the non-linear viscosity coefficients are negative or zero and become sources of energy instead of sinks, and third, an inconsistency in the capping of the viscosities for the viscous regime prevents the stresses from remaining on or within the yield curve. We then propose solutions to these issues that ensure numerical convergence for high-resolution sea ice models. We test these solutions in an idealized experiment and show that they lead to improved numerical convergence and efficiency.

The paper is structured as follows: Section 2 reviews the original TD and PL viscous-plastic rheologies (Zhang & Rothrock, 2005). Section 3 discusses the problems and solutions in detail. Section 4 describes an idealized experiment used to compare the new and original rheology. Section 5 compares the numerical convergence of the original and new formulations. Conclusions follow in Section 6.

2 Sea ice model

2.1 The sea ice viscous-plastic rheological model

Following general practice, we simulate sea ice as a (vertically integrated) 2D viscous-plastic material. The ice velocities are calculated from the sea ice momentum equations:

$$\rho h \frac{\partial \vec{u}}{\partial t} = -\rho h f \vec{k} \times \vec{u} + \vec{\tau}_a + \vec{\tau}_o - \rho h \nabla \Phi_s + \nabla \cdot \boldsymbol{\sigma}, \quad (1)$$

where ρ is the ice density, h is the mean sea ice thickness, $\vec{u}$ is the ice drift velocity field, f is the Coriolis parameter, $\vec{k}$ is the vertical unit vector, $\vec{\tau}_a$ is the surface air stress, $\vec{\tau}_o$ is the ocean drag, $\nabla \Phi_s$ is the acceleration due to the gradient of geopotential (i.e., sea

surface) height, and $\boldsymbol{\sigma}$ is the vertically integrated internal ice stress tensor defined by the sea ice VP constitutive equations.

We use the constitutive equation for the elliptical yield curve (Hibler, 1979, 1977):

$$\sigma_{ij} = 2\eta\dot{\epsilon}_{ij} + (\zeta - \eta)\dot{\epsilon}_{kk}\delta_{ij} - \frac{P}{2}\delta_{ij}, \quad (2)$$

where ζ and η are the bulk and shear viscosities, and $\dot{\epsilon}_{ij}$ are the strain rates defined as $\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$. Different rheologies have different non-linear relationships between the viscosities and the strain rates.

The local ice strength P is defined as a function of the mean ice thickness h and concentration A , as

$$P = P^* h e^{-C^*(1-A)}, \quad (3)$$

where P^* is the compressive strength of 1 m thick ice and C^* is a model parameter defining the ice strength dependence on ice concentration.

Equation (2) can also be written in stress invariant form as

$$\sigma_I = 2\zeta\dot{\epsilon}_I - \frac{P}{2}, \quad (4)$$

$$\sigma_{II} = 2\eta\dot{\epsilon}_{II}, \quad (5)$$

where the stress invariants are

$$\sigma_I = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \text{ and } \sigma_{II} = \frac{1}{2}\sqrt{(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2}, \quad (6)$$

and the strain rates invariants are

$$\dot{\epsilon}_I = \frac{1}{2}(\dot{\epsilon}_{11} + \dot{\epsilon}_{22}) \text{ and } \dot{\epsilon}_{II} = \frac{1}{2}\sqrt{(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + 4\dot{\epsilon}_{12}^2}. \quad (7)$$

With the factor $\frac{1}{2}$ for $\dot{\epsilon}_I$ and $\dot{\epsilon}_{II}$, we follow the definition of the strain rate invariants given in Zhang and Rothrock (2005).

2.2 Original formulations

Following Zhang and Rothrock (2005), the equation of the teardrop (TD) and the parabolic lens (PL) yield curves can be written as

$$\frac{\sigma_{II}}{P} = -\left(\frac{\sigma_I}{P} - k_t\right)\left(1 + \frac{\sigma_I}{P}\right)^q, \quad (8)$$

where $q = \frac{1}{2}$ for the TD and $q = 1$ for the PL yield curve. P is the local isotropic compressive strength and $k_t (= T/P)$ is the tensile factor such that T is the local isotropic

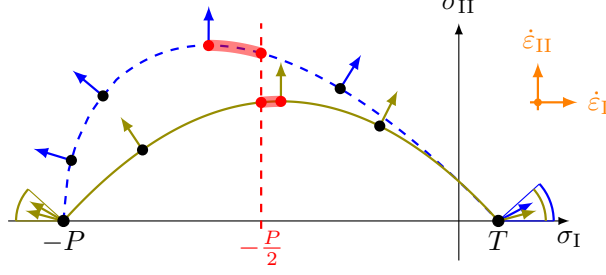


Figure 1. Representation of the teardrop and parabolic yield curves with a normal flow rule in invariant stress space (σ_I, σ_{II}) with $k_t = 0.1$. Regions of the yield curve where bulk viscosities are negative are shown as a thick red (see Sec. 3.1). The angles at the tips show the range of flow rule grouped at this point.

120 ice tensile strength (König Beatty & Holland, 2010). Equation (8) can be rewritten as

$$F = y + (x - k_t)(1 + x)^{\frac{1}{2}} = 0, \text{ for the TD,} \quad (9)$$

$$F = y + (x - k_t)(1 + x) = 0, \text{ for the PL,} \quad (10)$$

where

$$x = \frac{\sigma_I}{P}, \quad y = \frac{\sigma_{II}}{P}. \quad (11)$$

Equations (9) or (10), together with the associated flow rule conditions,

$$\dot{\epsilon}_I = \lambda \frac{\partial F}{\partial x} \frac{\partial x}{\partial \sigma_I} = \frac{\lambda}{P} \frac{\partial F}{\partial x}, \quad \dot{\epsilon}_{II} = \lambda \frac{\partial F}{\partial y} \frac{\partial y}{\partial \sigma_{II}} = \frac{\lambda}{P} \frac{\partial F}{\partial x}, \quad (12)$$

constitute systems of three equations and three unknowns $(\sigma_I, \sigma_{II}, \lambda)$ or (u, y, λ) . The solution of these systems can be written as

$$x_{TD} = \frac{\sigma_I}{P} = \frac{-[6 - 3k_t - 2l^2] + 2l\sqrt{l^2 + 3(1 + k_t)}}{9}, \quad (13)$$

$$y_{TD} = \frac{\sigma_{II}}{P} = -(x_{TD} - k_t)(1 + x_{TD})^{\frac{1}{2}}, \quad (14)$$

$$\lambda = P\dot{\epsilon}_{II}, \quad (15)$$

and

$$x_{PL} = \frac{\sigma_I}{P} = \frac{1}{2}(l - 1 + k_t), \quad (16)$$

$$y_{PL} = \frac{\sigma_{II}}{P} = -(x_{PL} - k_t)(1 + x_{PL}), \quad (17)$$

$$\lambda = P\dot{\epsilon}_{II}, \quad (18)$$

where $l = \frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}}$. Note that a second root for x_{TD} is discarded because it leads to flow rules pointing inward of the yield curve (not shown). The bulk and shear viscosities ζ and η

can be calculated from the above constitutive equations (Eqs. 4 – 5) as,

$$\left. \begin{aligned} \zeta_{\text{TD}} &= \frac{\sigma_{\text{I}} + P/2}{2\dot{\epsilon}_{\text{I}}} = \frac{x_{\text{TD}} + 1/2}{2\dot{\epsilon}_{\text{I}}} P, \\ \eta_{\text{TD}} &= \frac{\sigma_{\text{II}}}{2\dot{\epsilon}_{\text{II}}} = \frac{y}{2\dot{\epsilon}_{\text{II}}} P = \frac{-(x_{\text{TD}} - k_t)(1 + x_{\text{TD}})^{\frac{1}{2}}}{2\dot{\epsilon}_{\text{II}}} P, \end{aligned} \right\} \text{for the TD,} \quad (19)$$

$$(20)$$

and

$$\left. \begin{aligned} \zeta_{\text{PL}} &= \frac{\sigma_{\text{I}} + P/2}{2\dot{\epsilon}_{\text{I}}} = \frac{x_{\text{PL}} + 1/2}{2\dot{\epsilon}_{\text{I}}} P, \\ \eta_{\text{PL}} &= \frac{\sigma_{\text{II}}}{2\dot{\epsilon}_{\text{II}}} = \frac{y}{2\dot{\epsilon}_{\text{II}}} P = \frac{-(x_{\text{PL}} - k_t)(1 + x_{\text{PL}})}{2\dot{\epsilon}_{\text{II}}} P, \end{aligned} \right\} \text{for the PL.} \quad (21)$$

$$(22)$$

In the limit where deformations tend to zero, the viscous coefficients η and ζ become infinite. To avoid this, ζ is capped with a maximum value, ζ_{max} . In contrast to the elliptical yield curve in the standard VP model, η is not a function of ζ and is also capped at a maximum value η_{max} ($= \zeta_{\text{max}}$, J. Zhang, personal communication).

In Zhang and Rothrock (2005), the following conditions are used for the narrow tip of the TD and PL yield curves:

$$\sigma_{\text{I}} = k_t P \text{ for } l > 1, \text{ for the TD and PL,} \quad (23)$$

$$\sigma_{\text{I}} = -P \text{ for } l < -1, \text{ for the PL.} \quad (24)$$

The number l is the link to the orientation of the flow rule, as it is the ratio of divergence and shear. The reason why stress states with the flow rule oriented such as $|l| \geq 1$ are not allowed is unclear. In addition, the conditions (23, 24) are not required for the mathematical derivation of Eqs. (13) and (16). In fact, they lead to a discontinuity of the yield curve, where no stress condition is defined, and shear viscosities that are equal to zero. This has consequences for the numerical convergence of the momentum equations (See Sect. 3.2). Note that for stress states with $|l| > 1$ no LKFs form in the sea ice VP model (see Appendix B, Ringeisen et al., 2019). On the one hand, this may have been the reason that Zhang and Rothrock (2005) introduced the conditions (23, 24). On the other hand, the elliptical yield curve, which serves as our reference for VP sea ice models, allows stress states with $|l| > 1$.

3 Problems and solutions

The formulation (13), (19), (20) — and (16), (21), (22) — of the constitutive equations, as defined in Zhang and Rothrock (2005), leads to three problems.

3.1 Negative bulk viscosity

In Eq. (19) and Eq. (21), the bulk viscosity ζ is negative when the numerator and denominator have different signs, i.e., for:

$$-\frac{(2-k_t)P}{3} < \sigma_I < -\frac{P}{2}, \text{ for the TD,} \quad (25)$$

$$-\frac{P}{2} < \sigma_I < -\frac{(1-k_t)P}{2}, \text{ for the PL.} \quad (26)$$

These regions are marked in red on Fig. 1. In Zhang and Rothrock (2005), the negative ζ are capped at zero ($\zeta_{\min} = 0$, J. Zhang, personal communication). This leads to stress states outside of the yield curve near its apex (see region [1] on Figure 2a).

Negative bulk viscosities appear because the constitutive relation Eq. (2) (or Eqs. 4–5) is derived for a yield curve that is symmetric around $\sigma_I = \frac{P}{2}$ with zero tensile strength ($k_t = 0$) but does not apply to the non-symmetrical teardrop or symmetrical parabolic lens yield curves with normal flow rules but non-zero tensile strength. Rewriting Eqs. (13) and (16) as a linear function of the divergence $\dot{\epsilon}_I$ and a strain-rate independent part (the pressure term), we obtain,

$$\sigma_I = 2 \underbrace{\frac{P}{9\dot{\epsilon}_{II}} \left(\frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}} + \sqrt{\left(\frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}} \right)^2 + 3(1+k_t)} \right)}_{\zeta_{TD}} \dot{\epsilon}_I - \frac{2-k_t}{3}P \text{ for the TD,} \quad (27)$$

$$\sigma_I = 2 \underbrace{\frac{P}{4\dot{\epsilon}_{II}}}_{\zeta_{PL}} \dot{\epsilon}_I - \frac{1-k_t}{2}P \text{ for the PL.} \quad (28)$$

By comparing this formulation to Eq (4), new definitions for ζ_{TD} and ζ_{PL} emerge:

$$\zeta_{TD} = \frac{P}{9\dot{\epsilon}_{II}} \left(\frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}} + \sqrt{\left(\frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}} \right)^2 + 3(1+k_t)} \right) \quad (29)$$

$$\zeta_{PL} = \frac{P}{4\dot{\epsilon}_{II}}. \quad (30)$$

and the pressure term in its original form ($\frac{P}{2}$) becomes ($\frac{2-k_t}{3}P$) and ($\frac{1-k_t}{2}P$). These new formulations for ζ_{TD} , ζ_{PL} , and σ_I solve the problem of negative ζ without the need of capping at zero, compared to Eqs. (19) and (21). In this formulation, we let the constitutive equations for σ_I and the definition of ζ_{TD} and ζ_{PL} emerge naturally from the system of equations defined by the yield curve and the normal flow rule, without the need to use the constitutive equation of the elliptical yield curve without tensile strength Eq (4). Note that, for the PL, the pressure term is the same as for the elliptical yield curve with tensile strength (König Beatty & Holland, 2010). The σ_{II} constitutive equation and the

shear viscosity η is still computed with Eqs. (5), (20) for the TD, and Eqs. (5), (22) for the PL.

3.2 Zero shear viscosity

The conditions (23 and 24) for $l = \frac{\dot{\epsilon}_I}{\dot{\epsilon}_{II}}$ are not necessary for the derivation of the constitutive equations. A consequence of these conditions is a discontinuity on the yield curve with undefined stress, a void (region [2] on Fig. 2a). We can replace these conditions by new ones that force the stress states to stay on the pointy tips of the yield curves and avoid $\sigma_{II} < 0$:

$$\sigma_I = \min(\sigma_I, T), \text{ for the TD and PL} \quad (31)$$

$$\sigma_I = \max(\sigma_I, -P), \text{ for the PL.} \quad (32)$$

However, when σ_I is capped like this, the shear viscosity η is identically zero, with $\sigma_I = -P$ and $\sigma_I = k_t P = T$ (Eqs. 20 and 22), and consequently $\sigma_{II} = 0$. A zero shear viscosity η , while not an energy source as for the negative bulk viscosity, still means less dissipation and may lead to numerical instabilities. Note that even small shear viscosities are sufficient to stabilize the model during large deformations.

The following regularizing conditions on x ensure that the TD and PL yield curves are continuous and that $\eta > 0$:

$$\sigma_I = \min(\sigma_I, \alpha T) \text{ for the TD and PL} \quad (33)$$

$$\sigma_I = \max(\sigma_I, -P + \alpha T) \text{ for the PL,} \quad (34)$$

where $\alpha (\lesssim 1, \text{ e.g., } \alpha = 0.95)$ ensures that η is never equal to zero; i.e., $\sigma_I \neq T$ or $-P$. However, these new conditions truncate the pointy tips of the TD and PL, making them square. With $\alpha \simeq 1$, this truncation is barely noticeable and but it improves the numerical convergence significantly.

3.3 Mixed modes of viscous and plastic deformation

When using the teardrop and parabolic lens yield curve with a normal flow rule, viscous creep and plastic deformation are allowed to occur independently from one another. For instance, plastic shear deformation is allowed while there is creep in convergence. Practically, this means that viscosities ζ and η are capped to ζ_{max} and η_{max} independently, in contrast to the elliptical yield curve where η is function of ζ and both

are capped simultaneously. While this behavior is not physically inconsistent, it leads to stresses outside of the yield curve, and stress states that are inside the yield curve that are plastic in one mode or the other. This makes numerical convergence more difficult for a general yield curve.

To overcome this issue, we impose limits of ζ and η as

$$\zeta = \min \left(\zeta, \zeta_{\max} \min \left(1, \frac{\zeta}{\eta} \right) \right), \quad (35)$$

$$\eta = \min \left(\eta, \eta_{\max} \min \left(1, \frac{\eta}{\zeta} \right) \right), \quad (36)$$

where $\zeta_{\max} = \eta_{\max}$ are model parameters corresponding to ζ_{max} for the elliptical yield curve (Hibler, 1979). With this new relation, we ensure that all stress states inside the yield curve represent viscous deformations and that all stress on the yield curve represent plastic deformation. This new formulation has the disadvantage that it leads to different maximum viscosity depending on where the viscous stress state is relative to the yield curve.

3.4 Comparison of the original and new formulation

Figure 2a shows plastic and stress states with the original and new formulation of the teardrop yield curve. To create these stress states, we create a random field of deformation rates $\dot{\epsilon}_{ij}$ which contains all combinations of shear, compression, and tension. We then apply the constitutive equation to compute the stresses σ_{ij} . Note that the stress states of both formulation are therefore computed with the same strain rates. The magnitude of the random strain rates is set to ensure both viscous and plastic states.

With the original simulation, the stress states for which $\zeta = 0$ are outside of the yield curve and gather along the $\sigma_I = \frac{P}{2}$ line — region [1] — and there are no stress states on the yield curve close to the tip — region [2]. These two features do not appear in our modified formulation of the yield curve. The same comparison can be made with the parabolic lens yield curve (not shown).

Figure 2b shows the viscous capping process. Before viscous capping, the stresses are all on the yield curve (green). When ζ , or η , are capped independently the stresses move horizontally, or vertically, or towards the center if both viscosities are large enough to be capped (orange lines). When the viscosities are capped together following Eqs. (35–

36), the stresses move towards the center of the teardrop (blue lines) ($\sigma_I = -\frac{2-k_t}{3}$). Note that for Fig. 2b, we already use our solutions for η and ζ from Sect. 3.1 and 3.2.

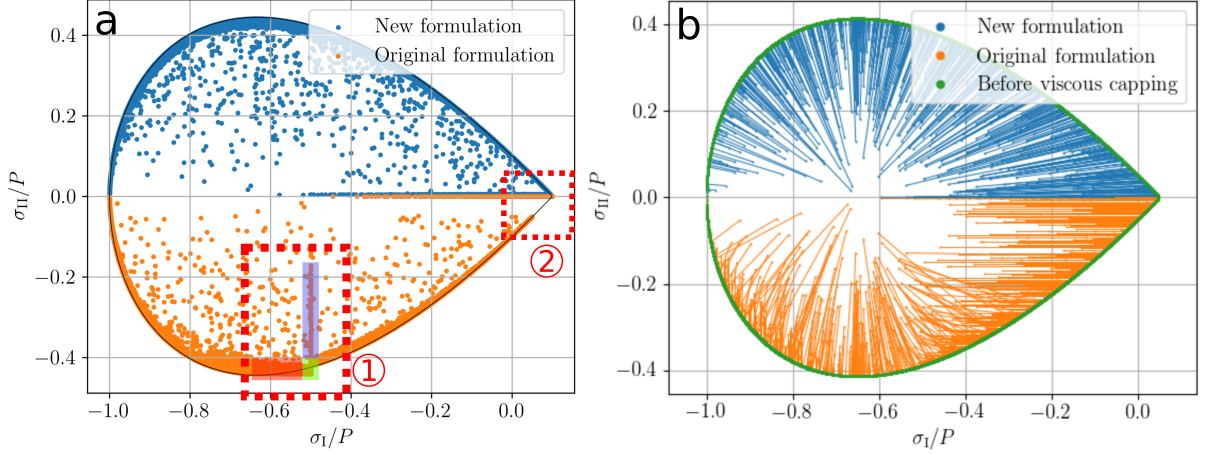


Figure 2. (a) Stress states with the original (orange, lower half plane) and the proposed (blue, upper half plane) formulation of the teardrop yield curve for a random distribution of strain rates. The tensile factor $k_t = 0.1$ and the parameter $\alpha = 0.95$ (see Sect. 3.2). The red boxes [1] and [2] illustrate the issues described in Sect. 3.1 and 3.2. In box [1], the three colors indicate three behaviors: Stress states marked in red exceed the yield curve because of the viscous capping in ζ , these stresses are displaced horizontally toward the $\frac{P}{2}$ line. In green, some stress states are outside of the yield curve as the bulk viscosity is capped at zero. And in blue, stress states with $\zeta = 0$ move inside the ellipse along the $\frac{P}{2}$ line as η is capped for viscous deformation. (b) The lines show the trajectories of the stresses from before viscous capping to after viscous capping in the original (orange, lower half plane) and new formulation (blue, upper half plane) of the viscous capping process (Sect. 3.3). When bulk and shear viscosities are capped independently, the stresses move horizontally or vertically. By capping the viscosities together following Eqs. (35–36), we ensure that the stresses move towards the center of the TD yield curve ($\frac{2-k_t}{3}$).

215

216 4 Experimental setup

217 We use a uniaxial loading experiment in the MIT general circulation model (MITgcm,
218 Campin et al., 2021). A piece of sea ice of 60 km by 250 km is embedded in an 100 km
219 by 260 km domain with a constant grid spacing of 1 km (see Fig. 3). The sea ice is $h =$
220 1 m thick and at $A = 100\%$ concentration. The rectangle of ice is in contact with the

southern border and centered in the domain laterally. Hence, there is 20 km of open water on the East and West, and 10 km of open water in the North. We impose a uniform southward surface stress of 0.15 N m^{-2} . We use a no-slip boundary condition for the southern boundary ($u = 0$ and $v \geq 0$). The other boundaries use periodic boundary condition. Since there is open water to the east, west and north side of the domain, the simulation results are robust with respect to the exact choice of boundary conditions on those boundaries. The timestep is 10 s and the experiment total length is $18\,000 \text{ s} = 5 \text{ h}$. In all simulations below, we use a tensile factor $k_t = 0.05$, unless specified otherwise.

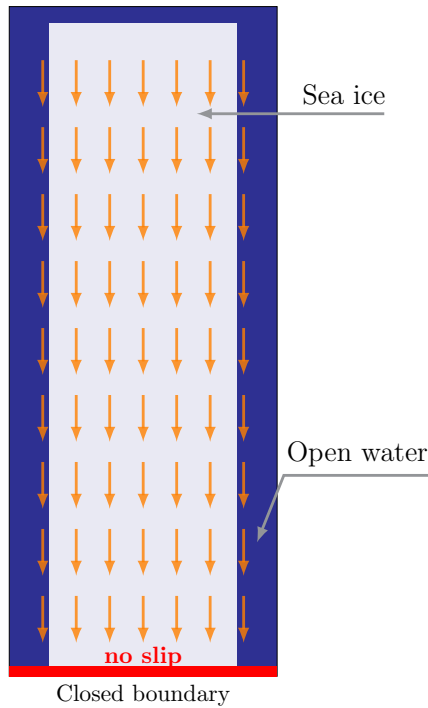


Figure 3. Experimental setup of the idealized experiment. The orange arrows represent the uniform surface stress directed towards the closed boundary in red.

We use a Picard solver with 10 outer-loop iterations (or pseudo-timesteps) (Zhang & Hibler, 1997), unless otherwise specified. For each of the solver outer-loop iterations, we use an LSR solver to solve the linearized problem until the solution reaches a 10^{-6} m s^{-1} accuracy or 500 iterations, whichever comes first. These conditions are realistic for pan-arctic sea ice simulations with 1-2 km resolution (see for example N. Hutter & Losch, 2020).

5 Results and discussion

Figure 4a shows the L2-norm residual and the number of linear (LSR) iterations for each non-linear loop with the TD yield curve in the idealized experiment described in Sect. 4. The different colors show the residuals with and without the modifications described in Sect. 3, and with the standard elliptical yield curve VP rheology for reference. In the original formulation, the model does not converge; i.e. the residual norm increases and stays high. With the new formulation, the model converges and reaches a residual norm similar to the one of the elliptical yield curve with normal flow rule. The modifications improve the convergence by more than one order of magnitude whilst decreasing the total number of linear iterations (thus the total simulation time) by nearly two orders of magnitude (Fig.4a). With the new formulation, the stress states are mostly

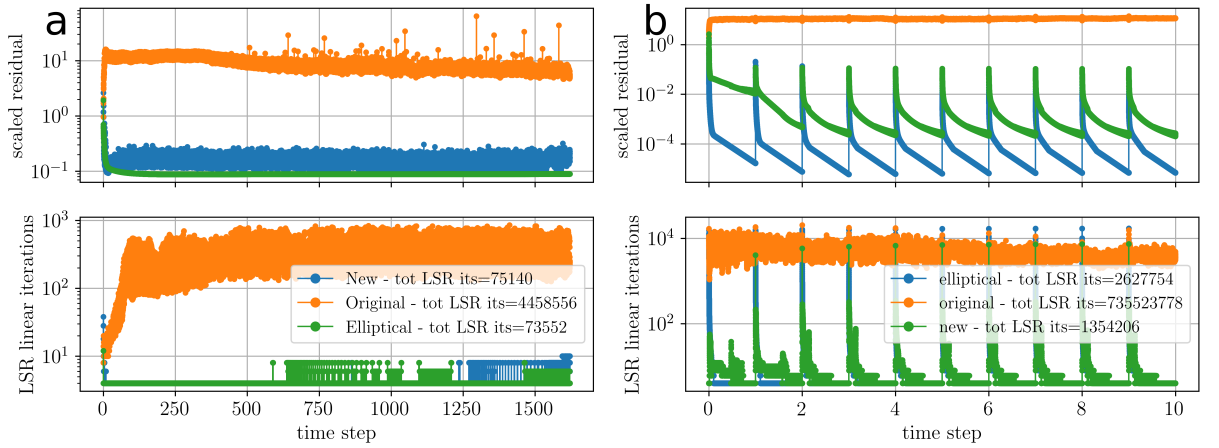


Figure 4. (a) L2-norm of the residual and number of linear LSR iterations for each non-linear iterations in each timestep with the teardrop yield curve without (orange) and with (blue) modifications, and with the elliptical yield curve for reference. The number in the legend indicates the total number of linear iterations used the simulation. We use here realistic solver settings for a 1 km resolution simulation: 10 outer-loops and 10^{-6} or 500 iterations for the LSR. (b) Same as (a), but only the 10 first timesteps when the number of non-linear iterations is increased to 15 000 and the LSR condition for convergence is changed to 10^{-9} m s^{-1} or 15 000 iterations.

on or within the yield curve, and the linear structures along $\sigma_{\text{I}} = -P/2$ and $\sigma_{\text{II}} = 0$ disappear (Fig. 5).

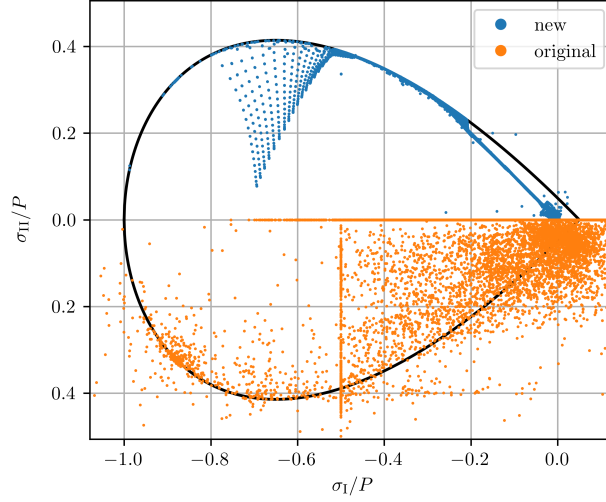


Figure 5. Stress states corresponding to the last timestep of the simulations shown on Fig. 4a. For the new formulation, the stresses form a line following the $\sigma_{II} = -\sigma_I$ axis, as expected for uniaxial compression (Ringeisen et al., 2019).

Figure 4b shows the ten first time steps of the same experiments but with stricter numerical convergence requirements (the solver’s details are listed in the figure caption). Even with a large number of non-linear iterations, the original formulation does not converge and keeps a residual around 10^1 . The modified teardrop yield curve can reach a residual of almost 10^{-4} and continues to decrease. However The rate of decrease is slower than for the elliptical yield curve (Lemieux & Tremblay, 2009). The reasons for the slower convergence with the TD are unclear but may be caused by varying viscosity inside the yield curve.

For the elliptical yield curve with $P = \text{const}$, the stress states within the yield curve always have the same viscosities ($\zeta = \zeta_{\text{max}}$ and $\eta = \frac{\zeta_{\text{max}}}{e^2}$). With the TD and PL yield curves, the viscosities are defined independently and the transition from plastic to viscous deformations also has to be treated independently (see Sect. 3.3). For small stresses (i.e., σ_I and $\sigma_{II} \simeq 0$), this leads to unphysical gradients of shear viscosity. These gradients lead to spatial variations in stresses, thus small scale variations in the viscous deformations. These variations in deformations make the numerical convergence more difficult for the TD compared to the elliptical yield curve. This behavior leads to the cloud of points close to $\sigma_I = 0$ on Fig. 5.

With only two outer-loops (Zhang & Hibler, 1997; Zhang, 2020), the numerical convergence of both the original and new formulation is almost the same and the residual stays around $Res \simeq 10^1$, but the number of linear iterations of the new formulation is reduced (not shown).

6 Conclusions

A new formulation of the teardrop (TD) and parabolic lens (PL) yield curves with normal flow rule addresses three problems in its original formulation (Zhang & Rothrock, 2005). Two of these problems have a common cause: zero bulk or shear viscosity. The zero bulk viscosity ζ is a consequence of the assumption that the constitutive equation Eq. (2) presented in Hibler (1979) is valid for all yield curves, including asymmetrical ones with respect to a average internal pressure $\frac{P}{2}$ and with non-zero tensile strength. When letting the constitutive equation emerge naturally from the yield curve equation and the normal flow rule conditions, a similar relation is found, except for the pressure term with a new scaling factor different from $\frac{1}{2}$ that is a function of k_t . With this modification negative bulk viscosities of in the original formulation no longer appear and additional limiting from below is unnecessary. The zero shear viscosity η is a consequence the condition that keeps stress at the pointy tips (at $\sigma_I = T$ or $-P$) of the yield curves. These conditions of the original formulation are independent of k_t , leading to a discontinuity on the yield curve. The identically zero shear viscosity leads to instabilities. This problem can be fixed by cutting the tip before it reaches the σ_I axis, ensuring a continuous yield curve and non-zero viscosities.

The third problem, of a lesser importance, is linked to the capping of the viscosities for the transition from plastic to viscous states. We reformulate the maximum limit on the bulk and shear viscosities to ensure that both are capped consistently and not independently. By doing so, we avoid stress states that represent half-viscous and half-plastic deformation. The direct consequence of theses changes is an improved numerical convergence and stress states that are on or within the yield curve. The most important was to avoid negative and zero viscosities.

These problems seem to have gone unnoticed because of the generally small number of solver outer-loops used in sea ice models. Sea ice models that use these yield curves generally use a very limited number of non-linear iterations (2 pseudo-timesteps, Zhang

& Hibler, 1997). In this case, issues in the model’s formulation remain hidden behind the unconverged solution. By investigating the formulations of sea ice models in idealized simulations where numerical convergence can be attempted, problems can be identified, investigated, and solved.

The shape of the teardrop yield curve resembles the shape of the Mohr–Coulomb yield curve, and eliminates non-differentiable points when compared with yield curves in other models (Ip et al., 1991; Tremblay & Mysak, 1997; Hibler & Schulson, 2000; Dansereau et al., 2016; Rampal et al., 2019). This makes the teardrop yield curve an interesting alternative for general use in the community. The question whether the teardrop yield curve is an interesting alternative to solve the overestimation of intersection angles in sea ice models (N. C. Hutter et al., 2021, in review) is left for future work.

Open Research - Availability Statement

The rheological models described in this paper are implemented in the sea ice package of the MIT general circulation model (MITgcm) version 67z (Marshall et al., 1997; Losch et al., 2010; Campin et al., 2021). The model’s code and simulations data used in Section 5 are available in the Zenodo archive <https://doi.org/10.5281/zenodo.6091690> (Ringeisen et al., 2022).

Acknowledgments

The authors thank Nils Hutter and Elise Droste for their comments on this manuscript.

This project has been supported by the Deutsche Forschungsgemeinschaft (DFG) through the International Research Training Group “Processes and impacts of climate change in the North Atlantic Ocean and the Canadian Arctic” (grant no. IRTG 1904 ArcTrain). This project a contribution to the Natural Science and Engineering Research Council - Discovery Program and the Grant & Contributions from Environment and Climate Change Canada awarded to Tremblay.

References

Bouchat, A., Hutter, N. C., Chanut, J., Dupont, F., Dukhovskoy, D. S., Garric, G., ... al, e. (2021). Sea Ice Rheology Experiment (SIREx), Part I: Scaling and statistical properties of sea-ice deformation fields. *Earth and Space*

- 324 *Science Open Archive*, 36. Retrieved from [https://doi.org/10.1002/](https://doi.org/10.1002/essoar.10507397.1)
 325 [essoar.10507397.1](https://doi.org/10.1002/essoar.10507397.1) doi: 10.1002/essoar.10507397.1
- 326 Campin, J.-M., Heimbach, P., Losch, M., Forget, G., Edhill3, Adcroft, A., ...
 327 Dussin, R. (2021, June). *MITgcm/MITgcm: checkpoint67z*. Zenodo.
 328 Retrieved 2021-07-28, from <https://zenodo.org/record/4968496> doi:
 329 10.5281/ZENODO.4968496
- 330 Coon, M. D., Maykut, G., A., Pritchard, R. S., Rothrock, D. A., & Thorndike, A. S.
 331 (1974, May). Modeling The Pack Ice as an Elastic-Plastic Material. *AIDJEX*
 332 *BULLETIN*, No. 24 (Numerical Modeling Report), 1–106.
- 333 Dansereau, V., Weiss, J., Saramito, P., & Lattes, P. (2016, July). A Maxwell elasto-
 334 brittle rheology for sea ice modelling. *The Cryosphere*, 10(3), 1339–1359.
 335 Retrieved 2016-10-07, from <http://www.the-cryosphere.net/10/1339/2016/>
 336 doi: 10.5194/tc-10-1339-2016
- 337 Drucker, D. C. (1950). Some implications of work hardening and ideal plasticity.
 338 *Quarterly of Applied Mathematics*, 7(4), 411–418.
- 339 Dukowicz, J. K. (1997, March). Comments on “Stability of the Viscous–Plastic Sea
 340 Ice Rheology”. *Journal of Physical Oceanography*, 27(3), 480–481. Retrieved
 341 2021-11-16, from [https://journals.ametsoc.org/view/journals/phoc/](https://journals.ametsoc.org/view/journals/phoc/27/3/1520-0485_1997_027_0480_cosotv_2.0.co_2.xml)
 342 [27/3/1520-0485_1997_027_0480_cosotv_2.0.co_2.xml](https://journals.ametsoc.org/view/journals/phoc/27/3/1520-0485_1997_027_0480_cosotv_2.0.co_2.xml) (Publisher: Ameri-
 343 can Meteorological Society Section: Journal of Physical Oceanography) doi:
 344 10.1175/1520-0485(1997)027<0480:COSTV>2.0.CO;2
- 345 Hibler, W. D. (1977, September). A viscous sea ice law as a stochastic average of
 346 plasticity. *Journal of Geophysical Research*, 82(27), 3932–3938. Retrieved
 347 from <http://onlinelibrary.wiley.com/doi/10.1029/JC082i027p03932/>
 348 **abstract** doi: 10.1029/JC082i027p03932
- 349 Hibler, W. D. (1979, July). A Dynamic Thermodynamic Sea Ice Model. *Jour-
 350 nal of Physical Oceanography*, 9(4), 815–846. Retrieved 2016-10-07, from
 351 [http://journals.ametsoc.org/doi/abs/10.1175/1520-0485%281979%](http://journals.ametsoc.org/doi/abs/10.1175/1520-0485%281979%29009%3C0815%3AADTSIM%3E2.0.CO%3B2)
 352 [29009%3C0815%3AADTSIM%3E2.0.CO%3B2](http://journals.ametsoc.org/doi/abs/10.1175/1520-0485%281979%29009%3C0815%3AADTSIM%3E2.0.CO%3B2) doi: 10.1175/1520-0485(1979)
 353 009<0815:ADTSIM>2.0.CO;2
- 354 Hibler, W. D., & Schulson, E. M. (2000, July). On modeling the anisotropic failure
 355 and flow of flawed sea ice. *Journal of Geophysical Research: Oceans*, 105(C7),
 356 17105–17120. Retrieved 2017-02-02, from <http://onlinelibrary.wiley.com/>

- doi/10.1029/2000JC900045/abstract doi: 10.1029/2000JC900045
- Hutchings, J. K., Heil, P., & Hibler, W. D. (2005, December). Modeling Linear Kinematic Features in Sea Ice. *Monthly Weather Review*, 133(12), 3481–3497. Retrieved 2017-02-09, from <http://journals.ametsoc.org/doi/abs/10.1175/MWR3045.1> doi: 10.1175/MWR3045.1
- Hutter, N., & Losch, M. (2020, January). Feature-based comparison of sea ice deformation in lead-permitting sea ice simulations. *The Cryosphere*, 14(1), 93–113. Retrieved 2020-02-24, from <https://www.the-cryosphere.net/14/93/2020/> doi: <https://doi.org/10.5194/tc-14-93-2020>
- Hutter, N., Zampieri, L., & Losch, M. (2019, February). Leads and ridges in Arctic sea ice from RGPS data and a new tracking algorithm. *The Cryosphere*, 13(2), 627–645. Retrieved 2020-03-21, from <https://www.the-cryosphere.net/13/627/2019/> (Publisher: Copernicus GmbH) doi: <https://doi.org/10.5194/tc-13-627-2019>
- Hutter, N. C., Bouchat, A., Dupont, F., Dukhovskoy, D. S., Koldunov, N. V., Lee, Y. J., ... al, e. (2021). Sea Ice Rheology Experiment (SIREx), Part II: Evaluating simulated linear kinematic features in high-resolution sea-ice simulations. *Earth and Space Science Open Archive*, 35. Retrieved from <https://doi.org/10.1002/essoar.10507396.1> doi: 10.1002/essoar.10507396.1
- Ip, C. F., Hibler, W. D., & Flato, G. M. (1991). On the effect of rheology on seasonal sea-ice simulations. *Annals of Glaciology*, 15, 17–25.
- Koldunov, N. V., Danilov, S., Sidorenko, D., Hutter, N., Losch, M., Goessling, H., ... Jung, T. (2019). Fast EVP Solutions in a High-Resolution Sea Ice Model. *Journal of Advances in Modeling Earth Systems*, 11(5), 1269–1284. Retrieved 2020-02-10, from <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2018MS001485> doi: 10.1029/2018MS001485
- Kwok, R. (2001). Deformation of the Arctic Ocean Sea Ice Cover between November 1996 and April 1997: A Qualitative Survey. In J. P. Dempsey & H. H. Shen (Eds.), *IUTAM Symposium on Scaling Laws in Ice Mechanics and Ice Dynamics* (pp. 315–322). Dordrecht: Springer Netherlands. doi: 10.1007/978-94-015-9735-7
- König Beatty, C., & Holland, D. M. (2010, January). Modeling Landfast Sea Ice by Adding Tensile Strength. *Journal of Physical Oceanography*, 40(1), 185–

198. Retrieved 2017-05-15, from <http://journals.ametsoc.org/doi/abs/10.1175/2009JP04105.1> doi: 10.1175/2009JPO4105.1
- 392 Lemieux, J.-F., & Tremblay, B. (2009, May). Numerical convergence of viscous-
393 plastic sea ice models. *Journal of Geophysical Research: Oceans*, 114(C5).
394 Retrieved 2018-11-05, from <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2008JC005017> doi: 10.1029/2008JC005017
- 396 Losch, M., & Danilov, S. (2012). On solving the momentum equations of dynamic
397 sea ice models with implicit solvers and the elastic-viscous-plastic technique.
398 *Ocean Modelling*, 41, 42–52.
- 399 Losch, M., Menemenlis, D., Campin, J.-M., Heimbach, P., & Hill, C. (2010). On the
400 formulation of sea-ice models. Part 1: Effects of different solver implementa-
401 tions and parameterizations. *Ocean Modelling*, 33(1–2), 129–144. Retrieved
402 2017-01-31, from <https://www.sciencedirect.com/science/article/pii/S1463500309002418> doi: 10.1016/j.ocemod.2009.12.008
- 403 Marshall, J., Adcroft, A., Hill, C., Perelman, L., & Heisey, C. (1997, March). A
404 finite-volume, incompressible Navier Stokes model for studies of the ocean on
405 parallel computers. *Journal of Geophysical Research: Oceans*, 102(C3), 5753–
406 5766. Retrieved 2017-02-27, from <http://onlinelibrary.wiley.com/doi/10.1029/96JC02775/abstract> doi: 10.1029/96JC02775
- 407
408
- 409 Maykut, G. A. (1978). Energy exchange over young sea ice in the cen-
410 tral Arctic. *Journal of Geophysical Research: Oceans*, 83(C7), 3646–
411 3658. Retrieved 2020-06-02, from <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/JC083iC07p03646>
412 (_eprint: <https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/JC083iC07p03646>)
413 doi: 10.1029/JC083iC07p03646
- 414
- 415 Notz, D., & Community, S. (2020). Arctic Sea Ice in CMIP6. *Geophysical*
416 *Research Letters*, 47(10), e2019GL086749. Retrieved 2021-11-24, from
417 <https://onlinelibrary.wiley.com/doi/abs/10.1029/2019GL086749>
418 (_eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1029/2019GL086749>)
419 doi: 10.1029/2019GL086749
- 420 Palmer, A. C., Maier, G., & Drucker, D. C. (1967). Normality relations and convex-
421 ity of yield surfaces for unstable materials or structural elements.
- 422 Pritchard, R. S. (2005). Stability of sea ice dynamics models: Viscous-plastic

- 423 rheology, replacement closure, and tensile cutoff. *Journal of Geophysical Re-*
 424 *search: Oceans*, 110(C12). Retrieved 2020-10-29, from [https://agupubs](https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2003JC001875)
 425 [.onlinelibrary.wiley.com/doi/abs/10.1029/2003JC001875](https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2003JC001875) (_eprint:
 426 <https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/2003JC001875>) doi:
 427 10.1029/2003JC001875
- 428 Rampal, P., Dansereau, V., Olason, E., Bouillon, S., Williams, T., Korosov, A., &
 429 Samaké, A. (2019, September). On the multi-fractal scaling properties of sea
 430 ice deformation. *The Cryosphere*, 13(9), 2457–2474. Retrieved 2020-08-28,
 431 from <https://tc.copernicus.org/articles/13/2457/2019/> (Publisher:
 432 Copernicus GmbH) doi: <https://doi.org/10.5194/tc-13-2457-2019>
- 433 Ringeisen, D., Losch, M., & Tremblay, B. (2022, February). *MITgcm configuration*
 434 *and simulations for the test of the teardrop yield curve*. Zenodo. Retrieved
 435 2022-02-15, from <https://zenodo.org/record/6091690> (Language: eng) doi:
 436 10.5281/zenodo.6091690
- 437 Ringeisen, D., Losch, M., Tremblay, L. B., & Hutter, N. (2019, April). Simu-
 438 lating intersection angles between conjugate faults in sea ice with different
 439 viscous–plastic rheologies. *The Cryosphere*, 13(4), 1167–1186. Retrieved
 440 2019-05-08, from <https://www.the-cryosphere.net/13/1167/2019/> doi:
 441 <https://doi.org/10.5194/tc-13-1167-2019>
- 442 Ringeisen, D., Tremblay, L. B., & Losch, M. (2021, June). Non-normal flow rules
 443 affect fracture angles in sea ice viscous–plastic rheologies. *The Cryosphere*,
 444 15(6), 2873–2888. Retrieved 2021-06-29, from [https://tc.copernicus.org/](https://tc.copernicus.org/articles/15/2873/2021/)
 445 [articles/15/2873/2021/](https://tc.copernicus.org/articles/15/2873/2021/) (Publisher: Copernicus GmbH) doi: 10.5194/
 446 tc-15-2873-2021
- 447 Rothrock, D. A. (1975, November). The energetics of the plastic deformation of
 448 pack ice by ridging. *Journal of Geophysical Research*, 80(33), 4514–4519. Re-
 449 trieved 2016-10-10, from [http://onlinelibrary.wiley.com/doi/10.1029/](http://onlinelibrary.wiley.com/doi/10.1029/JC080i033p04514)
 450 [JC080i033p04514/abstract](http://onlinelibrary.wiley.com/doi/10.1029/JC080i033p04514) doi: 10.1029/JC080i033p04514
- 451 Schulkes, R. M. S. M. (1996, February). Asymptotic Stability of the Viscous–Plastic
 452 Sea Ice Rheology. *Journal of Physical Oceanography*, 26(2), 279–283. Re-
 453 trieved 2021-11-16, from [https://journals.ametsoc.org/view/journals/](https://journals.ametsoc.org/view/journals/phoc/26/2/1520-0485_1996_026_0279_asotvs_2_0_co_2.xml)
 454 [phoc/26/2/1520-0485_1996_026_0279_asotvs_2_0_co_2.xml](https://journals.ametsoc.org/view/journals/phoc/26/2/1520-0485_1996_026_0279_asotvs_2_0_co_2.xml) (Publisher:
 455 American Meteorological Society Section: Journal of Physical Oceanography)

- doi: 10.1175/1520-0485(1996)026<0279:ASOTVS>2.0.CO;2
- Tremblay, L.-B., & Mysak, L. A. (1997, November). Modeling Sea Ice as a Granular Material, Including the Dilatancy Effect. *Journal of Physical Oceanography*, 27(11), 2342–2360. Retrieved 2017-02-24, from [http://journals.ametsoc.org/doi/abs/10.1175/1520-0485\(1997\)027%3C2342:MSIAAG%3E2.0.CO;2](http://journals.ametsoc.org/doi/abs/10.1175/1520-0485(1997)027%3C2342:MSIAAG%3E2.0.CO;2)
- doi: 10.1175/1520-0485(1997)027<2342:MSIAAG>2.0.CO;2
- Tsamados, M., Feltham, D. L., & Wilchinsky, A. V. (2013, January). Impact of a new anisotropic rheology on simulations of Arctic sea ice. *Journal of Geophysical Research: Oceans*, 118(1), 91–107. Retrieved 2016-11-08, from <http://onlinelibrary.wiley.com/doi/10.1029/2012JC007990/abstract>
- doi: 10.1029/2012JC007990
- Zhang, J. (2020). Sea ice properties in high-resolution sea ice models. *Journal of Geophysical Research: Oceans*, n/a(n/a), e2020JC016689. Retrieved 2020-12-14, from <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2020JC016686> (_eprint: <https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/2020JC016686>) doi: <https://doi.org/10.1029/2020JC016686>
- Zhang, J., & Hibler, W. D. (1997, April). On an efficient numerical method for modeling sea ice dynamics. *Journal of Geophysical Research: Oceans*, 102(C4), 8691–8702. Retrieved 2016-11-23, from <http://onlinelibrary.wiley.com/doi/10.1029/96JC03744/abstract> doi: 10.1029/96JC03744
- Zhang, J., & Rothrock, D. A. (2005, August). Effect of sea ice rheology in numerical investigations of climate. *Journal of Geophysical Research: Oceans*, 110(C8), C08014. Retrieved 2017-12-18, from <http://onlinelibrary.wiley.com/doi/10.1029/2004JC002599/abstract> doi: 10.1029/2004JC002599