

A Spectral element method to compute Earth's free core nutation

Mian Zhang¹ and Chengli Huang¹

¹Shanghai Astronomical Observatory (CAS)

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Abstract

The Free Core Nutation (FCN) is a rotational mode related to non-alignment of the rotation axis of the core and of the mantle. There is a big gap between the observed FCN period (about 430 Sidereal days) and the theoretically calculated period (ranging from 450 Sd to 470 Sd). We propose a spectral element method to compute the period of FCN and obtain a good result, which is 437 Sd. **Keywords:** FCN, Spectral Element Method

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Mian Zhang^{1,2}, Cheng-li Huang^{1,2}

¹Key Laboratory of Planetary Sciences, Shanghai Astronomical Observatory, Chinese Academy of Sciences, Nandan Rd. 80, Shanghai, 200030 China

²School of Astronomy and Space Sciences, University of Chinese Academy of Sciences, 19A Yuquan Road, Beijing 100049, China

Key Points:

- We propose a spectral element method to compute the FCN, and the result is better than the theoretical periods by traditional methods.

Abstract

The Free Core Nutation (FCN) is a rotational mode related to non-alignment of the rotation axis of the core and of the mantle. There is a big gap between the observed FCN period (about 430 Sidereal days) and the theoretically calculated period (ranging from 450 Sd to 470 Sd). We propose a spectral element method to compute the period of FCN and obtain a good result, which is 437 Sd.

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Plain Language Summary

The Free Core Nutation (FCN) is a free oscillation of Earth. It can be observed by the celestial methods, and it carries the information of Earth's deep interior. The theoretical computation has a deviation greater than 4.5%. Lots of researches use different geophysical factors to explain the deviation, which are all factitious and unsatisfactory. We propose a new method to compute the period of FCN, and get a deviation about 1.4%. It shows the computation method is the key factor to explain the deviation.

1 Introduction

The Free Core Nutation (FCN) is a rotational normal mode related to non-alignment of the rotation axis of the core and of the mantle. FCN has a long period (about 430 Sidereal days, Vondrák & Ron, 2020) seen from a celestial reference frame and is a retrograde mode; FCN is also a Nearly-Diurnal Free Wobble(NDFW) seen from a terrestrial reference frame. It can be obtained by processing the observed VLBI data of Earth's rotation and superconducting gravimeter data of Earth's tides. FCN connects Earth's deep interior and the celestial observation of Earth's rotation. It depends on (therefore reflects on) the physics and dynamics of the core and the mantle, especially near the core-mantle boundary(CMB). Hence, FCN is a very important tool for people to study the earth's deep interior.

There are 3 approaches for theoretical computing FCN. The first is the angular momentum method which was proposed by Hough (1895), who designed an earth model composed of a homogeneous rigid shell and an incompressible homogeneous fluid core. This model is transformed into an oblate by Earth's rotation, and is called Hough-Poincaré

model. From angular momentum conservation law, FCN period can be written as M. Rochester et al. (1974).

$$\nu = -e_c \Omega \left(1 - \frac{A_c}{A}\right)^{-1}, \quad (1)$$

where e_c is the flattening of CMB, Ω is the rotation speed of Earth, A_c and A are the equatorial moments of inertia of the core and of the whole earth respectively. Hough predicted a retrograde nearly-diurnal wobble associated with FCN. However, his prediction wasn't accepted at his time, until Jeffreys (1926) proved the existence of the fluid core by seismology data in 1926. The angular momentum method has the virtue of simplicity, but its FCN result derives much from the real one due to the simple model. For example, M. Rochester et al. (1974) obtained 350 Sd for the FCN period.

The second approach is the linear momentum approach. Smith (1974) applied the method to solve normal modes to solve FCN. He extended and applied the Generalized Spherical Harmonics (GSH) based on previous studies, e.g. Phinney and Burridge (1973) (see C.-L. Huang and Liao (2003) for corrections and comment), to transform vectors and tensors in the governing equations for the small periodic oscillations of an oblate spheroidal rotating elastic isotropic earth model from an ellipsoidal domain to an equivalent spherical domain. Along this approach the resultant periods of FCN ranged from 450 Sd to 460 Sd (Wahr, 1981; Dehant, 1990; C.-l. Huang et al., 2001; Rogister, 2001) of the Preliminary Reference Earth Model (PREM, Dziewonski & Anderson, 1981). There is still a gap between this and the observed.

There are many publications trying to interpret this gap by various assumptions. Gwinn et al. (1986), Dehant and Defraigne (1997) explained this discrepancy by non-hydrostatic ellipticity of the CMB. C.-l. Huang et al. (2001) showed that the resultant would accord with the observed by modifying ϵ_{CMB} from 0.002547 to 0.002666 (a 4.7% increase). The excess over the hydrostatic equilibrium value for ϵ_{CMB} estimated by Mathews et al. (2002) was between 3.7% and 3.9%. Buffett et al. (2002) tried to interpret this by the geomagnetic torque on the CMB.

The third approach is the variational approach (Johnson & Smylie, 1977; Moon, 1982; Smylie et al., 1992). Jiang and Smylie (1996) got the period about 450 Sd by this approach. The FCN mode is computed based upon a variational principle of the liquid outer core of the Earth. This variational principle is numerically implemented by a finite el-

ement approach. However, their work just focused on the fluid core. We follow this idea and apply a spectral element method on the whole earth model. Spectral element method is a spectral method working on multiple subdomains(Karniadakis & Sherwin, 2013). We don't adjust the hydrostatic equilibrium figure of the Earth, and finally obtain the FCN eigenperiod: 437 Sd.

2 Equations and boundary conditions

In solid crust, mantle and inner core, we take

$$\rho_0 \omega^2 \vec{u} - 2i\rho_0 \omega \vec{\Omega} \times \vec{u} + \rho_0 \nabla V_1 + \rho_0 \nabla(\vec{u} \cdot \vec{g}_0) - \rho_0 \vec{g}_0 (\nabla \cdot \vec{u}) + \nabla \cdot \vec{S} = 0 \quad (2)$$

as the equation governing the small periodic oscillations of an oblate spheroidal rotating elastic isotropic Earth model, disturbed from hydrostatic equilibrium(F. A. Dahlen, 1972). In eq(2), ρ_0 , V_1 and $\vec{g}_0$ are density, additional potential, and gravity in an equilibrium configuration respectively; $\vec{\Omega}$ is Earth's angular velocity relative to the moving mean vernal equinox, which we take 86164.01 seconds in this paper. And the mass element dm in an equilibrium configuration experiences a small displacement $\vec{u}$ with an oscillation's angular frequency ω . In eq(2) is the Lagrangian variation of the Cauchy stress tensor is

$$\vec{S} = \lambda(\nabla \cdot \vec{u}) \vec{I} + \mu[\nabla \vec{u} + (\nabla \vec{u})^T], \quad (3)$$

where λ , μ are Lamé parameters. In fluid core,

$$\rho_0 \omega^2 \vec{u} - 2i\rho_0 \omega \vec{\Omega} \times \vec{u} - \nabla p_1 + \rho_0 \nabla V_1 + \rho_1 \vec{g}_0 = 0 \quad (4)$$

is the equation governing the isentropic small oscillations of an inviscid liquid core given by the conservation laws for mass, momentum, gravitational flux and entropy(M. Rochester, 1989). Additional density ρ_1 and additional pressure p_1 are defined as

$$\rho_1 = -\nabla \cdot (\rho_0 \vec{u}), \quad (5)$$

and

$$p_1 = -\vec{u} \cdot \nabla p_0 + \alpha^2 \rho_1 + \alpha^2 \vec{u} \cdot \nabla \rho_0, \quad (6)$$

90 where p_0 is pressure in an equilibrium configuration, and α is compressional wave
 91 speed. The tensor in eq(4) is

$$\vec{S} = -(p_1 + \vec{u} \cdot \nabla p_0) \vec{I} . \quad (7)$$

Poisson's equation,

$$\nabla^2 V_1 = 4\pi G \nabla \cdot (\rho_0 \vec{u}) , \quad (8)$$

92 holds true in both fluid and solid layers.

93 Continuation on the boundary between solid and fluid layers requires

$$\{\hat{n} \cdot \vec{u}\}_-^+ = 0 \quad (9)$$

$$\{\hat{n} \cdot \vec{S}\}_-^+ = 0 \quad (10)$$

$$\{V_1\}_-^+ = 0 \quad (11)$$

$$\{\hat{n} \cdot [\nabla V_1 - 4\pi G \rho_0 \vec{u}]\}_-^+ = 0 . \quad (12)$$

94 And continuation on the boundary between solid layers requires

$$\{\vec{u}\}_-^+ = 0 \quad (13)$$

$$\{\hat{n} \cdot \vec{S}\}_-^+ = 0 \quad (14)$$

$$\{V_1\}_-^+ = 0 \quad (15)$$

$$\{\hat{n} \cdot [\nabla V_1 - 4\pi G \rho_0 \vec{u}]\}_-^+ = 0 . \quad (16)$$

95 The boundary conditions on the free surface require

$$\{\hat{n} \cdot [\nabla V_1 - 4\pi G \rho_0 \vec{u}]\}_-^+ = 0 \quad (17)$$

$$\{\hat{n} \cdot \vec{S}\}_-^+ = 0 \quad (18)$$

$$\{V_1\}_-^+ = 0 . \quad (19)$$

96 **3 Multiple subdomain spectral method**

97 Spectral method can solve the above governing equations. Suppose that an unknown
 98 function $u(x)$ satisfies a differential equation:

$$L[u(x)] = D, \quad (20)$$

99 where L is a linear differential operator. Spectral method represents $u(x)$ as a truncated
100 series:

$$u(x) \approx u_N(x) = \sum_{n=0}^N c_n \eta_n(x), \quad (21)$$

101 where $\eta_n(x)$ are basis functions and c_n are their coefficients. This series is then put into
102 the differential equation(20):

$$L\left[\sum_{n=0}^N c_n \eta_n(x)\right] = D. \quad (22)$$

103 By the Galerkin method the above equation turns into a group of equations:

$$\int_V \iota_j(x) L\left[\sum_{n=0}^N c_n \eta_n(x)\right] dx = \int_V \iota_j(x) D dx, \quad (23)$$

104 where $\iota_j(x)$ are trial functions. By solving eq(23) with the boundary conditions, we can
105 get an approximate solution of the unknown function $u(x)$, which is $u_N(x)$.

106 For a complex earth model, only one global domain is not enough to represent some
107 characteristics, such as the densities and the toroidal displacement fields between fluid
108 core and solid mantle. So the whole global domain is partitioned into K disjoint sub-
109 domains. In No. k subdomain an unknown function $u^{(k)}(x)$ is expressed as

$$u^{(k)}(x) \approx \sum_{n=0}^N c_n^{(k)} \eta_n^{(k)}(x), \quad (24)$$

110 where $\eta_n^{(k)}(x)$ are basis functions of No. k subdomain and $c_n^{(k)}$ are their coefficients. So
111 eq(23) turns into K groups of equations:

$$\int_{V^{(k)}} \iota_j^{(k)}(x) L^{(k)}\left[\sum_{n=0}^N c_n^{(k)} \eta_n^{(k)}(x)\right] dx = \int_{V^{(k)}} D^{(k)} dx, \quad (25)$$

112 where $\iota_j^{(k)}(x)$ are trial functions in No. k subdomain and $L^{(k)}$ are their linear operators.
113 Eq(25) will create a $K(N+1) \times K(N+1)$ matrix. Suppose that there are M boundary
114 conditions:

$$B_i[\sum_{k=1}^K u^{(k)}(x)] = E_i, \quad i = 1 \cdots M. \quad (26)$$

We use Tau method(Karniadakis & Sherwin, 2013) to combine these boundary conditions with eqs(25). Tau method replaces M equations in eqs(25) with M boundary conditions in eqs(26). The unknown functions $u^{(k)}(x)$ in all K subdomains can be obtained by solving the new $K(N+1) \times K(N+1)$ matrix. The global $u(x)$ is the union of $u^{(k)}(x)$: $\cup_{i=1}^K u^{(k)}(x)$, which is like a sheaf in category theory. This spectral method on multiple subdomains is a kind of spectral element method(Boyd, 2001; Karniadakis & Sherwin, 2013), and is called stratified Galerkin method here as Galerkin's method is chosen to convert the continuous operator problem to the discrete problem.

4 Integrations in volumes

To solve the governing equations, Smith (1974) transformed these equations into a group of Partial Differential Equations(PDEs), and integrated the vector of variables from Earth's center to surface with some certain initial values by Runge-Kutta method. Compared to Smith's numerical computation, our approach is to directly integrate these governing equations containing variables that have no given value, which are manipulated as symbols.

Although the hydrostatic equilibrium figure is an ellipsoid, we still solve these equations in spherical coordinates, which makes symbolic operations of vector spherical harmonics and tensors more complex and tedious. We adopt a linear operator method similar to Rogister and Rochester (2004). However, Kopal (1980) recommended a non-orthogonal coordinate system named 'Clairaut' coordinates for the astrophysical research. The coordinate surfaces of this non-orthogonal coordinate system consist with the equilibrium surfaces of an equipotential ellipsoid. Wu (1993), Seyed-Mahmoud and Moradi (2014) used Clairaut coordinates to study the dynamics of the fluid core. Rogister and Rochester (2004), M. G. Rochester et al. (2014) and Crossley and Rochester (2014) applied Clairaut coordinates to the linear momentum approach accurate to second order in the ellipticity.

Smith (1974) applied ESD to deal with integration in the first-order approximated ellipsoid. ESD transforms an ellipsoid into a sphere, then parameters are modified with ellipticity. PDEs don't have θ or ϕ explicitly, then the vector of variables is integrated

along the radius r 's direction. However, ESD approach is difficult to deal with asymmetric models, for instance, it is difficult and complex to transform a surface with a Y_3^0 component or a Y_2^2 component to a spherical surface. We don't adopt the ESD, and integrate the governing equations in asymmetric shells directly. Suppose that an asymmetric shell has an inner boundary:

$$r = R_{in} + \sum_{n,m} \xi_n^m Y_n^m(\theta, \phi), \quad (27)$$

and an outer boundary:

$$r = R_{out} + \sum_{n,m} \Xi_n^m Y_n^m(\theta, \phi), \quad (28)$$

where R_{in} , R_{out} , ξ_n^m , Ξ_n^m are all constants, and $Y_n^m(\theta, \phi)$ are spherical harmonics. If $\xi_n^m = 0$ and $\Xi_n^m = 0$, then this shell is a spheric shell; if only ξ_2^0 and Ξ_2^0 are not equal to 0, then this shell is a first-order ellipsoidal shell.

The volume between the inner and the outer boundaries is

$$\iiint dV = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{R_{in} + \sum_{n,m} \xi_n^m Y_n^m(\theta, \phi)}^{R_{out} + \sum_{n,m} \Xi_n^m Y_n^m(\theta, \phi)} r^2 \sin \theta dr d\theta d\phi. \quad (29)$$

The integral of a vector equation $\vec{\mathbf{E}}\mathbf{q}$ in this volume with a trial function vector $\vec{\Lambda}_{(i,j,k)}$ is

$$\begin{aligned} & \iiint \vec{\Lambda}_{(i,j,k)}^* \cdot \vec{\mathbf{E}}\mathbf{q} dV \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{R_{in} + \sum_{n,m} \xi_n^m Y_n^m(\theta, \phi)}^{R_{out} + \sum_{n,m} \Xi_n^m Y_n^m(\theta, \phi)} \vec{\Lambda}_{(i,j,k)}^* \cdot \vec{\mathbf{E}}\mathbf{q} r^2 \sin \theta dr d\theta d\phi, \end{aligned} \quad (30)$$

where the asterisk symbol (*) in superscript is the complex conjugate operator. For the governing equations eq(2) and eq(4), $\vec{\Lambda}_{(i,j,k)}$ can be $\eta_i(r)\vec{R}_j^k(\theta, \phi)$, $\eta_i(r)\vec{S}_j^k(\theta, \phi)$, and $\eta_i(r)\vec{T}_j^k(\theta, \phi)$, which $\vec{R}_j^k(\theta, \phi)$, $\vec{S}_j^k(\theta, \phi)$ and $\vec{T}_j^k(\theta, \phi)$ are radial, consoidal, and toroidal vector harmonics respectively[cite dahlen tramp].

The integral of a scalar equation $\mathbf{E}\mathbf{q}$ with a trial function $\sigma_{(i,j,k)}$ is

$$\begin{aligned} & \iiint \sigma_{(i,j,k)} * \mathbf{E}\mathbf{q} dV \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \int_{R_{in} + \sum_{n,m} \xi_n^m Y_n^m(\theta, \phi)}^{R_{out} + \sum_{n,m} \Xi_n^m Y_n^m(\theta, \phi)} \sigma_{(i,j,k)} * \mathbf{E}\mathbf{q} r^2 \sin \theta dr d\theta d\phi. \end{aligned} \quad (31)$$

For Laplace's equation, the trial function $\sigma_{(i,j,k)}$ is $\eta_i(r)Y_j^k(\theta, \phi)$.

5 Integrations on surfaces

Now we will discuss how to solve boundary conditions on an asymmetric boundary. Suppose a boundary surface is described by

$$r(\theta, \phi) = r_0 + \sum_{n,m} \kappa_n^m Y_n^m(\theta, \phi), \quad (32)$$

then the radius vector of a point at (r, θ, ϕ) in this surface is

$$\vec{r} = r(\theta, \phi) \hat{r}, \quad (33)$$

and the normalized normal vector $\hat{n}$ is

$$\hat{n} = \frac{\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}}{\left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right|}. \quad (34)$$

The area of this boundary surface is

$$\oint dS = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right| d\theta d\phi. \quad (35)$$

The surface integral of a boundary condition is

$$\begin{aligned} & \oint \hat{n} \cdot \square dS \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\hat{n} \cdot \square) \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right| d\theta d\phi \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right) \cdot \square d\theta d\phi, \end{aligned} \quad (36)$$

where $\square$ is a vector. As

$$\begin{aligned} \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} &= r^2 \sin \theta \hat{r} - r r_{\theta} \sin \theta \hat{\theta} - r r_{\phi} \hat{\phi} \\ &= r^2 \sin \theta \left[\hat{r} - \frac{1}{r} (r_{\theta} \hat{\theta} + \frac{r_{\phi}}{\sin \theta} \hat{\phi}) \right] \\ &= [r(\theta, \phi)]^2 \sin \theta [\hat{r} - \nabla r(\theta, \phi)], \end{aligned} \quad (37)$$

so eq(36) becomes

$$\begin{aligned} & \oint \hat{n} \cdot \square dS \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \{[\hat{r} - \nabla r(\theta, \phi)] \cdot \square\} [r(\theta, \phi)]^2 \sin \theta d\theta d\phi. \end{aligned} \quad (38)$$

170 The box symbol $\square$ is a vector continuation boundary condition: $\square = \Delta \vec{u} = \vec{u}^+ -$
 171 $\vec{u}^-$. Then multiply a trial function $\sigma_{(i,j,k)}$:

$$\begin{aligned} 0 &= \oint \sigma_{(i,j,k)} \hat{n} \cdot \Delta \vec{u} dS \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sigma_{(i,j,k)} \{[\hat{r} - \nabla r(\theta, \phi)] \cdot \Delta \vec{u}\} [r(\theta, \phi)]^2 \sin \theta d\theta d\phi, \end{aligned} \quad (39)$$

172 where $\sigma_{(i,j,k)} = Y_j^k(\theta, \phi) \eta_i(r)$.

173 Now suppose $\square$ is a tensor continuation boundary condition: $\square = \delta \vec{T} = \vec{T}^+ - \vec{T}^-$,
 174 then eq(38) turns into

$$\begin{aligned} 0 &= \oint \vec{\Lambda}_{(i,j,k)} \cdot [\hat{n} \cdot \delta \vec{T}] dS \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \vec{\Lambda}_{(i,j,k)} \cdot \{[\hat{r} - \nabla r(\theta, \phi)] \cdot \delta \vec{T}\} [r(\theta, \phi)]^2 \sin \theta d\theta d\phi. \end{aligned} \quad (40)$$

175 For a scalar continuation condition:

$$\delta a = a^+ - a^- = 0, \quad (41)$$

176 it is a little bit tricky, for it is impossible to get a linear form of its surface integral in
 177 explicit formula, as there is a $|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}|$ in the denominator in eq(34). So multiply eq(41)
 178 by $|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}| \sin \theta$:

$$|\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}| \sin \theta * \delta a = 0, \quad (42)$$

179 then the boundary condition(41) turns into

$$\begin{aligned} 0 &= \oint \sigma_{(i,j,k)} |\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}| \sin \theta * \delta a * |\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}| dS \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sigma_{(i,j,k)} |\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}|^2 * \delta a * \sin \theta * d\theta d\phi, \end{aligned} \quad (43)$$

180 where

$$\begin{aligned}
 & \left| \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} \right|^2 \\
 &= r^4 \sin^2 \theta + r^2 [r_\theta \sin \theta]^2 + r^2 r_\phi^2 \\
 &= r^4 (1 - \cos^2 \theta) + r^2 [r_\theta \sin \theta]^2 + r^2 r_\phi^2.
 \end{aligned} \tag{44}$$

6 Earth model

PREM is adopted here excluding the ocean. We divide this model into 12 layers according to PREM to describe the parameters such as density and Lamé parameters; and 3 layers to describe the variables such as the displacement vector field and the additional potential scalar. The 3 layers are the inner core, the outer core, and the mantle(with crust). Then the earth model is modified by one order ellipticity by rotation. We solve the hydrostatic equilibrium figure by a more prototypic equation instead of Clairaut's equation, which is (Moritz, 1990; C. Huang et al., 2019)

$$\begin{aligned}
 & \left(\frac{1}{r_0^n} \frac{d\epsilon_n^m}{dr_0} + \frac{n}{r_0^{n+1}} \epsilon_n^m \right) \int_0^{r_0} \rho q^2 dq \\
 & - \int_{r_0}^R \rho \frac{d}{dq} \left(\frac{\epsilon_n^m}{q^{n-2}} \right) dq + \frac{5\omega^2}{12\pi G} \Big|_{\text{if } n=2, m=0} = 0.
 \end{aligned} \tag{45}$$

This integro-differential equation is solved by our spectral element method. For a first-order approximated ellipsoid, an equipotential surface in the new equilibrium figure is

$$r_S = r_0 [1 + \epsilon_2^0 P_2(\cos \theta)], \tag{46}$$

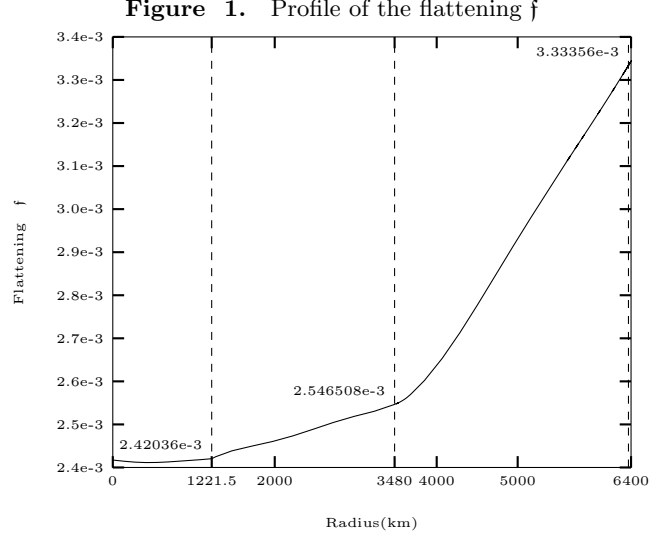
where r_0 is the radius of the surface in the old equilibrium figure. The flattening f is

$$f = \frac{a-b}{a}, \tag{47}$$

where a , b are the equatorial radius and the polar radius respectively, and it can prove that in first-order approximation (Moritz, 1990)

$$\epsilon_2^0 = -\frac{2}{3}f. \tag{48}$$

Figure(6) shows the profile of the flattening with respect to r . The flattening are 1/392.70 and 1/299.98 at CMB and the surface respectively.



196 After deformation, the density ρ , the Lamé parameters λ and μ , and the gravita-
 197 tional potential ψ are decomposed into spherical parts and non-spherical parts:

$$\rho(r_S) = \rho_0(r_S) + \delta\rho(r_S) \quad (49)$$

$$\lambda(r_S) = \lambda_0(r_S) + \delta\lambda(r_S) \quad (50)$$

$$\mu(r_S) = \mu_0(r_S) + \delta\mu(r_S) \quad (51)$$

$$\psi(r_S) = \psi_0(r_S) + \delta\psi(r_S), \quad (52)$$

198 where $\rho_0(r_S)$, $\lambda_0(r_S)$, $\mu_0(r_S)$, and $\psi_0(r_S)$ are the parameters before deformation. Dahlen(cite
 199 Dahlen1968) gave the non-spherical parts as

$$\delta\rho(r_S) = \epsilon_2^0 r_S \frac{\partial \rho_0(r_S)}{\partial r_S} P_2(\cos \theta) \quad (53)$$

$$\delta\lambda(r_S) = \epsilon_2^0 r_S \frac{\partial \lambda_0(r_S)}{\partial r_S} P_2(\cos \theta) \quad (54)$$

$$\delta\mu(r_S) = \epsilon_2^0 r_S \frac{\partial \mu_0(r_S)}{\partial r_S} P_2(\cos \theta) \quad (55)$$

$$\delta\psi(r_S) = \epsilon_2^0 r_S \frac{\partial \psi_0(r_S)}{\partial r_S} P_2(\cos \theta). \quad (56)$$

200 Then the gravity $\vec{g}$ in the new equilibrium configuration is

$$\vec{g} = \nabla \left\{ \psi + \frac{1}{3} \Omega^2 r_S^2 [1 - P_2(\cos \theta)] \right\}. \quad (57)$$

7 Boundary conditions at the center

For this FCN computation the displacement field $\vec{u}$ is truncated as $\vec{T}_1^1 + (\vec{R}_2^1 + \vec{S}_2^1) + \vec{T}_3^1$. Vector spherical harmonics are expanded in power series in each subdomain, for instance, $\vec{S}_2^1$ is expanded as

$$\sum_{i=0}^{r_{max}} a_i r^i \vec{S}_2^1, \quad (58)$$

where r_{max} is the max power order, and a_i are unknown coefficients. $\vec{S}_2^1$ has the same expansion forms in the inner core, the fluid core, and the mantle, but the coefficients are different, for instance, a_i can be written as $a_i^{(IC)}$, $a_i^{(OC)}$ and $a_i^{(MT)}$ respectively. Similarly, we can get the expansions of $\vec{T}_1^1$, $\vec{R}_2^1$, $\vec{T}_3^1$ and V_1 .

The boundary condition in the center is required to be regular. This is a vague statement. For the free-oscillation equations, Crossley (1975) expanded the variables as power series,

$$y_i(r) = r^\alpha \sum_{\nu=0}^{\infty} A_{i,\nu} r^\nu, \quad i = 1, 2, \dots, 6, \quad (59)$$

then substituted the power series into the partial differential equations, and got one independent initial solution by picking α and a set of $A_{i,\nu}$ making PDEs finite in the center. Then 3 independent solutions combined into one general solution. Our first approach is similar to the Crossley's in some sense, which is to increase the lower limit of eq(58) to satisfy the finitude of $(\frac{1}{r})^k$ in the governing equation. For example, when a term of the displacement field $\vec{u}$ with vector spherical harmonic basis $\vec{S}_2^1$ is substituted into the equation, there are M terms with $(\frac{1}{r})^{k_1}$, $(\frac{1}{r})^{k_2}$, ..., $(\frac{1}{r})^{k_M}$, and $k_1, k_2, \dots, k_M$ are all non-negative integers. Suppose k_M is the largest, the lower limit $i = 0$ in eq(58) should increase to $i = k_M$. Note that the different vector spherical harmonic bases may have the different k_M , for instance, $\vec{T}_1^1$ and $\vec{S}_2^1$ have the different k_M . After pre-computation, we take these k_M : 0 for $\vec{T}_1^1$, 2 for $\vec{R}_2^1$, 2 for $\vec{S}_2^1$, and 2 for $\vec{T}_3^1$. In our previous paper (Zhang & Huang, 2019) we took this approach, and got FCN's period: -431 Sd for the $r_{max} = 4$ instance.

Our second approach is to establish several algebraic equations of the coefficients. For instance, eq(2) has 4 set algebraic equations for the vector spherical harmonics: $\vec{T}_1^1$, $\vec{R}_2^1$, $\vec{S}_2^1$, and $\vec{T}_3^1$. There are 2 algebraic equations in each set, for example, there are 2 equa-

228 tions corresponding with $\frac{1}{r}\vec{S}_2^1$ and $\frac{1}{r^2}\vec{S}_2^1$ respectively in $\vec{S}_2^1$ set. The equation for $\frac{1}{r}\vec{S}_2^1$ takes
 229 the form:

$$z_{i_1} + z_{i_2} + \dots + z_{i_N} = 0. \quad (60)$$

230 This equation comes from putting the expansions of the displacement field $\vec{u}$:

$$\sum_{i=0}^{r_{max}} a_i r^i \vec{T}_1^1 + \sum_{i=0}^{r_{max}} b_i r^i \vec{R}_2^1 + \sum_{i=0}^{r_{max}} c_i r^i \vec{S}_2^1 + \sum_{i=0}^{r_{max}} d_i r^i \vec{T}_3^1 \quad (61)$$

231 and the expansion of the additional potential V_1 :

$$\sum_{i=0}^{r_{max}} e_i r^i Y_2^1 \quad (62)$$

232 into eq(2) and filtering the terms with $\frac{1}{r}\vec{S}_2^1$. We use the second approach in this paper.

233 8 Matrix

234 It is very complex and difficult to expand massive mathematical expressions and
 235 to integrate them in an asymmetric model. So we write a computer algebra system in
 236 Common Lisp to implement these functions. After tedious symbol computations, we can
 237 get a large matrix. The row represents the ordinal of a trial function, and the column
 238 represents the ordinal of an unknown coefficient. Finally there is still an unknown quan-
 239 tity: ω in the matrix. ω is a eigenfrequency so that the determinant of the matrix must
 240 be zero. It is difficult to compute the determinant of a large matrix, so we use Singu-
 241 lar Value Decomposition(SVD) algorithm to get an equivalent determinant. We use the
 242 reliable SVD routine of Linear Algebra PACKage(LAPACK).

243 For instance, Suppose $r_{max} = 5$, we will get three 124×124 matrices $\mathbb{M}_0$, $\mathbb{M}_1$ and
 244 $\mathbb{M}_2$, then the final matrix $\mathbb{M}$ is

$$\mathbb{M} = \mathbb{M}_0 + \omega \mathbb{M}_1 + \omega^2 \mathbb{M}_2. \quad (63)$$

245 To find a normal mode is to find an ω that satisfies the below equation:

$$\mathbb{M}_0 + \omega \mathbb{M}_1 + \omega^2 \mathbb{M}_2 = 0. \quad (64)$$

246 If ω_* is a solution, then the determinant of the matrix:

$$\mathbb{M} = \mathbb{M}_0 + \omega_* \mathbb{M}_1 + \omega_*^2 \mathbb{M}_2 \quad (65)$$

247 must be zero. However, it is virtually impossible to compute the determinant of a $124 \times$
 248 124 matrix. An alternative way is to compute its condition number, and find the max-
 249 ima. The matrix $\mathbb{M}$ is decomposed by SVD into

$$\mathbb{M} = \mathbb{U} \mathbb{S} \mathbb{V}^*, \quad (66)$$

250 where $\mathbb{U}$ and $\mathbb{V}$ are unitary, and $\mathbb{S}$ is a diagonal matrix with non-negative real numbers
 251 on the diagonal. Suppose $\mathfrak{s}_{max}$ is the largest diagonal entry and $\mathfrak{s}_{min}$ is the smallest, then
 252 define the condition number $\mathfrak{c}$ as

$$\mathfrak{c} := \log \mathfrak{s}_{max} - \log \mathfrak{s}_{min}. \quad (67)$$

253 So to find the zero points of the determinant of the $\mathbb{M}$ is to find the maxima of $\mathfrak{c}$ when
 254 searching ω .

255 Figure(8) shows this condition number $\mathfrak{c}$ with respect to angular frequency ω rang-
 256 ing from $7.29e-5$ to $7.31e-5$ where $r_{max} = 5$. There are 2 peaks in this range, which
 257 means there are 2 possible zero determinants, in other words, 2 possible normal modes.

Figure 2. Condition number $\mathfrak{c}$ with angular frequency ω

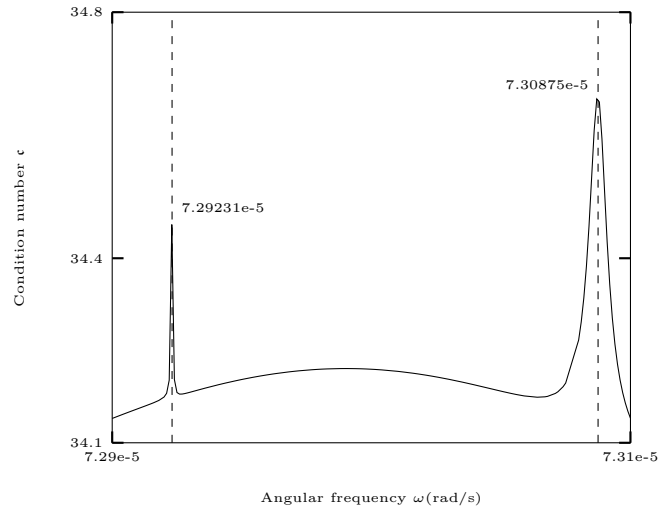


Table 1. Periods of TOM

r_{max}	Angular Frequency(rad/s)	Period (Sd)	Deviation
3	7.29232e-5	0.99997	-0.003%
4	7.29231e-5	0.99997	-0.003%
5	7.29231e-5	0.99997	-0.003%
6	7.29232e-5	0.99997	-0.003%
7	7.29240e-5	0.99996	-0.004%

9 Result

Tilt-over Mode(TOM) has the same period with the rigid earth, which is exact 1 Sd. We use TOM to evaluate the accuracy of our numerical solutions. Table(1) shows the period of TOM with respect to r_{max} order. All the terms in the equations and the boundary conditions are accurate to $(\epsilon_2^0)^2$ in our computation. When $3 \leq r_{max} \leq 7$, the absolute value of the deviation between the calculated and the observed is less than 0.004%. When $r_{max} \geq 8$, we can't find an extremum of the equivalent determinant, because the power series has a disadvantage which is not normalized. For instance, if an unknown function is written as

$$u(x) = \sum_{i=0}^N a_i r^i, \quad (68)$$

the computation will encounter numerical overflow when i is large. If we put a factor R which is large than r ,

$$u(x) = \sum_{i=0}^N a_i \left(\frac{r}{R}\right)^i, \quad (69)$$

$\left(\frac{r}{R}\right)^i$ will approach zero quickly. We try medium R , however, the result is not satisfactory. We now work on writing a new version of our computer algebra system which will have a new architecture, replace power series with Chebyshev's polynomials and support multithreads.

Table 2. Periods of FCN

r_{max}	Angular Frequency(rad/s)	Period (Sd)
3	7.3098e-5	-412
4	7.3088e-5	-437
5	7.3088e-5	-437
6	7.3088e-5	-437
7	7.3088e-5	-437

Table 3. Periods of FCN with $\vec{T}_1^1 + (\vec{R}_2^1 + \vec{S}_2^1) + \vec{T}_3^1 + (\vec{R}_4^1 + \vec{S}_4^1) + \vec{T}_5^1$

r_{max}	Angular Frequency(rad/s)	Period (Sd)
4	7.3088e-5	-437
5	7.3089e-5	-434
6	7.3089e-5	-434

For the deviations of TOM is about -0.003% , we take 5 significant digits in FCN. Table(2) shows the results of FCN. When $r_{max} = 3$, the result doesn't seem to be good, because the basis functions are not enough. From $r_{max} = 4$ to $r_{max} = 7$, the angular frequency of FCN are $7.3088e - 5$ and the corresponding period in the celestial reference frame is -437 Sd. There is a minor difference(about 1.4%) between this result and our previous result: -431 Sd(Zhang & Huang, 2019), because we take the second approach to deal with the boundary condition in the center here, and Zhang and Huang (2019) took the first approach.

If the displacement field $\vec{u}$ is truncated as $\vec{T}_1^1 + (\vec{R}_2^1 + \vec{S}_2^1) + \vec{T}_3^1 + (\vec{R}_4^1 + \vec{S}_4^1) + \vec{T}_5^1$, and the second approach for the boundary condition in the center is applied, we will get the angular frequency of FCN: $7.3088e - 5$, $7.3089e - 5$, and $7.3089e - 5$ Sd for $r_{max} = 4$, $r_{max} = 5$, and $r_{max} = 6$ respectively. Table(3) shows these results.

10 Discussion

There are some possible reasons for the good computed result. First the Galerkin method avoids the derivatives of density and Lamé parameters. For Lamé parameters, their derivatives are in the term $\nabla \cdot \vec{S}$ in the governing equation(2), the Galerkin method multiplies a vector trial function $\vec{X}$ on the both sides of eq(2). It is easy to prove that(F. Dahlen & Tromp, 1998)

$$\begin{aligned} \int_V \vec{X} \cdot (\nabla \cdot \vec{S}) dV &= \int_V \nabla \cdot (\vec{X} \cdot \vec{S}) dV - \int_V \nabla \vec{X} : \vec{S} dV \\ &= \int_S \hat{n} \cdot (\vec{X} \cdot \vec{S}) dS - \int_V \nabla \vec{X} : \vec{S} dV \\ &= \int_S (\vec{X} \cdot \hat{n} \cdot \vec{S}) dS - \int_V \nabla \vec{X} : \vec{S} dV. \end{aligned} \quad (70)$$

From above equivalent, the integral of $\nabla \cdot \vec{S}$ turns into the integral of $\vec{S}$, so the derivatives of Lamé parameters are eliminated. For density, its derivative is in the term $4\pi G \nabla \cdot (\rho_0 \vec{u})$ in Poisson's equation(8); the Galerkin method multiplies a trial function f on the both sides of eq(8). It is easy to prove that

$$\int_V f * 4\pi G \nabla \cdot (\rho_0 \vec{u}) dV = \int_S f * 4\pi G \rho_0 \hat{n} \cdot \vec{u} dS - \int_V 4\pi G \rho_0 \nabla f \cdot \vec{u} dV. \quad (71)$$

So the derivative of the density in the term $\nabla \cdot (\rho_0 \vec{u})$ is eliminated. The boundary conditions (10) (9) can substitute the surface integrals $\int_S (\vec{X} \cdot \hat{n} \cdot \vec{S}) dS$ and $\int_S f * 4\pi G \rho_0 \hat{n} \cdot \vec{u} dS$, which become the natural boundary conditions. Seyed-Mahmoud (1994) used the natural boundary conditions to deal with the rotational modes in fluid core.

These parameters were reconstructed by inversion of seismology data and normal modes(Dziewonski & Anderson, 1981). For PREM, the layered structure and the rude initial profiles of density and Lamé parameters were reconstructed from seismology data. Then these parameters were modified by the data of normal modes. Dziewonski and Anderson (1981) applied the Rayleigh's principle and the perturbation theory on normal modes, and his approach was just like the Galerkin method which replaced trial functions with the original displacement field $\vec{u}$, and modified boundary conditions into natural boundary conditions. This approach can invert the parameters as a whole, but may remove some details and smooth the profile curves. Our approach focuses on whole characteristics too, and can neglect some details of the derivatives, for instance, some small jumps in the density profile. On the other hand, some fine details may affect linear mo-

mentum approach. Rogister and Rochester (2004) used Clairaut coordinates, and had the similar advantage that the ODEs governing free oscillations of a rotating hydrostatic earth model contained no derivatives of material properties.

The second reason is that we don't adopt the ESD. The ESD transforms a first-order approximated ellipsoid into a sphere. For a point $\vec{P}(r_p, \theta_0, \phi_0)$ in some equipotential ellipsoidal surface, the corresponding point in ESD is $\vec{r}(r_0, \theta_0, \phi_0)$, which satisfies

$$r_p = r_0 - \frac{2}{3}\epsilon(r_0)P_2(\cos \theta). \quad (72)$$

This is actually a coordinate transformation; thus coordinates change and so do vectors, tensors and metric. Moreover, the governing equations should also change, and the original governing equations don't usually hold true in the new coordinates. Generally speaking, the governing equations in an original coordinates should be rewritten in Hamilton form $H(p, q)$, where p and q are generalized momentums and coordinates respectively. In the new coordinates, the new Hamilton form is $K(P, Q)$, where P and Q are generalized momentums and coordinates respectively. It is not strict to solve $H(p, q)$ with P, Q , and a rigorous way is to use $K(P, Q)$.

However, the two reasons are possible, and real reasons need more research.

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