A generalized interpolation material point method for shallow ice shelves. Part I: shallow shelf approximation and ice thickness evolution

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Abstract

We develop a generalized interpolation material point method (GIMPM) for the shallow shelf approximation (SSA) of ice flow. The GIMPM, which can be viewed as a particle version of the finite element method, is used here to solve the shallow shelf approximations of the momentum balance and ice thickness evolution equations. We introduce novel numerical schemes for particle splitting and integration at domain boundaries to accurately simulate the spreading of an ice shelf. The advantages of the proposed GIMPM-SSA framework include efficient advection of history or internal state variables without diffusion errors, automated tracking of the ice front and grounding line at sub-element scales, and a weak formulation based on well-established conventions of the finite element method with minimal additional computational cost. We demonstrate the numerical accuracy and stability of the GIMPM using 1-D and 2-D benchmark examples. We also compare the accuracy of the GIMPM with the standard material point method (sMPM) and a reweighted form of the sMPM. We find that the grid-crossing error is very severe for SSA simulations with the sMPM, whereas the GIMPM successfully mitigates this error. While the grid-crossing error can be reasonably reduced in the sMPM by implementing a simple material point reweighting scheme, this approach it not as accurate as the GIMPM. Thus, we illustrate that the GIMPM-SSA model is viable for the simulation of ice sheet-shelf evolution and enables boundary tracking and error-free advection of history or state variables, such as ice thickness or damage.

A generalized interpolation material point method for shallow ice shelves. Part I: shallow shelf approximation and ice thickness evolution

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Key Points

- Our material point method for shallow ice flow enables error-free advection of history variables, such as damage, and natural tracking of the ice front
- The method can be readily implemented into existing finite element software (here, Elmer/Ice), and is suitable for large-scale application
- The method is verified by reproducing analytical solutions for steady-state flow and front evolution, and by simulating an idealized marine ice sheet

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1 Abstract

2 We develop a generalized interpolation material point method (GIMPM) for the shallow shelf 3 approximation (SSA) of ice flow. The GIMPM, which can be viewed as a particle version of the 4 finite element method, is used here to solve the shallow shelf approximations of the momentum 5 balance and ice thickness evolution equations. We introduce novel numerical schemes for 6 particle splitting and integration at domain boundaries to accurately simulate the spreading of an 7 ice shelf. The advantages of the proposed GIMPM-SSA framework include efficient advection 8 of history or internal state variables without diffusion errors, automated tracking of the ice front 9 and grounding line at sub-element scales, and a weak formulation based on well-established 10 conventions of the finite element method with minimal additional computational cost. We 11 demonstrate the numerical accuracy and stability of the GIMPM using 1-D and 2-D benchmark 12 examples. We also compare the accuracy of the GIMPM with the standard material point method 13 (sMPM) and a reweighted form of the sMPM. We find that the grid-crossing error is very severe 14 for SSA simulations with the sMPM, whereas the GIMPM successfully mitigates this error. 15 While the grid-crossing error can be reasonably reduced in the sMPM by implementing a simple 16 material point reweighting scheme, this approach it not as accurate as the GIMPM. Thus, we 17 illustrate that the GIMPM-SSA model is viable for the simulation of ice sheet-shelf evolution 18 and enables boundary tracking and error-free advection of history or state variables, such as ice 19 thickness or damage.

20 Plain Language Summary

Ice shelves largely govern the evolution of the Antarctic ice sheet by buttressing grounded ice
flow into the ocean. This buttressing is sensitive to changes in ice thickness, upstream ice flow,

23 ice front position, contact with bedrock, fracture-induced weakening, and calving. The current 24 generation of ice flow models are particularly ill-suited for capturing those processes associated 25 with fracture and boundary tracking because they solve equations using exclusively mesh-based 26 methods. For large-deformation ice flow, these methods produce diffusion errors when advecting 27 history variables (e.g. fracture variables) and often rely on overly-approximate or cumbersome 28 schemes to track the ice front position. Here, we introduce a shallow shelf ice flow model based 29 on the material point method, a particle variation of the finite element method where boundaries 30 are naturally tracked and advection errors are avoided. A mesh is only needed when solving the 31 momentum equations, where the particles serve as moving integration points. As part of our 32 implementation, we introduce new schemes for particle splitting and integration at domain 33 boundaries, and we test several shape functions for mapping between the particles and the grid. 34 We demonstrate the accuracy and stability of the method with a series of benchmark examples.

35 1. Introduction

36 The fate of the entire Antarctic Ice Sheet is largely controlled by ice shelf dynamics. Over 80% 37 of the Antarctic Ice Sheet drains into the ocean through floating ice shelves (Pritchard et al., 38 2012), where nearly all mass loss of the ice sheet occurs due to roughly equal contributions from 39 basal melting and tabular calving (Depoorter et al., 2013; Rignot et al., 2013; Paolo et al., 2015). 40 However, ice shelves also provide resistance to the flow of upstream grounded ice into the 41 ocean, which primarily arises from contact with the walls of the bays in which they typically 42 reside and localized grounding at pinning points such ice rises and ice rumples. Any loss of this 43 resistance, or buttressing, results in an increased flux of grounded ice flow into the ocean, 44 thereby contributing to sea level rise (Dupont & Alley, 2005).

45 On decade to century timescales, the magnitude of ice shelf buttressing is controlled by 46 ice front evolution (i.e. fluctuations in contact with bay walls/pinning points), fracture or 47 thermomechanical weakening (e.g. Borstad et al., 2013; Sun et al., 2017), changes in ice 48 thickness such as thinning from ocean-driven basal melt (Pritchard et al., 2012; Cook et al., 49 2016), and response of the upstream grounded ice that feeds the shelf. Ideally, these four 50 processes should be represented in a fully coupled manner that accounts for the complex 51 feedbacks between them. For example, ice shelf thinning from basal melt has been associated 52 with increased fracture (Shepherd et al., 2003; Liu et al., 2015), and fracture determines the ice 53 front position through tabular calving. Calved icebergs can then alter local ocean properties and 54 circulation within the ice shelf cavity and wherever they drift, which in turn, may affect basal 55 melting rates (e.g. Robinson et al., 2010; Stern et al., 2015, 2016; Cougnon et al., 2017). Further, 56 a more general motivation for developing an integrated representation of these processes stems 57 from the lack of basal friction in ice shelves, which causes a highly nonlocal stress regime where 58 altering stress in one part of the shelf can affect stresses throughout the shelf (Cuffey & Paterson, 59 2013). Therefore, it is important that we develop advanced numerical models and methods to 60 enable realistic simulation of these processes controlling large-scale ice shelf evolution, and thus 61 gain a better understanding of Antarctic Ice Sheet dynamics and improve projections of sea level 62 rise.

Current large-scale ice flow models have difficulty in capturing the simultaneous
processes of front evolution, fracture, and thinning owing to the differences in the modeling
frameworks that are effective at describing each process separately. Because large-scale ice flow
is associated with extreme deformations, it is typically modeled within an Eulerian framework,
where velocity is calculated as the ice flows through a fixed region in space. Typically, Eulerian

68	models calculate flow velocity on a fixed mesh over time. However, some processes such as ice
69	mass transport or fracture (represented by damage), are not well-suited to the Eulerian approach
70	due to the artificial diffusion or dispersion inherent to Eulerian advection schemes. For example,
71	this artificial or numerical diffusion smears sharp edges and therefore compromises the accuracy
72	of damage advection and evolution. Furthermore, Eulerian approaches require separate schemes
73	to approximate ice front evolution, such as level-set (Bondzio et al., 2016) or volume of fluid
74	methods (Jouvet et al., 2008; Albrecht et al., 2011). In contrast, a Lagrangian approach, where
75	the position of mesh nodes update with flow, avoids numerical diffusion and naturally tracks ice
76	front evolution. However, Lagrangian or updated Lagrangian methods are only well-suited for
77	small deformation ice flow, such as within 2-D flow-band models for the propagation of
78	individual crevasses over short timescales (Duddu et al., 2013; Duddu & Waisman, 2013;
79	Jiménez et al., 2017). Use of Lagrangian methods to model entire ice shelf-sheet systems could
80	result in mesh degradation or tangling owing to the large deformations. Simple remeshing
81	schemes are not ideal because they also introduce artificial diffusion.
82	These limitations of traditional Eulerian and Lagrangian schemes may be overcome using
83	material point methods, which are formulated in a hybrid Eulerian-Lagrangian framework that
84	simultaneously allows large deformation flow, error-free advection of history variables, and
85	boundary tracking. The material point method (MPM) was originally introduced by Sulky et al.
86	(1994, 1995) for solid mechanics, as an adaptation of the particle-in-cell (PIC) (Evans et al.,
87	1957) and fluid-implicit-particle (FLIP) (Brackbill et al., 1988) methods. Henceforth, we will
88	refer to this original version as the standard MPM (sMPM). In the sMPM, the material domain is
89	discretized into a set of material points, or particles, that provide a Lagrangian description. Each
90	material point has a mass, volume, position, velocity, stress, and any history variables or other

material properties of the constitutive model. A background Eulerian mesh/grid is also defined,
which extends beyond the initial domain defined by the material points, and typically remains
fixed throughout the simulation. Grid cells containing material points constitute the "active"
mesh on which the equations of motion are solved in a similar manner as the finite element
method (FEM), but with material points serving as moving integration points. The mesh solution
is then used to update material point variables and positions.

97 Many variants of the sMPM have been formulated that retain the basic procedure, but 98 exhibit higher accuracy. These variants are largely motivated by the need to mitigate the well-99 known "cell-crossing error" in sMPM. This error arises from mapping between the material point 100 and the background grid using linear shape functions, which have discontinuous gradients 101 between grid cells so that abrupt transfers of stiffness occur as material points cross cell 102 boundaries or become unevenly distributed between neighboring cells. The first sMPM variant to 103 mitigate this error was the generalized interpolation material point method (GIMPM) developed 104 by Bardenhagen and Kober (2004), which convolves the linear nodal shape functions with 105 characteristic functions associated with each material point to result in continuous gradients 106 between grid cells (see Supplementary Figure S1). Other common variants of sMPM that modify 107 the shape functions to have continuous gradients include the convected particle domain 108 interpolation (CPDI) methods (Sadeghirad et al., 2011, 2013) and dual-domain material point 109 (DDMP) methods (Zhang et al., 2011). These variants of the sMPM have found diverse 110 applications for modeling impact, fracture, and granular media behavior; for a more detailed 111 literature review we refer the reader to Dunatunga and Kamrin (2015) and Coombs et al. (2020). 112 Material point methods have also been used to model certain components of the cryosphere,

including sea ice dynamics (Sulsky, 2007), snow (Stomakhin, 2013), and avalanches (Gaume etal., 2018).

115 Here, we develop an implementation of the GIMPM for simulating shallow-shelf ice 116 flow. To our knowledge, this is the first ever implementation of MPMs for shallow ice flow. Our 117 GIMPM formulation solves the momentum and mass balance equations for ice flow and 118 thickness evolution, and enables natural tracking of the ice front and grounding line. In Part II 119 (Huth et al., 2020), we incorporate an anisotropic nonlocal creep damage model (Pralong & 120 Funk, 2005; Duddu & Waisman, 2012) for fracture propagation. This paper solely focuses on the 121 description and verification of the GIMPM in simulating shallow ice flow, ice thickness 122 evolution, and ice-ocean boundary treatment. We solve for ice flow velocities using the Shallow 123 Shelf Approximation, or Shelfy-Stream Approximation (SSA), a 2-D vertically-integrated flow 124 model that is appropriate for large-scale ice shelf and ice stream flow, where horizontal 125 velocities can be considered vertically invariant (MacAyeal, 1989). The SSA constitutes the only 126 equations solved using the background Eulerian mesh/grid, while history variables such as ice 127 thickness and damage are updated explicitly and efficiently on each material point directly. The 128 primary advantage of our GIMPM formulation is that advection of all variables only involves 129 updating the material point positions, thus our Lagrangian advection scheme is computationally 130 inexpensive and avoids the artificial diffusion errors associated with Eulerian schemes. 131 Furthermore, the positions of the material points allow us to establish and track the ice front and 132 grounding line at sub-grid scales. We implemented our model within the open-source finite 133 element ice flow model Elmer/Ice (Gagliardini et al., 2013), by modifying the Elmer SSA solver 134 to implement GIMPM integration schemes and by introducing several modules for tracking and 135 evolving the set of material points.

136 In the following sections, we will detail the derivation of our method and quantify its 137 accuracy and stability for 1-D and 2-D ice flow simulations, including front advection. We will 138 illustrate that the GIMPM-SSA formulation is effective for: advecting history or internal state 139 variables without diffusion, maintaining the steady-state grounding lines of marine ice sheets, 140 and tracking ice front evolution on century timescales. To ensure numerical accuracy, we 141 formulate novel schemes for enforcing the conditions at the ice front and outflow boundaries, as 142 well as for determining ice thickness at material points due to particle splitting. This paper is 143 organized as follows: in Section 2 we review the SSA equations and their numerical 144 discretization using the FEM and the GIMPM; in Section 3 we provide the details of our 145 numerical implementation related to grid and particle variable updates; in Section 4 we present 146 schemes for boundary treatment and error control; in Section 5 we provide examples that test the 147 accuracy and numerical performance of the GIMPM-SSA formulation; in Section 6 we provide a 148 brief discussion on the pros and cons of the GIMPM, and finally, in Section 7 we make a few 149 concluding remarks.

150 2. Governing Equations

In this section, we will briefly describe the governing equations of ice flow based on the Shallow Shelf Approximation (SSA), followed by the numerical discretization using the finite element and generalized material point methods. We will use indicial notation for vectors and tensors to describe the strong and weak forms of the governing equations and use matrix notation to present the corresponding discretized linear system. We will use Einstein's summation convention only for spatial indices, where repeated indices imply summation. For brevity, we occasionally avoid indicial notation, and use bold face letters to denote vectors, tensors and matrices.

158 2.1. Shallow Shelf Approximation

159 Ice shelves and ice streams can be modeled under the assumption of plug flow, where horizontal 160 velocities and strain rates are constant over depth. Consequently, the incompressible Stokes 161 equations are modified to exclude vertical shear and vertically integrated to derive the SSA that 162 describes the horizontal force or momentum balance as

$$\frac{\partial T_{ij}}{\partial x_i} + (\tau_{\rm b})_i = \rho g H \frac{\partial s}{\partial x_i}, \qquad (1)$$

163 where spatial indices $i, j \in \{1,2\}$ correspond to the horizontal plane, $\mathbf{x} = x_i \hat{e}_i$ denotes the in-164 plane spatial coordinates, \hat{e}_i are the basis vectors for the Cartesian coordinate system, ρ is the ice 165 density, g is the acceleration due to gravity, H is the ice thickness, s is the top surface elevation, 166 $\bar{\eta}$ is the depth-averaged effective viscosity, and τ_b is the basal traction described by a friction 167 law. For simplicity, here we assume the friction law as

$$(\tau_{\rm b})_i = \hat{\beta} v_i \,, \tag{2}$$

168 where $\hat{\beta}$ is a friction parameter and v_i are the horizontal velocities for which the SSA is solved. 169 Note that the above equation can used to specify several friction laws currently available in 170 Elmer/Ice (Gagliardini et al., 2013) by defining $\hat{\beta}$ to be dependent on velocity, and in some 171 cases, pressure. In (1), the two-dimensional vertically-integrated stress tensor **T** is defined as 172 (Bueler and Brown, 2009; Morland, 1987):

$$\mathbf{T} = 2\bar{\eta}H \begin{bmatrix} 2\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{\partial v_1}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2} \end{bmatrix},$$
(3)

173 which may alternatively be expressed in terms of strain rate $\dot{\varepsilon}_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ as

$$\mathbf{T} = 2\bar{\eta}H \begin{bmatrix} 2\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} & \dot{\varepsilon}_{12} \\ \dot{\varepsilon}_{21} & \dot{\varepsilon}_{11} + 2\dot{\varepsilon}_{22} \end{bmatrix}.$$
 (4)

174 The constitutive relation for ice flow relates deviatoric stress, σ_{ij}^D , to strain rate as

$$\sigma_{ij}^D = 2\eta \dot{\varepsilon}_{ij} , \qquad (5)$$

175 where the effective viscosity, η , follows the Norton-Hoff flow law (Glen, 1955; Nye, 1957):

$$\eta = \frac{1}{2} B \dot{\varepsilon}_{\rm e}^{(1-n)/n} \,. \tag{6}$$

176 In the above equation, *n* is the flow law exponent, $\dot{\varepsilon}_{e}$ is the scalar second invariant or effective

177 strain rate $\dot{\varepsilon}_{e} = \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ji}/2}$, and *B* is a flow rate factor dependent on temperature and ice fabric.

178 The depth-averaged effective viscosity used in the SSA takes the same form as (6), but uses only

179 in-plane strain components to determine $\dot{\varepsilon}_e$ and a depth-averaged rate factor, $\bar{B} = \frac{1}{H} \int_b^s B \, dz$,

180 where b is the vertical coordinate of the ice basal surface. We take z as positive in the upward

181 direction, where z = 0 corresponds to sea level.

A boundary condition at the ice front is set according to the seawater pressure at the ice
terminus opposing ice flow

$$\sigma_{ij}\hat{n}_j = \begin{cases} \rho_w gz \ \hat{n}_i & \text{for } z < 0\\ 0 & \text{for } z \ge 0 \end{cases}$$
(7)

184 where $\sigma_{ij} = \sigma_{ij}^D - p\delta_{ij}$ is the Cauchy stress, *p* is the hydrostatic pressure, δ_{ij} is the Kronecker's 185 delta, and \hat{n} is the unit (outward) normal to the ice front. Equation (7) is depth-integrated for 186 implementation into the SSA as (Morland and Zainuddin, 1987):

$$\int_{b}^{s} \sigma_{ij} \hat{n}_j \, dz = -\frac{1}{2} \rho_{\rm w} g b^2 \hat{n}_i \,. \tag{8}$$

187 Appropriate Dirichlet conditions for velocity are set at all other boundaries.

The procedure for deriving the weak form of the SSA and discretization using the sMPM or the GIMPM is similar to that using the FEM, so we briefly review the procedure using the FEM first for clarity. Full details of this procedure can be found in the literature (e.g. Weis, 2001; Greve and Blatter, 2009; Lipscomb et al., 2019). The weak form of the SSA is derived using the Bubnov-Galerkin method of weighted residuals by multiplying (1) by an arbitrary smooth test function w(x) and integrating over the domain. After applying the divergence theorem and introducing the boundary conditions, we obtain:

$$\int_{\Omega} T_{ij} \frac{\partial w_i}{\partial x_j} d\Omega + \int_{\Omega} w_i \rho g H \frac{\partial s}{\partial x_i} d\Omega - \int_{\Omega} w_i (\tau_{\rm b})_i d\Omega - \int_{\Gamma_k} w_i T_{ij} \hat{n}_j d\Gamma - \int_{\Gamma_{\rm cf}} \frac{1}{2} w_i (\rho g H^2 - \rho_w g b^2) \hat{n}_i d\Gamma = 0 , \qquad (9)$$

196 where Ω and Γ_{cf} represent the area of the ice domain and calving front boundary, respectively, 197 and Γ_k represents the boundaries where Dirichlet conditions for velocity are applied. 198 All variables in (9), including the test function w, are represented continuously on the 199 mesh/grid using nodal shape functions. For example, velocity at a spatial location x and time t is 200 defined as

$$\boldsymbol{\nu}(\boldsymbol{x},t) = \sum_{I=1}^{N_n} \boldsymbol{\nu}_I(t) \boldsymbol{\phi}_I(\boldsymbol{x}) , \qquad (10)$$

where the nodes of the mesh are x_I , $I = 1, ..., N_n$, $\phi_I(x)$ is the nodal shape function associated with node *I*, and N_n is the number of nodes of the chosen finite element (here we use 4-noded quadrilateral elements, so $N_n = 4$). Substituting the continuous representations for v and w from (10) into (9), and noting that the test functions are arbitrary, a linear system can be assembled and solved for horizontal velocities \boldsymbol{v} . The element tangent stiffness matrix **K** and residual force vector \boldsymbol{f} can be split into components and expressed as follows:

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{bmatrix}, \tag{11}$$

207 where v_1 and v_2 are the vectors of nodal velocity components, the vectors f_1 and f_2 contain the

208 gravitational forcing, and the element submatrices of the tangent matrix are given by

$$K_{11IJ} := \int_{\Omega^{E}} 2\overline{\eta} H \left(2 \frac{\partial \phi_{I}(x)}{\partial x_{1}} \frac{\partial \phi_{J}(x)}{\partial x_{1}} + \frac{1}{2} \frac{\partial \phi_{I}(x)}{\partial x_{2}} \frac{\partial \phi_{J}(x)}{\partial x_{2}} \right) d\Omega + \int_{\Omega^{E}} \hat{\beta} \phi_{I}(x) \phi_{J}(x) d\Omega ,$$

$$K_{22IJ} := \int_{\Omega^{E}} 2\overline{\eta} H \left(2 \frac{\partial \phi_{I}(x)}{\partial x_{2}} \frac{\partial \phi_{J}(x)}{\partial x_{2}} + \frac{1}{2} \frac{\partial \phi_{I}(x)}{\partial x_{1}} \frac{\partial \phi_{J}(x)}{\partial x_{1}} \right) d\Omega + \int_{\Omega^{E}} \hat{\beta} \phi_{I}(x) \phi_{J}(x) d\Omega , \qquad (12)$$

$$K_{12IJ} := \int_{\Omega^{E}} 2\overline{\eta} H\left(\frac{\partial \phi_{I}(\boldsymbol{x})}{\partial x_{2}} \frac{\partial \phi_{J}(\boldsymbol{x})}{\partial x_{1}} + \frac{1}{2} \frac{\partial \phi_{I}(\boldsymbol{x})}{\partial x_{1}} \frac{\partial \phi_{J}(\boldsymbol{x})}{\partial x_{2}}\right) d\Omega,$$

$$K_{21IJ} := \int_{\Omega^{E}} 2\overline{\eta} H\left(\frac{\partial \phi_{I}(\boldsymbol{x})}{\partial x_{1}} \frac{\partial \phi_{J}(\boldsymbol{x})}{\partial x_{2}} + \frac{1}{2} \frac{\partial \phi_{I}(\boldsymbol{x})}{\partial x_{2}} \frac{\partial \phi_{J}(\boldsymbol{x})}{\partial x_{1}}\right) d\Omega,$$

where *I* and *J* denote the nodal indices of the element and Ω^E is the domain of the element, as indicated by the superscript, '*E*'. The right-hand size of (11) is given as

$$f_{1I} \coloneqq \int_{\Omega^{E}} \phi_{I}(\mathbf{x}) \rho g H \frac{\partial s}{\partial x_{1}} d\Omega ,$$

$$f_{2I} \coloneqq \int_{\Omega^{E}} \phi_{I}(\mathbf{x}) \rho g H \frac{\partial s}{\partial x_{2}} d\Omega .$$
(13)

211 At boundary elements for the calving front,

$$f_{iI} = \int_{\Gamma_{cf}^{E}} \frac{1}{2} \phi_{I}(\mathbf{x}) (\rho g H^{2} - \rho_{w} g b^{2}) \hat{n}_{i} d\Gamma = 0.$$
 (14)

Following standard finite element procedure, the integrals in (12)-(14) are evaluated using

213 Gaussian quadrature. Variables H, ρ , b, B, and ∇v must be mapped to the Gauss points from the

nodes, where B and ∇v are used to calculate the depth-averaged effective viscosity $\overline{\eta}$.

215 2.3. Weak form and discretization using the GIMPM

The original formulation of the GIMPM (Bardenhagen and Kober, 2004) was derived using the Petrov-Galerkin method, wherein the test function \boldsymbol{w} and the trial function \boldsymbol{v} belong to different function spaces. In the GIMPM, each material point or particle is assigned with a particle characteristic function, χ_p , that must satisfy partition of unity in the reference or undeformed configuration

$$\sum_{p} \chi_p(\mathbf{x}, t = 0) = 1 \ \forall \ \mathbf{x} ,$$
(15)

Note that partition of unity is also a requirement for the element shape functions. We choose χ_P to be the commonly used 'hat' function with value one within the material point domain Ω_p and zero outside as

$$\chi_p(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \Omega_p ,\\ 0, & \text{otherwise.} \end{cases}$$
(16)

Note that if $\chi_p(\mathbf{x})$ is chosen as a Dirac delta function, then the sMPM is retrieved. We assign a rectangular domain to each material point over which χ_p is defined, which we will refer to as the GIMPM domain of a material point. To satisfy (15) and (16), the initial material domain must be discretized into material points so that no gaps or overlapping with neighboring GIMPM domains occurs. The GIMPM requires using a regular background grid of rectangular elements. We perform the initial discretization by evenly subdividing the domain Ω into GIMPM domains Ω_p by introducing a specified number of material points for each active background grid cell. In this formulation, we will update the lengths of the GIMPM domains due to deformation (see Section 3), with the goal of maintaining partition of unity over time precisely, to the extent possible. The area associated with a material point, A_p , is then defined as

$$A_p = \int_{\Omega_p} \chi_p(\mathbf{x}) d\Omega \,, \tag{17}$$

where Ω_p is the support area of the particle characteristic function and Ω is the area of the overall ice domain. Most literature on material point methods generalizes the formulation to 3-D by using volume (V_p) rather than area (A_p), but we use A_p here because the SSA is inherently 2-D. The values of material point variables may be initialized by integrating properties of the continuum body against the particle characteristic functions. For example, the initial value of material point property, h_p^0 , may be expressed as an area-averaged form of the initial continuum field $h^0(x)$ as

$$h_p^0 = \frac{1}{A_p^0} \int_{\Omega^0} h^0(x) \chi_p(x) d\Omega^0, \qquad (18)$$

where superscript '0' indicates the initial time step. The validity of Equation (18) is a
consequence of the partition of unity. Consistently, at any future time step *m*, the particle
characteristic functions may be used as a basis to represent the material property throughout the
computational domain

$$h^{m}(\mathbf{x}) = \sum_{p=1}^{N_{p}} h_{p}^{m} \chi_{p}(\mathbf{x}),$$
(19)

where N_p is the number of material points in the domain. We use (19) to formulate continuous representations of vertically-integrated stress (T_{ij}) , gravitational driving force $(\rho g H \frac{\partial s}{\partial x_i})$, and basal traction $(\tau_b)_i$, which we substitute into (9) so that the integrals over the area of the entire 248 ice domain become sums of integrals over material points. Further substituting in the Co

249 continuous representations for w using (10), and utilizing that the test functions are arbitrary, we

250 obtain the weak form as

$$\sum_{p=1}^{N_p} (T_{ij})_p \frac{\partial S_{Ip}}{\partial x_j} A_p + \sum_{p=1}^{N_p} S_{Ip} \rho_p g H_p \left(\frac{\partial s}{\partial x_i}\right)_p A_p - \sum_{p=1}^{N_p} S_{Ip} [(\tau_b)_i]_p A_p - \int_{\Gamma_k} \phi_I T_{ij} \hat{n}_j d\Gamma - \int_{\Gamma_{cf}} \frac{1}{2} \phi_I (\rho g H^2 - \rho_w g b^2) \hat{n}_j d\Gamma = 0,$$
(20)

where S_{Ip} are the GIMPM weighting functions corresponding to node *I* (of the fixed background Eulerian mesh) evaluated at a material point location as

$$S_{Ip} = S_I(\boldsymbol{x}_p) = \frac{1}{A_p} \int_{\Omega_p} \chi_p(\boldsymbol{x}) \phi_I(\boldsymbol{x}) \, d\Omega \,, \qquad (21)$$

and with the gradient defined as

$$\frac{\partial S_{Ip}}{\partial x_i} = \frac{1}{A_p} \int_{\Omega_p} \chi_p(\mathbf{x}) \frac{\partial \phi_I(\mathbf{x})}{\partial x_i} d\Omega \,. \tag{22}$$

254 We take the nodal shape functions ϕ_I to be element-wise linear Lagrange interpolants.

255 These shape functions at the two nodes of the linear element can be defined as

$$\phi_{1} = \phi_{1}(\mathbf{x}) = \frac{1-\xi}{2},$$

$$\phi_{2} = \phi_{2}(\mathbf{x}) = \frac{1+\xi}{2},$$
(23)

where ξ is the local (or isoparametric) coordinate (between -1 and +1) of any material or integration point within the parent element. For illustration, we plot the both the linear and GIMPM shape functions in 1-D in Supplementary Figure S1. The GIMPM shape function exceeds the boundaries of a single element and is C1 continuous by smoothing the discontinuous gradient observed in the linear shape functions, which mitigates cell-crossing error. Material points influence all elements that their GIMPM domains overlap. Note that in practice, we evaluate the convolutions in (21) and (22) as the overlap of χ_p and the linear shape functions within each element rather than for each node individually, which allows us to assemble stiffness matrices is a similar manner to the FEM (see, e.g., Charlton et al., 2017). Additionally, we note that replacing the linear shape functions with higher order interpolants would be problematic as the latter are not positive throughout the element. Further substituting into (20) the C₀ continuous representations for \boldsymbol{v} using (10) yields element sub-matrices that are computed by summing over material points:

$$K_{11IJ} := \sum_{p=1}^{n_p} A_p 2\bar{\eta}_p H_p \left(2 \frac{\partial \phi_{Ip}}{\partial x_1} \frac{\partial S_{Jp}}{\partial x_1} + \frac{1}{2} \frac{\partial \phi_{Ip}}{\partial x_2} \frac{\partial S_{Jp}}{\partial x_2} \right) + \sum_{p=1}^{n_p} A_p \hat{\beta}_p \phi_{Ip} S_{Jp},$$

$$K_{22IJ} := \sum_{p=1}^{n_p} A_p 2\bar{\eta}_p H_p \left(2 \frac{\partial \phi_{Ip}}{\partial x_2} \frac{\partial S_{Jp}}{\partial x_2} + \frac{1}{2} \frac{\partial \phi_{Ip}}{\partial x_1} \frac{\partial S_{Jp}}{\partial x_1} \right) + \sum_{p=1}^{n_p} A_p \hat{\beta}_p \phi_{Ip} S_{Jp},$$

$$K_{12IJ} := \sum_{p=1}^{n_p} A_p 2\bar{\eta}_p H_p \left(\frac{\partial \phi_{Ip}}{\partial x_2} \frac{\partial S_{Jp}}{\partial x_1} + \frac{1}{2} \frac{\partial \phi_{Ip}}{\partial x_1} \frac{\partial S_{Jp}}{\partial x_2} \right),$$

$$K_{21IJ} := \sum_{p=1}^{n_p} A_p 2\bar{\eta}_p H_p \left(\frac{\partial \phi_{Ip}}{\partial x_1} \frac{\partial S_{Jp}}{\partial x_2} + \frac{1}{2} \frac{\partial \phi_{Ip}}{\partial x_2} \frac{\partial S_{Jp}}{\partial x_1} \right),$$
(24)

where n_p is the number of material points in the element and we use the same shorthand for the linear shape functions evaluated at a material point location, $\phi_{Ip} = \phi_I(\mathbf{x}_p)$, as defined for the

271 GIMPM functions in (21). Similarly, the components of the body force vector are computed as

$$\boldsymbol{f}_{1} \coloneqq \sum_{p=1}^{n_{p}} S_{Ip} \rho_{p} g H_{p} \left(\frac{\partial s}{\partial x_{1}}\right)_{p} A_{p} ,$$

$$\boldsymbol{f}_{2} \coloneqq \sum_{p=1}^{n_{p}} S_{Ip} \rho_{p} g H_{p} \left(\frac{\partial s}{\partial x_{2}}\right)_{p} A_{p} .$$
(25)

272 Comparing Equations (24) and (25) with (12) and (13), we can notice the subtle differences

between the GIMPM and the standard FEM. We address the numerical treatment of the ice frontboundary for GIMPM/sMPM in Section 4.1.

275 By replacing
$$S_{Jp}$$
 and $\frac{\partial S_{Jp}}{\partial x_i}$ with ϕ_{Jp} and $\frac{\partial \phi_{Jp}}{\partial x_i}$, respectively, in (24) – (25) we can obtain

the sMPM. However, errors in the sMPM can accumulate if the sum of material point area (weights) within an element significantly varies from the element area or if the distribution of material points within the element becomes irregular. In Section 5, we show how increasing the number of material points can mitigate this error to a certain extent, but ultimately, an alternative material point weighting is required to more evenly distribute material point weights between elements and maintain accuracy (Gonzalez Acosta et al., 2017). The new weight, W_p , is a function of both the element area, A_E , and material point areas, A_p , as given by

$$W_p = \frac{A_{\rm E}}{\sum_{p=1}^{n_p} A_p}.$$
(26)

Thus, in the reweighted sMPM, W_p replaces A_p as the integration weights in (24) and (25). This reweighting can also be used with the GIMPM, where A_p becomes the area of overlap between a GIMPM domain and the element. However, reweighted GIMPM is mostly unnecessary because material point weight is already smoothly distributed between neighboring elements unless severe overlaps or gaps develop between neighboring GIMPM domains. We largely avoid these errors in our simulation studies, and therefore do not apply the reweighting to our GIMPM simulations here.

290 **3.** Numerical Implementation

291

292 cell as described in Section 2.2. The unknown variables, namely ice flow velocity and ice 293 thickness, are defined directly on the material points; whereas, the external parameters such as 294 bedrock elevation, the basal friction parameter, and accumulation/ablation rates are defined on 295 the background fixed mesh. For simplicity, each simulation presented here uses a constant flow 296 rate factor B and density ρ for all material points. However, these quantities can be treated as 297 spatially-varying and history-dependent. 298 In this section, we detail the numerical procedure for a typical computational cycle, 299 according to the simplified representation given in Figure 1. The cycle begins with a series of 300 parameter mappings between the material points and the grid (Figure 1a and 1b), which are 301 needed in preparation for solving the SSA. The mappings are used to initialize the SSA grid

At time t = 0, the ice domain is discretized into a specified number of material points per grid





Figure 1. MPM-SSA numerical procedure. Grid processes (Eulerian) are highlighted in red and material point processes (Lagrangian) are highlighted in blue.

303 (25). The SSA is solved using an iterative routine (Figure 1c), where material point viscosity is 304 updated alongside grid velocity until convergence. Subsequently, the grid solution is used to 305 update material point positions, velocities, and geometric parameters (Figure 1d). Finally, 306 material point history variables are updated (Figure 1e), which only includes ice thickness in this 307 study. To improve readability, we will use matrix notation for vectors and tensors to avoid 308 showing spatial indices and show only node and particles indices to explain the mapping 309 between nodes and particles.

310 3.1. SSA Initialization: Grid parameters

To allow a solution of the SSA, the velocity field and thickness are initialized on the grid by mapping from the material points (Figure 1a), where the thickness on the grid is subsequently converted to surface elevation. The gradients of surface elevation and velocity are mapped to material points for the SSA matrix assembly. The initialized grid velocity is further required as part of the update routine for material point velocity (Section 3.4). The velocity mapping from particles to nodes is performed using a formula that enforces momentum conservation:

- -

$$\boldsymbol{\nu}_{I} = \frac{\sum_{p}^{N_{p}} m_{p} \boldsymbol{\nu}_{p} S_{Ip}}{\sum_{p}^{N_{p}} m_{p} S_{Ip}},$$
(27)

317 where m_p is the material point mass

$$m_p = \rho_p H_p A_p \,. \tag{28}$$

318 Ice thickness is mapped to the grid from material points as

$$H_{I} = \frac{\sum_{p}^{N_{p}} H_{p} S_{Ip} A_{p}}{\sum_{p}^{N_{p}} S_{Ip} A_{p}},$$
(29)

319 where the denominator is necessary to normalize the interpolation. After each mapping, nodal 320 values of velocity or thickness at Dirichlet boundaries are overwritten with the values specified 321 by the essential boundary condition. Nodal surface elevations are calculated from the nodal ice 322 thicknesses as

$$s_I = b_I + H_I \,, \tag{30}$$

where the nodal elevation of the ice base, b_I , is computed as the maximum value of the bedrock elevation (z_{bed}) or the ice base elevation according to hydrostatic equilibrium as

$$b_I = \max\left\{ (z_{\text{bed}})_I, z_{\text{sea}} - H_I\left(\frac{\rho}{\rho_w}\right) \right\},\tag{31}$$

325 and $z_{sea} = 0$ is the sea level.

326 3.2. SSA Initialization: Material point parameters

The second half of the SSA initialization procedure is focused on updating material point variables (Figure 1b). Surface height and velocity gradients are determined at any material point *p* by mapping from the nodes. The friction parameter ($\hat{\beta}_p$), bedrock elevation (z_{bed_p}), and rate factor (B_p) must also be defined at the material point level, which may require mapping from the nodes as well. Any scalar grid property, h_I , may be interpolated to the material points as

$$h_p = \sum_{I}^{N_n} h_I S_{Ip} \,. \tag{32}$$

332 Similarly, for gradients, the mapping is

$$\nabla h_p = \sum_{I}^{N_n} h_I \nabla S_{Ip} \,. \tag{33}$$

333 Lastly, material points are marked as grounded or floating. Defining grounding status at material 334 points (or at Gauss points in the FEM) rather than at nodes during the SSA solution has been 335 shown to provide a more accurate estimate of grounding line dynamics during the SSA solution 336 (Seroussi et al., 2014). However, to be consistent with Elmer/Ice conventions, we also define 337 grounding status at the nodal level as part of the procedure to define the sub-element scale grounding line. If node I has $b_I = (z_{bed})_I$, it is marked as grounded; otherwise, it is floating. If a 338 339 material point belongs to an element whose surrounding elements have a mix of grounded and 340 floating nodes, then that it is clear that material point is near the grounding line, and its 341 grounding status is determined using the same procedure used for the nodes. Otherwise, it 342 inherits the grounding status of its surrounding nodes.

343 3.3. SSA Solution

The SSA is solved implicitly using an "iteration on viscosity" scheme where we update material point viscosity, $\bar{\eta}_p$, each iteration until convergence (MacAyeal, 1989) (Figure 1c). This is done by mapping the gradients of nodal velocity solution from the previous iteration to material points using (33), which are converted to strain-rates to calculate $\bar{\eta}_p$ using (6). We achieve quick convergence of the SSA solution using the Biconjugate Gradient Stabilized (BiCGSTAB) method, Incomplete LU preconditioning, and a combination of Picard and Newton iterations.

350 3.4. Material Point Updates

351 Upon completion of the SSA, grid velocities are used to update material point velocities,

352 position, and geometric properties (Figure 1d).

Velocities and position: To update velocity and position, material point methods typically adopt
 the approach of the FLuid Implicit Particle (FLIP) method. For velocity, this update is given as

$$\boldsymbol{v}_{p}^{m+1} = \boldsymbol{v}_{p}^{m} + \Delta t \sum_{l}^{N_{n}} \boldsymbol{a}_{l}^{m} S_{lp} = \boldsymbol{v}_{p}^{m} + \sum_{l}^{N_{n}} (\boldsymbol{v}_{l}^{m+1} - \boldsymbol{v}_{l}^{m}) S_{lp} , \qquad (34)$$

where a_I^m is the acceleration at time step *m* at node *I*, and \boldsymbol{v}_I^m is the nodal velocity previously interpolated to the grid from the material points before the SSA solution in Equation (27). This material point position update is

$$\boldsymbol{x}_{p}^{m+1} = \boldsymbol{x}_{p}^{m} + \Delta t \sum_{I}^{N_{n}} \boldsymbol{v}_{I}^{m+1} S_{Ip} \,.$$
(35)

358 In practice, the FLIP update scheme can introduce noise that results from the mismatch between 359 the number of material points and grid nodes, so our code also includes the update scheme 360 XPIC(k), an algorithm that can remove FLIP noise using a set of k additional projections 361 (Hammerguist and Nairn, 2017). Lower orders of k may introduce undesired damping, while 362 higher orders of k are computationally expensive. Note that the additional projections required 363 for XPIC(k) can accumulate a small amount of error in conjunction with our boundary treatment 364 at the ice front (Section 4.1), but this error can be avoided by always using FLIP updates within 365 k elements of the ice front. While the simulations in this paper are relatively insensitive to the 366 update scheme chosen, Nairn et al. (2017) demonstrated that XPIC(5) yields sharp and stable 367 crack propagation in damage simulations.

368 *Geometric properties*: All updates to material point geometric properties, which include area and 369 the lengths defining the GIMPM domain, depend on the deformation gradient, a fundamental

370 kinematic quantity that characterizes the deformation at a material point based on its current

371 (deformed) and reference (undeformed) spatial coordinates. The material point deformation

372 gradients (\mathbf{F}_{p}) are tracked over time, and are updated as

$$\boldsymbol{F}_{p}^{m+1} = \left(\boldsymbol{I} + \Delta t \nabla \boldsymbol{v}_{p}^{m+1}\right) \boldsymbol{F}_{p}^{m} .$$
(36)

373 where I is the second-order identity tensor. In the sMPM, the determinant of F_p is used to update 374 the material point area as

$$A_p^{m+1} = \det(F_p^{m+1})A_p^0.$$
(37)

In the GIMPM, material point area is calculated as the product of the lengths defining therectangular GIMPM domain.

377 Our implementation currently includes two schemes to update GIMPM domain lengths. 378 The lengths should be updated carefully in order to minimize overlap or separation of GIMPM 379 domains over time, and thus maintain partition of unity as precisely as possible throughout the 380 domain. The first scheme updates the lengths of a GIMPM domain such that the resulting rectangular domain approximates the quadrilateral domain that would be obtained if the position 381 382 of each corner of the GIMPM domain was updated individually (Coombs et al., 2020). In 383 practice, this "corner-tracking" update scheme may be simplified to tracking the midpoints \widehat{x}_p of 384 the GIMPM domain edges as

$$\widehat{\boldsymbol{x}}_{p} = \widehat{\boldsymbol{x}}_{p}^{m} + \Delta t \sum_{l} \boldsymbol{\nu}_{l}^{m+1} \boldsymbol{\phi}_{l}(\widehat{\boldsymbol{x}}_{p}^{m}) \,.$$
(38)

385 The GIMPM domain lengths can be obtained using the maximum and minimum extents of \hat{x}_p as

$$\left(l_p^{m+1}\right)_i = \frac{1}{2} \left[\max\left(\widehat{x}_p\right)_i - \min\left(\widehat{x}_p\right)_i \right], \tag{39}$$

followed by a correction that guarantees proper volume (area in 2-D) as

$$(l_p^{m+1})_i = (l_p^{m+1})_i \left[\frac{\det (F_{kl}^{m+1}) \Pi_{j=1}^{n_D} (l_p^0)_j}{\Pi_{j=1}^{n_D} (l_p^{m+1})_j} \right]^{1/n_D}.$$

$$(40)$$

where n_D is the dimension of the problem ($n_D = 2$ in our case). More detailed derivations of the above scheme can be found in Coombs and others (2020). This corner-tracking scheme performs well in minimizing overlap or separation of GIMPM domains over time in any flow regime, but cannot be used at outflow boundaries where a GIMPM domain may only partially overlap the active background grid, assuming velocities beyond the active grid are unknown (see Section 4.1). For these material points, we instead use the second update scheme, given by

$$\left(l_p^{m+1}\right)_i = \left(l_p^0\right)_i U_{ii}^{m+1} \text{ (no implied sum on i)},$$
(41)

where $(l_p^0)_i$ are the original domain lengths and $U_{ij} = \sqrt{F_{ki}^T F_{kj}}$ is the symmetric material 393 394 stretch tensor, so that the deformation gradient rotated into the original Cartesian reference frame 395 (Charlton et al., 2017). Although the "stretch-tensor" update scheme can be used instead of the 396 "corner-tracking" scheme in the entire domain, we caution that it is less capable of minimizing 397 overlap or separation of GIMPM domains under large shearing deformation. Because the 398 "stretch-tensor" scheme is sufficiently accurate under stretching and rotation and is 399 computationally more efficient than the "corner-tracking" scheme, the former scheme is suitable 400 for simulations without large shearing deformations. Here, we only use the stretch-tensor scheme 401 for the 1-D flow-band simulations, noting that the corner-tracking scheme gives identical results. 402 In all the 2-D simulations, we employ the corner-tracking scheme.

403 3.5. *History Variable Updates*

The computational cycle finishes by updating the history variables on the material points (Figure 1e). Here, we only consider ice thickness (H_p) , which is updated explicitly according to the Lagrangian description of surface mass conservation for a column of ice at time step m + 1 as

$$H_p^{m+1} = H_p^m + (\dot{b}_p^m - \nabla \cdot \boldsymbol{v}_p^m H_p^m) \Delta t , \qquad (42)$$

407 where \dot{b}_p^m (m a-1) is the sum of the basal and surface accumulation rates. We add damage as a 408 history variable in Part II (Huth et al., 2020).

409 4. Boundary treatment and splitting

410 Boundary conditions in MPMs may be applied at the edges of the active computational grid as in 411 the FEM. However, special treatment is required at inflow boundaries to properly introduce new 412 material points to the domain, and at outflow boundaries where material points must be 413 eliminated or GIMPM domains may partially overlap the boundary. Further treatment is also 414 needed at the moving ice front boundary to avoid integration errors, as this boundary may not 415 align with element edges. We detail our boundary treatment in this section. In addition, we detail 416 our material point splitting scheme, which mitigates additional integration errors that may arise 417 under tension, where the resolution of material points per grid cell decreases over time as the 418 area of the material points grows.

419 4.1. Boundary treatment

420 *Inflow boundaries:* Since material points at inflow boundaries advect downstream, a scheme is
421 needed to ensure that they are replaced by inflow of new material points. For the simple



Figure 2. MPM-SSA boundary treatment. (a) At inflow boundaries (dotted red), an additional domain (shaded blue) is specified upstream and seeded with extra material points. Velocities are specified throughout the additional domain so that the extra material points advect into the primary domain at the correct velocity. (b) A material point with a GIMPM domain overlapping an outflow boundary (red) is split into sub-particles during grid-to-material point mappings. The sub-particles separately receive the interpolation, which is subsequently consolidated back to the original material point. (c) At the ice front, grid cells partially full with material points (yellow) are integrated using the FEM, where the boundary condition is assigned at the element edges (dashed red) that mark the transition between active and inactive (grey-striped) grid cells.

422 simulations in this paper, we incorporate inflow boundaries by seeding additional material points 423 on a domain that extends beyond the boundaries. Velocities on the extra domain and inflow 424 boundary are set so that the additional material points flow smoothly into the primary domain at 425 the velocity specified by the boundary condition. This scheme is illustrated in Figure 2a, where 426 the material point GIMPM domains are dotted grey, the inflow boundary is indicated by the 427 dotted red line, and elements belonging to the additional inflow domain are highlighted in blue. 428 We note that a more efficient, but complicated, scheme may be implemented, where in the 429 "inflow elements" refill with material points automatically as they become empty (Zhao et al., 430 2019).

431 *Outflow boundaries:* At outflow boundaries, material points exit the domain and are removed 432 from the simulation. In the GIMPM, material points with GIMPM domains that overlap an 433 outflow boundary will not receive a full interpolation during grid to material point mappings by 434 default, assuming parameter values are unknown beyond the active portion of the background 435 grid. In Figure 2b (left side), a material point GIMPM domain is shown overlapping an outflow 436 boundary (red). For each active element that the GIMPM domain overlaps, our treatment is to 437 temporarily introduce a sub-particle with a GIMPM domain matching the area of overlap 438 between the original material point domain and the element (Figure 2b, right side). The sub-439 particles receive the interpolation, with the original material point then receiving the average of 440 sub-particle values weighted by the area of their subdomains.

441 *Ice front boundary:* While the position of the ice front is naturally tracked by material points 442 positions, it will rarely align with element edges. Applying the ice front stress boundary 443 condition along element edges results in large integration errors if the element is not sufficiently 444 full of material points. To mitigate this issue, we forgo material point integration for elements 445 containing the ice front. Instead, we employ Gauss quadrature (i.e. the finite element method) 446 within the element, which allows us to enforce the boundary condition at the element edge that 447 divides active and inactive elements. An illustration of this treatment is shown in Figure 2c, 448 where inactive elements are shaded with grey stripes, partially-full ice front elements being 449 approximated with the FEM are shaded with yellow, and the ice front boundary is indicated by 450 the dashed red line. Note that using FEM at the ice front requires mapping the material point 451 history variables used in the SSA solution to the element nodes in the same manner as (29). We 452 show in Section 5 that using the FEM at the ice front is sufficiently accurate; however, 453 alternative solutions exist if desired. One approach would be to apply the boundary condition

along line segments or B-splines that approximate the sub-element scale location of the ice front
(Bing et al., 2019). If using the sMPM, an additional option would be to adjust or replace the
mesh to always align with the position of the ice front, and maintain sMPM integration within
the element and FEM boundary conventions. This treatment would be possible because sMPM is
not restricted to a rectilinear grid.

459 4.2. Material Point Splitting

460 The highly tensile regime of ice shelves tends to cause material points to elongate or grow over 461 time. Material points can be split as necessary to maintain a desired resolution of material points per grid cell. For the GIMPM, we initiate splitting when the domain length l_p exceeds a given 462 463 threshold. We implement a similar procedure for the sMPM, where a pseudo-domain length is 464 tracked using the accumulated strain of a material point in Cartesian directions (Ma et al., 2009). 465 The splitting threshold cannot exceed the length of a grid cell, and can vary across the domain if, 466 for example, greater material point resolution is desired near the grounding line. For splitting in direction i, the two split material point coordinates, ${}^{s_1}(x_p)_i$ and ${}^{s_2}(x_p)_i$ are set to 467

$${}^{s_{1}}(x_{p})_{i} = (x_{p})_{i} + \frac{1}{4}(l_{p})_{i},$$

$${}^{s_{2}}(x_{p})_{i} = (x_{p})_{i} - \frac{1}{4}(l_{p})_{i}.$$
(43)

Each new material point is then assigned half the current $(l_p^{m+1})_i$ and initial $(l_p^0)_i$ domain length corresponding to the splitting direction *i*, from the parent material point being split. For a unidirectional split, the non-split current and initial domain lengths are inherited from the parent material point without modification. The deformation gradient and velocities of the parent

$${}^{s}H_{p} = H_{p} + \frac{\partial H_{p}}{\partial x_{i}} \Big[{}^{s} \big(x_{p} \big)_{i} - \big(x_{p} \big)_{i} \Big].$$

$$(44)$$

472 material point are transferred directly to the new material points, but direct transfer of thickness 473 may cause visible thickness oscillations in areas of steep thickness gradients. We propose to 474 mitigate these oscillations by instead reassigning thickness to each split material point as where the thickness gradient, $\frac{\partial H_p}{\partial x}$, must be interpolated from the grid. Figure 3 gives the 475 476 thicknesses for a subset of material points at the end of the steady state flow-band test described 477 in Section 5 (GIMPM at 5 km grid resolution and 4 material points per cell), both with and 478 without adjusting thickness according to (44). By using (44), the thickness oscillations from 479 splitting are almost fully eliminated.



Figure 3. Thickness for a subset of material points at the end of the steady state flow-band test with and without adjusting thickness according to its gradient during splitting. This simulation used the GIMPM at 5 km grid resolution and 4 material points per cell.

480 5. Examples

In this section, we consider several examples using the GIMPM and sMPM for SSA simulations to validate and test the methods. We quantify error in modeled stress and front propagation versus analytical solutions in 1-D, and further demonstrate front propagation in 2-D. We then test the methods on an idealized marine ice sheet to show that they can maintain steady state grounding line positions over time and can advect passive scalar fields without artificial diffusion.

487 5.1. Flow-band test case: steady state

488 We test our GIMPM-SSA framework against a flow-band model that gives the analytical steady 489 state for a longitudinally unconfined ice shelf with a constant flux at the upstream inflow 490 boundary. The flow-band model is formulated under the assumption of unidirectional flow, and 491 is therefore inherently 1-D. In practice, we model the flow-band in 2-D, where the domain is 1 element wide, but unidirectional flow is still enforced (i.e. $(v_2)_p = 0$). This experiment was 492 493 previously used to verify a finite-difference front-tracking scheme (Albrecht et al., 2011), and we 494 use the same values for ice density ($\rho = 910 \text{ kg m-3}$), seawater density ($\rho_w = 1028 \text{ kg m-3}$), and the flow rate factor ($B = 1.9 \times 10^8$ Pa s1/3). The flux at the upstream boundary is given as $Q_0 =$ 495 496 v_0H_0 , where we take the velocity, $v_0 = 300$ m a-1 and the thickness, $H_0 = 600$ m. The solution 497 for the spreading rate is given as

$$\frac{\partial v_1}{\partial x_1} = \left(\frac{\rho g}{4B} \left(1 - \frac{\rho}{\rho_w}\right) H\right)^3 = C H^3 , \qquad (45)$$

where all flow is along the x_1 -axis (Weertman, 1957). The analytical deviatoric stress can be calculated using (5) and (6). The thickness and velocity profiles are obtained from conservation of mass and momentum are given by $H(x_1) = \left(\frac{4c}{Q_o}x_1 + \frac{1}{H_0^4}\right)^{\frac{1}{4}}$ and $v_1(x_1) = \frac{Q_0}{H(x_1)}$, respectively (van der Veen, 2013).

503 We first test the ability of the GIMPM-SSA model to maintain the given steady state. We consider a domain that spans from the inflow boundary at $x_1 = 0$ to a fixed ice front $x_1 = 250$ 504 505 km. The two-dimensional steady-state geometry corresponding to this configuration is shown in 506 Figure 4. The initial material point locations fully cover this domain, as well an additional 507 domain beyond the inflow boundary that must be included to enforce the inflow boundary condition. Note that the analytical solution does not include an ice front, as $H(x_1) = 0$, but 508 509 including an ice front at any location on the domain will not change the steady-state upstream provided the ice front boundary conditions (8) are assigned. Setting the ice front at $x_1 = 250$ 510 511 km gives a realistic thickness at the ice front of ~219 m. We test the sMPM and the GIMPM at 512 varying material point and grid resolutions, and with and without the reweighting given by (26).



Figure 4. Analytical steady state ice geometry for the flow-band test and the GIMPM solution at 300 years using a 2.5 km resolution grid and 9 material points per cell.

Each trial is initialized with the analytical solutions for thickness and velocity, and run forward for 300 years using one-month time steps. The threshold material point length at which splitting is initiated is set to 1.5 times the original length. The length after splitting is then 0.75 times the original length, which due to the purely tensile flow regime, therefore constitutes the lower bound on potential lengths that will develop throughout the simulation.

518 Figure 5 shows the deviatoric stress and velocities for the material point initially located closest to $x_1 = 0$ km as it advects to its final location of ~177 km over 300 years. These figures 519 520 use a 2.5 km mesh, which unless otherwise indicated, are initialized with 9 material points per 521 cell (3×3 in 2-D). Figure 5a compares the result that does not use the reweighting scheme from 522 (26) with the analytical result. Stresses fluctuate widely due to uneven material point weighting 523 between elements, which results in inaccurate velocities, positions, and thicknesses. Figure 5b 524 gives the velocities from the sMPM when using 9 and 16 material points per cell and the 525 reweighted sMPM when using 4 material points per cell. It is evident that increasing the material 526 point resolution in the sMPM may slightly mitigate the error, but it increases the computational e 527 xpense and is not nearly as accurate as reweighted sMPM. The reweighted sMPM ensures a 528 smoother transition of the stiffness matrix between elements and even with just 4 material points 529 per cell, and yields results that almost exactly match the analytical solution. The severity of the 530 error without the reweighting scheme is not common to all MPM simulations, and is likely due 531 to the highly nonlocal stress regime of the SSA. As there appears to be very little tolerance for 532 this type of error, the reweighting scheme from (26) appears to be essential for accurate SSA 533 simulations using the sMPM.



Figure 5. Results from the steady state flow-band test for the material point initially located closest to 0 km, where 9 material points are initialized per 2.5 km grid cell. (a) Deviatoric stress using the unweighted sMPM. (b) Velocities corresponding to (a) compared to the velocities obtained using 16 material points per cell, as well as the velocities using the reweighted sMPM with only 4 material points per cell. (c) Deviatoric stress using the reweighted sMPM and the GIMPM, which closely match the analytical result. (d) Detail of the boxed region in (c). The discontinuities for the reweighted sMPM are caused by the grid-crossing error, and are largely alleviated using the GIMPM.

The stress response using the reweighted sMPM and the GIMPM are given in Figure 5c, and show significant improvement over the sMPM in Figure 5a. Note that the reweighting scheme has no effect when implemented with the GIMPM, as no gaps or overlaps of the GIMPM domains develop in the test case. The fit with the analytical solution is less accurate where $x_1 <$ ~40 km, as ice shelf surface slopes are high and therefore finer mesh resolution is needed for improved accuracy. In generally, the GIMPM is more accurate than the reweighted sMPM, as the latter still does not fully alleviate cell crossing errors. This is evident in Figure 5d, which shows the zoom of the region within the grey box from Figure 5c. The GIMPM alleviates, but still cannot entirely eliminate the sharp stress discontinuities or oscillations as the material crosses cell boundaries.

We next perform a mesh convergence study to investigate the performance of the reweighted sMPM and the GIMPM, using error measures common to material point methods (e.g. Bing et al., 2019, Wang et al., 2019, Charlton et al., 2017). We evaluate the relative error in deviatoric stress between the modeled and analytical results for a material point at a time step as

$$R_{p} = \frac{\left| (\sigma_{11}^{D})_{p}^{n} - (\sigma_{11}^{D})_{p}^{a} \right|}{\left| (\sigma_{11}^{D})_{p}^{a} \right|},$$
(46)

where superscripts 'n' and 'a' indicate the numerical and analytical values, respectively. Global error (R_G) at a time step is then computed by summing over all material points as

$$R^{\rm G} = \frac{\sum_p R_p A_p}{\sum_p A_p},\tag{47}$$

Figure 6 gives the global error averaged over time for the steady state flow-band test for all combinations of material point and grid resolutions for the reweighted sMPM and the GIMPM. We find that the GIMPM is generally more accurate than the reweighted sMPM. While both methods show a reduction of error with increased grid refinement, the reweighted sMPM is less accurate with fewer material points per element (i.e. 4MPs vs. 9MPs or 16 MPs); whereas, with the GIMPM accuracy is effectively independent of number material points, because the grid-



Figure 6. Deviatoric stress error from the steady state flow-band test for the GIMPM and the reweighted sMPM.

557 5.2. Flow-band test case: front propagation

We also use the flow-band model to test the ability of our scheme to track the calving front. The analytical position of the ice front, xc, at time t can be found from the relation $Q_0 t =$ $\int_0^{x_c} H(x') dx'$ (Albrecht et al., 2011), and is given by

$$x_{c}(t) = \frac{Q_{0}}{4C} \left[\left(3Ct + \frac{1}{H_{0}^{3}} \right)^{\frac{4}{3}} - \frac{1}{H_{0}^{4}} \right].$$
(48)

We track the ice front over 300 years, setting the initial position of the ice front at $x_1 = 0$. Analytical versus modeled front position is plotted in Figure 7. The modeled front position is evaluated using the GIMPM with a 2.5 km grid and 9 material points per grid cell. The fit is nearly perfect, with a maximum deviation from the analytical position of only 26 meters. Both the reweighted sMPM and the GIMPM for all grid and material point resolutions achieve results nearly indistinguishable from the analytical front position over time, with none of the simulations deviating from the analytical front position by more than 10% of a single grid cell length. Thus, this study demonstrates that the GIMPM can accurately simulate the stresses, geometry, and ice



569 front position of an evolving ice shelf.

Figure 7. Ice front position using the GIMPM plotted against the analytical solution using 9 material points per 2.5 km grid cell.

570 5.3. Front advection in 2-D

571 To test our front propagation scheme in 2-D, we simulate the radial spread of an unconstrained 572 floating ice tongue. This benchmark example was considered under steady-state conditions in 573 previous studies (e.g. Morland and Zainuddin, 1987; Pegler and Worster 2012, 2013; Wearing et 574 al., 2020). Our aim here is to achieve only qualitatively consistent results, because there is no 575 analytical solution for 2-D diverging ice flow. The setup of this the simulation follows Example 576 1 in Wearing et al. (2020). We use sMPM rather than GIMPM, as this simulation is best 577 performed on a grid without perfectly rectilinear elements. However, recall that sMPM is a special case of GIMPM that uses the Dirac delta function instead of the hat function for χ_p , the 578 particle characteristic function. The grid is shown in Figure 8a. The curved upstream boundary 579 580 (red) corresponds to an arc extracted from a circle with 70 km radius with a central angle of 10°. 581 Flow is axisymmetric with respect to the vertical axis defined at the center of the circle, and we 582 set free slip conditions at the lateral boundaries by enforcing that the normal component of 583 velocity is zero. At the upstream boundary, a constant thickness of 400 m and an inflow velocity

of 500 m a-1 is enforced. We evenly initialize 9 material points per cell on an inflow domain beyond the upstream boundary (not shown), and allow the system to evolve until the ice front reaches the downstream edge of the computational grid, which occurs after 86.6 years.

587 The corresponding final thicknesses and positions of all material points are plotted in 588 Figure 8b. The thicknesses of all material points at any radial distance match very closely 589 regardless of their azimuthal position, reflecting that the simulation has achieved the expected 590 axisymmetric flow regime. Also plotted is the steady-state thickness profile as calculated using 591 the FEM under the assumption that the calving front is fixed at the downstream edge. While it is 592 encouraging that the two thickness profiles show similar trends, we emphasize that unlike in the 593 1-D case, we do not expect a simulation with a moving ice front to replicate the steady state flow 594 exactly. Some mismatch is expected because an unconstrained ice tongue experiences buttressing 595 that increases proportionately with ice tongue length (Wearing et al., 2010). This buttressing is 596 related to "hoop" stresses that must be overcome for flow to diverge laterally. In Figure 8b, the



Figure 8. (a) Background grid used to simulate the unconstrained 2-D spreading of an ice tongue from an upstream boundary (red) with a 70 km radius of curvature. **(b)** The ice thicknesses of all material points after growing the ice tongue from the upstream boundary for 86.6 years, and the steady-state thicknesse profile calculated from the FEM using the same ice front position. Material point thicknesses are slightly lesser towards the ice front due to the increased rate of spreading these material points experienced earlier in the simulation, when the ice tongue was shorter and buttressing was lesser.

material points towards the ice front are relatively thin compared to the steady state because they
endured larger rates of spreading earlier in the simulation when the ice tongue was short and
buttressing was lesser. This example demonstrates that material point methods can be used for 2D ice front tracking in a physically-consistent manner.

- 601 5.4. *Marine Ice Sheet Model Intercomparison Project (MISMIP+)*
- 602 Our final experiment tests the ability of our model to maintain the steady state from the
- 603 idealized, but more realistic, geometry detailed in the MISMIP+ (Asay-Davis et al., 2016). This
- 604 geometry is a 640 km \times 80 km marine ice sheet, spanning an ice divide at $x_1 = 0$ km to a
- 605 calving front at 640 km. At steady state, the grounding line is centered at $x_1 \sim 450$ km and $x_2 =$
- 606 40 km. At the lateral boundaries, $v_2 = 0$. The steady state grounding configuration where $x_1 > 0$
- 607 350 km is shown in Figure 9a. The grounding line lies on a retrograde slope, and is therefore
- 608 very sensitive to perturbations or error, so that the configuration is ideal for testing the accuracy
- 609 of the GIMPM. Furthermore, this is a very high shear regime, which is often problematic for the
- 610 MPMs. Therefore, we update GIMPM domains with the corner-tracking scheme from (38)–(40).



Figure 9. (a) Grounding line at 0 years after initiating the GIMPM/sMPM, where blue material points are grounded and red are floating. The configuration after 100 years is shown for **(b)** the GIMPM and **(c)** the sMPM

611 We initially determine the steady state using the FEM according to the recommended 612 values for the friction parameter, the viscosity parameter, the rate factor, densities, and surface 613 accumulation given for the MISMIP+. Afterwards, we continue the simulation using both the 614 GIMPM and the reweighted sMPM. We use a 0.5 km grid and initialize the simulation with 9 615 material points per cell. After 100 years, both the GIMPM (Figure 9b) and the reweighted sMPM 616 (Figure 9c) are able to maintain the sensitive initial grounding line position (Figure 9a). The 617 reweighted sMPM grounding line region, however, is slightly noisier than GIMPM that is only 618 visible in a close-up view of the image in Figure 9c (zoom not shown).

During the simulation, we also advect a passive scalar field to demonstrate how when using the GIMPM, this field can be advected without artificial diffusion. This field is initially assigned a value of unity along a series of 0.5 km vertical strips, which are each separated by 50 km in the x_1 -direction, and a value of zero elsewhere (Figure 10a). We chose the width of the strips to roughly correspond to the width of an ice shelf rift, which can range from zero to several



Figure 10. MISMIP+ advection of a passive scalar. Results shown for same area as Figure 9 around the grounding line for the GIMPM versus Discontinuous Galerkin (DG). The initial state is given in (a). The field at 5 years is shown for (b) the GIMPM and (d) DG. The field at 100 years is shown for (c) the GIMPM and (e) DG.

624 kilometers wide. The field was initially assigned on the grid, and interpolated to the material 625 points before the simulation. For comparison, we ran the same simulation using the 626 Discontinuous Galerkin (DG) method, the least-diffusive Eulerian advection method already 627 available in Elmer/Ice. The advected profiles of the scalar field obtained from the GIMPM are shown in Figure 10b after 5 years and Figure 10c after 100 years. The scalar field does not 628 629 experience artificial diffusion, and takes an arcuate shape over time that reflects the high shear 630 experienced from the lateral grounded margins. The results from the DG method are shown in 631 Figure 10d after 5 years and in Figure 10e after 100 years. Although the DG method produces a 632 similar arcuate profile for the scalar field as the GIMPM, the value of the scalar field is 633 diminished due to numerical diffusion over time. We initially set the maximum value of the 634 scalar as 1.0 in the rift regions. With the DG method, the furthest downstream values of the 635 scalar near the centerline of the y-domain (y=40 km) quickly diminish to ~0.7 over 5 years, and 636 the originally sharp vertical strip diffuses over a width about three times its initial width. By 100 637 years, the diffusion increases in severity, and the values of the scalar near the centerline 638 diminished to ~0.2. With the GIMPM, the maximum values of the scalar field are maintained at 639 1.0. Thus, this simulation study illustrates the superior performance of the GIMPM, based on a 640 hybrid Lagrangian-Eulerian framework, in alleviating numerical diffusion issues persistent with 641 the DG method in a purely Eulerian framework.

Figure 11 shows the maximum principal deviatoric stress, σ_{max}^{D} , for the MISMIP+ test obtained using the GIMPM and the reweighted sMPM after t = 100 years. The initial σ_{max}^{D} is given in Figure 11a, where the largest stresses are concentrated near the lateral grounding line. The GIMPM field after 50 years (Figure 11b) is almost identical to the initial field. The reweighted sMPM field at 50 years (Figure 11d) while mostly identical to the initial field,

however, is characterized by oscillations due to grid-crossing error, which also cause the noise in the grounding line configuration in Figure 9c. By 100 years, both the GIMPM (Figure 11c) and the reweighted sMPM (Figure 11e) stress fields develop some artifacts in the stress field near the grounding line, as material points tend to become poorly distributed under extreme shear (Figure 12). This type of error is a limitation of our current GIMPM implementation; we discuss potential approaches to alleviate it in Section 6.



 $\sigma_{
m max}^{
m D}$ (Mpa)

Figure 11. MISMIP+: Maximum principal deviatoric stresses (Mpa) for the **(a)** initial state, **(b)** GIMPM at 50 years, **(c)** GIMPM at 100 years, **(d)** reweighted sMPM at 50 years, **(e)** reweighted sMPM at 100 years. The arrow in **(c)** indicates where continual heavy shear eventually causes poorly distributed material points, as shown in detail in Figure 12.



Figure 12. MISMIP+: Poorly distributed material points develop in the GIMPM simulation after 100 years of heavy shear, where indicated by the arrow in Figure 11c. Underlying grid resolution is 0.5 km.

653 6. Discussion

654 Our current GIMPM or reweighted sMPM formulations should be sufficiently accurate for many 655 applications in which it is essential to accurately track the ice front or history variables, such as 656 damage (Huth et al., 2020). However, additional developments are needed to mitigate the 657 artifacts introduced due to intense distortion of material point domains in high shear regimes 658 over long timescales (Section 5.4). One approach is to reinitialize the material points 659 periodically, which in the simplest case would involve interpolating all material point properties 660 to a new set of material points. Although this approach risks some artificial diffusion, it may be 661 negligible if reinitialization is infrequent. However, more sophisticated schemes are also 662 available that reinitialize material points locally as needed, while minimizing artificial diffusion 663 (e.g., Yue et al., 2015).

664 Further development of our method will likely include implementing more robust shape 665 functions. For example, Convected Particle Domain Interpolation (CPDI) methods assemble 666 shape functions according to the shapes of the material point domains, and alleviate cell-crossing 667 error. Unlike the GIMPM, CPDI methods are not restricted to tracking rectangular domains, and 668 instead may track parallelograms (CPDI1; Sadeghirad et al., 2011) or the corners of the domains 669 individually (CPDI2; Sadeghirad et al., 2013). The CDPI1 method has been shown to perform 670 especially well under intense shearing (Wang et al., 2019), and may be appropriate for avoiding 671 the errors related to high-shear observed in our GIMPM simulations over long timescales. 672 However, we note that our current GIMPM formulation is computationally less expensive and 673 easier to implement into existing finite element codes. Implementing CPDI methods will require 674 substantial modifications to our current discretization scheme and boundary treatment.

675 As an alternative to using material point domain-tracking shape functions, it may also be 676 advantageous to consider techniques that eliminate cell-crossing error through other means, such 677 as the dual domain material point (DDMP) method (Zhang et al., 2011) or the use of spline-678 based shape functions (e.g. Stomakhin et al., 2013). All of these techniques, including CPDI, 679 share an additional advantage over the GIMPM in that they may be employed using non-uniform 680 meshes of varying element types, such as triangular meshes commonly used in major ice flow 681 codes (e.g. ISSM and Elmer/Ice). Analyzing the error, convergence qualities, and speed of these 682 methods in the context of ice shelf flow and fracture will constitute future research.

683 Although not illustrated in this paper, an additional advantage of our GIMPM-SSA model 684 is that complex 3-D multiphysics can be represented while still being efficient enough to couple 685 with Earth system models. Because horizontal velocities are vertically-invariant within the SSA 686 framework, 3-D processes can be approximated locally with each material point using a series of 687 vertical layers, and subsequently vertically integrated if needed for implementation into the next 688 SSA solution. While the same can be applied to mesh-based Eulerian methods, the associated 689 advection schemes are not only dissipative, but scale in computational expense with the number 690 of layers used. However, as for 2-D variables, advection of a 3-D field for material point 691 methods within a shallow shelf framework avoids artificial diffusion and only requires updating 692 material point locations. We employ this 3-D approach to model damage evolution in ice shelves 693 in Part II (Huth and others, 2020); however, other potential applications of the approach include 694 modeling coupled processes such as temperature evolution, firn compaction, fabric anisotropy, 695 and marine ice formation. We also note that a full 3-D implementation of material point methods 696 for full-Stokes models is also possible for studying individual glaciers, but it would be 697 prohibitively expensive for continental-scale ice sheets.

698 **7.** Conclusion

699 We presented the generalized interpolation material point method for shallow shelf ice flow, and 700 verified that this formulation can reproduce and maintain analytical solutions for steady state ice 701 flow and ice front advection. The advantages of this formulation include: 702 1. Error-free Lagrangian advection or transport without numerical diffusion or dispersion 703 2. Computationally inexpensive, explicit time updates for ice thickness and history variables. 704 such as damage 705 3. Natural tracking of the ice front and grounding line at sub-element scale 706 4. Accurate schemes for boundary treatment and redistribution of thickness during particle 707 splitting that facilitate simulations over long timescales. 708 5. Consistent with the well-established conventions of the finite element method for shallow 709 shelf ice flow 710 By choosing the particle characteristic functions to be either the Dirac delta or the 'hat' 711 functions, the present formulation can reproduce the existing implementations of sMPM and the 712 GIMPM. We demonstrated that the sMPM shape functions are very sensitive to cell-crossing 713 errors and uneven distributions of material points, likely due to the quasi-static and highly 714 nonlocal stress regime of the SSA. By simply modifying the shape functions with a reweighting 715 scheme in the sMPM, we can significantly decrease this sensitivity to cell-crossing errors. This 716 numerical error is almost entirely alleviated using the GIMPM without the reweighting scheme, 717 so it is more appropriate for many applications on timescales of decades to centuries. A major 718 advantage of the reweighted sMPM over the GIMPM is that it is applicable with adaptive and 719 non-uniform quadrilateral and triangular mesh discretization, which is ideal for accurately 720 resolving grounding line dynamics. Future work is necessary to mitigate errors in the GIMPM

721 associated with the intense distortion and gaps in the material point distribution observed in high 722 shear regimes over long timescales. Potential solutions for this error involve developing material 723 point reinitialization schemes, improving GIMPM domain updating schemes, and/or 724 implementing different shape functions. In addition, future developments should focus on 725 implementing additional physics to fully take advantage of the GIMPM-SSA treatment of history 726 variables; this could be particularly beneficial when parameterizing complex 3-D processes using 727 a series of vertical layers assigned to each material point. Thus, the GIMPM-SSA model can 728 potentially develop into a powerful tool for studying large-scale, coupled ice sheet processes 729 simultaneously, thus enabling the accurate prediction of ice sheet response to climate change and 730 eventually global sea level rise.

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Supporting Information for

A generalized interpolation material point method for shallow ice shelves. Part I: shallow shelf approximation and ice thickness evolution

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Figure S1

Introduction

This supporting information provides the shape functions of the standard and generalized interpolation material point methods.



Figure S1. The 1-D sMPM (left) and GIMPM (right) shape functions for a node positioned at $x_i = o$ with the length of an element given by h. The length of the material point domain used in the GIMPM convolution is h/2 (green shading).