Unsteady Ekman–Stokes dynamics: implications for surface-wave induced drift of floating marine litter

Christopher John Higgins[1], Jacques Vanneste[2], and Ton van den Bremer[3]

[1]University of Oxford
[2]University of Edinburgh
[3]Oxford University

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C. Higgins[1,2], J. Vanneste[2], T.S. van den Bremer[1]

[1]Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK
[2]School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3FD, UK

Key Points:

- Marine litter studies include surface wave transport by Stokes drift but have neglected wave-induced Eulerian-mean flows in the upper ocean.
- We present a model of the Eulerian-mean Ekman–Stokes response to time-varying Stokes drift for use in marine litter transport models.
- Using buoy data we show the unsteady Ekman–Stokes flow significantly alters both magnitude and direction of near-surface transport.

Corresponding author: Christopher Higgins, `christopher.higgins@keble.ox.ac.uk`

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Plain Language Summary

In transport models for floating marine litter, surface wave effects are often included by simply superimposing their Stokes drift (the small net drift induced by waves) upon wind-driven flows and currents. However, due to Earth's rotation and turbulent dissipation in the ocean's surface mixed layer, the Stokes drift also drives additional Eulerian-mean flows. To obtain the correct transport velocity, the wave-induced Eulerian-mean flow must be added to the Stokes drift. We develop a model that enables estimation of this wave-induced Eulerian-mean flow from measurements or predictions of the wave field and apply our model to buoy data. Accounting for the Eulerian-mean flow significantly alters predictions of transport of floating marine litter.

1 Introduction

Floating marine debris, including plastic pollution, has rapidly become one of the most pressing environmental problems (Eriksen et al., 2014), particularly for marine ecosystems (Lavender Law, 2017). Although consensus exists about the longevity of plastic in the marine environment (Andrady, 2011) and the relatively large buoyancy of a significant share of plastic produced (Geyer et al., 2017), with both factors contributing to their long-distance transport, the total plastic budget of the world's oceans is poorly understood. A significant mismatch exists between the estimated amount of land-generated plastic that enters coastal waters (5-12 million tonnes yr^{-1}, Jambeck et al. (2015)) and the estimated total amount of plastic floating at sea (less than 0.3 million tonnes, Cózar et al. (2014); Eriksen et al. (2014); van Sebille et al. (2015)). Similarly, the amount of plastics measured at sea over the last few decades (Lebreton et al., 2019; Ostle et al., 2019; Wilcox et al., 2020) has not kept pace with growth in global plastic production (Goldstein et al., 2012; Geyer et al., 2017). To understand this mismatch, an improved understanding of the physical processes governing the transport and dispersion is required (van Sebille et al., 2020). This letter focuses on one of these processes: surface waves.

As a particle undergoes its periodic motion beneath surface waves, it experiences a Lagrangian-mean velocity in the waves' direction known as Stokes drift (Stokes, 1847). More generally, Stokes drift is the difference between the average Lagrangian flow velocity of a fluid parcel and the average Eulerian flow velocity of the fluid measured at a fixed spatial location (e.g. Bühler (2014); van den Bremer & Breivik (2017)). Surface gravity waves on the open ocean are mostly caused by winds. At any location and time, the wave field is a superposition of waves that have been generated by earlier winds at another location. Wave models, such as WAM and WaveWatch-III (Tolman, 2009), have been developed to predict wave fields and thus Stokes drift (Webb & Fox-Kemper, 2011; Breivik et al., 2014).

A recent and growing body of literature is examining the role of Stokes drift in the transport and dispersion of floating plastic pollution. Iwasaki et al. (2017) showed that in the Sea of Japan, Stokes drift pushed microplastics closer to the coast. Delandmeter

& van Sebille (2019) and Onink et al. (2019) report a similar result in Arctic regions. Dobler et al. (2019) demonstrated that Stokes drift fundamentally changes transport patterns in the South Indian Ocean by shifting the convergence regions to the west, causing leakage into the South Atlantic rather than the South Pacific. Waves may also allow particles to cross strong circumpolar winds and currents (Fraser et al., 2018).

Crucially, the above studies have simply superimposed the Stokes drift obtained from the local wave field onto the Eulerian current field obtained from ocean general circulation models or observations. In doing so, they have ignored the fact that the Eulerian flow is itself modified by surface waves: on the rotating Earth, the Coriolis force associated with the Stokes drift drives an Eulerian-mean current in the turbulent upper-ocean boundary layer (Ursell, 1950; Hasselmann, 1970; Xu & Bowen, 1994; Lewis & Belcher, 2004), as noted in Onink et al. (2019). Together, the Stokes drift and this wave-induced Eulerian current form the Lagrangian velocity with which marine litter is transported. It is this wave-induced Eulerian current, which we call the Ekman–Stokes flow, that this letter examines.

We derive a model for computing the unsteady Eulerian-mean Ekman–Stokes response to a time-varying Stokes drift, taking into account the correct wave stress boundary condition and the Coriolis–Stokes forcing. We do so for the case of constant eddy viscosity in the turbulent upper-ocean layer and a quasi-monochromatic (or narrow-banded) wave field. The product of this letter is an Ekman–Stokes convolution kernel, which can readily be used to predict the wave-induced Eulerian-mean flow in the turbulent upper-ocean boundary layer and hence the Lagrangian transport of floating marine debris. This kernel is a low-computational-cost alternative to fully coupled general circulation and wave models, which include the effect of waves in both the Coriolis–Stokes forcing and the surface boundary condition (Breivik et al., 2015). Using sample wave field data from buoys, we show that accounting for the Eulerian-mean Ekman–Stokes response to a time-varying Stokes drift considerably alters the trajectories of drifting objects.

2 Unsteady Ekman–Stokes flow

We consider a homogeneous, incompressible ocean of constant depth d, described by horizontal coordinates x and y, and a vertical coordinate z measured upwards from the undisturbed water level. The governing equations are

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{f}\times\boldsymbol{u} = -\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{1a}$$

$$w\,|_{z=\eta} = \partial_t\eta + \boldsymbol{u}_{\mathrm{H}}|_{z=\eta}\cdot\boldsymbol{\nabla}_{\mathrm{H}}\eta, \qquad \hat{\boldsymbol{n}}\cdot\overleftrightarrow{\boldsymbol{\tau}}\cdot\hat{\boldsymbol{s}}|_{z=\eta} = 0, \tag{1b}$$

$$w\,|_{z=-d} = 0, \tag{1c}$$

where $z = \eta(x, y, t)$ denotes the free surface elevation, $\boldsymbol{u}$ is the three-dimensional velocity vector, $\boldsymbol{f}$ the Coriolis vector, $\boldsymbol{A}_{\mathrm{H}} \equiv (A_x, A_y, 0)$ the horizontal component of any $\boldsymbol{A}$, and $\overleftrightarrow{\boldsymbol{\tau}}$ the stress tensor with components $\tau_{ij} = -(p-p_0)\delta_{ij} + \nu(\partial_i u_j + \partial_j u_i)$, with p_0 the atmospheric pressure and ν is the turbulent eddy viscosity, both taken constant. The unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{s}}$ are normal and tangential to the free surface respectively, so the dynamic boundary condition is a stress-free condition.

2.1 Wave-averaged mean-flow equations

We assume the wave steepness is small, $\alpha \equiv kA \ll 1$, where A is the peak wave amplitude of η and k the peak wavenumber, and solve (1) to $O(\alpha^2)$ using a Stokes expansion $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots$, where the subscript denotes the order in α. We focus on deep-water waves ($kd \gg 1$).

Linear wave dynamics arises at $O(\alpha)$, where we ignore viscous effects, neglecting a thin vorticity boundary layer of thickness $\delta_\nu = \sqrt{2\nu/\omega}$ under the (generally satisfied) assumption $k\delta_\nu \ll 1$. Consequently, we ignore viscous damping of waves as they propagate. In contrast, we must retain the Coriolis force since, as demonstrated by Hasselmann (1970), $O(f/\omega)$ corrections put horizontal and vertical velocity components out of quadrature, with impact on the wave-averaged dynamics.

Integrating the $O(\alpha^2)$ equations over a wave period, we obtain the wave-averaged mean flow equations (e.g. Huang, 1979)

$$\partial_t \overline{u} - f v_{\mathrm{L}} = -\partial_x \overline{p} + \nu \nabla^2 \overline{u}, \qquad \partial_t \overline{v} + f u_{\mathrm{L}} = -\partial_y \overline{p} + \nu \nabla^2 \overline{v}, \tag{2a}$$

$$\partial_t \overline{w} = -\partial_z \overline{p} + \nu \nabla^2 \overline{w}, \qquad \partial_x \overline{u} + \partial_y \overline{v} = -\partial_z \overline{w}, \tag{2b}$$

where the overbar denotes a time average, $\boldsymbol{u}_{\mathrm{L}} = \overline{\boldsymbol{u}} + \boldsymbol{u}_{\mathrm{S}}$ is the Lagrangian (or particle-transport) velocity, with $\overline{\boldsymbol{u}} = \overline{\boldsymbol{u}_2}$ the Eulerian-mean velocity and $\boldsymbol{u}_{\mathrm{S}}$ the Stokes drift, and the horizontal component of the Coriolis vector introduces only higher-order corrections to the flow. Without the shear and pressure terms, equations (2a) and (2b) correspond to those considered by Hasselmann (1970). The Coriolis terms include the Coriolis–Stokes forcing $-f\hat{\boldsymbol{z}} \times \boldsymbol{u}_{\mathrm{S}}$ (Hasselmann, 1970; Polton et al., 2005), which drives an Eulerian 'anti-Stokes flow', cancelling the Stokes drift and exciting inertial oscillations, and explains Ursell (1950)'s prediction of zero net drift for periodic waves in a rotating frame.

We focus on the horizontal momentum equations (2a) in the Stokes layer, that is, the top $O(k^{-1})$-deep layer of the ocean where the Stokes drift and hence the Coriolis–Stokes forcing are localised. One of the boundary conditions is provided by averaging the condition of zero tangential stress in (1b) (Longuet-Higgins (1953), Ünlüata & Mei (1970), Xu & Bowen (1994) and Seshasayanan & Gallet (2019)); it is given by

$$\partial_z \overline{\boldsymbol{u}}_{\mathrm{H}}|_{z=0} = \partial_z \boldsymbol{u}_{\mathrm{SH}}|_{z=0}. \tag{3}$$

Examining the viscous but non-rotating case, Longuet-Higgins (1953) originally showed that vorticity is transported from the viscous boundary layers into the fluid interior, affecting the mass transport profile (Ünlüata & Mei, 1970; Xu & Bowen, 1994; Seshasayanan & Gallet, 2019). Additional Eulerian-mean wave-induced transport, known as boundary-layer streaming, occurs in the boundary layer (e.g. (Grue & Kolaas, 2017)). The contributions of Hasselmann (1970) and Longuet-Higgins (1953) (and the theory of wind-driven currents of Ekman (1905)) were unified by Xu & Bowen (1994) into a model of wave (and wind-) driven flow in finite-depth water.

In the Stokes layer, vertical gradients dominate over horizontal ones. It follows from (2b) that the vertical velocity component can be neglected and hence p is z-independent. Introducing the complex notation $\mathcal{U} = \overline{u} + i\overline{v}$ as in Huang (1979), we obtain the Ekman–Stokes equations

$$(\partial_t + if - \nu\partial_z^2)\mathcal{U} = -if\mathcal{U}_{\mathrm{S}}(\boldsymbol{x}, z, t), \quad \partial_z \mathcal{U} = \partial_z \mathcal{U}_{\mathrm{S}}(\boldsymbol{x}, z, t)\Big|_{z=0}, \quad \lim_{z\to-\infty} \mathcal{U} = 0, \tag{4a,b,c}$$

where the two boundary conditions follow from (3) and the requirement that the solution can be matched to a weak Eulerian flow outside the Stokes layer. The Eulerian Ekman–Stokes velocity solving (4) is driven by the Stokes drift in two ways, via the Coriolis–Stokes forcing in the fluid interior (Polton et al., 2005) and via the wave stress condition (4b).

Note that a surface wind stress could be added to the boundary condition (4b); by linearity, the wind-driven Ekman velocity would simply be superimposed in convolution form on the wave-driven velocity we obtain (for example, Madsen (1978) Eq. (21) for linearly-varying $\nu(z)$). Assuming the wind stress is greater than the wave stress so that the latter can be omitted, Lewis & Belcher (2004) derive solutions to (4) for non-constant viscosity, but do not account for time-dependence of the wave-induced Eulerian response arising from the time-variation of the surface wave field and the associated Stokes drift.

2.2 Solution by Laplace transform

We solve (4) by Laplace transform, assuming that the Stokes drift $\mathcal{U}_\mathrm{S}$ has a time-independent vertical structure $\exp(2kz)$, corresponding to a quasi-monochromatic wave field, but an otherwise arbitrary time dependence. Denoting the Laplace transform by a tilde, with

$$\tilde{g}(s) = \mathcal{L}\left\{g(t)\right\} = \int_0^\infty g(t)e^{-st}dt, \quad \tilde{g}(s) = \mathcal{L}^{-1}\left\{\tilde{g}(s)\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \tilde{g}(s)e^{st}ds, \quad \text{(5a,b)}$$

where γ is a real number such that the contour path of integration is in the region of convergence of $\tilde{g}(s)$, we find

$$\tilde{\mathcal{U}} = 2k\left(1 + \frac{if}{s+if-4k^2\nu}\right)\frac{\tilde{\mathcal{U}}_\mathrm{S}e^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} - \frac{if\tilde{\mathcal{U}}_\mathrm{S}e^{2kz}}{s+if-4k^2\nu}. \tag{6}$$

This is the sum of a particular solution – the second term – which can be interpreted as a partial anti-Stokes flow, varying over the Stokes depth $\delta_\mathrm{S} = (2k)^{-1}$, and a homogeneous solution – the first term – varying over the Ekman depth $\delta_\mathrm{E} = \sqrt{2\nu/f}$, which includes a contribution driven by the vertical shear of the Stokes drift through the boundary condition (4b) (second term in the brackets in (6)).

A special case of (6) occurs if $\mathcal{U}_\mathrm{S}$ approaches a steady value $\overline{\mathcal{U}_\mathrm{S}}$ as $t \to \infty$. Then $\overline{\mathcal{U}}$ tends to the time-independent solution (e.g. Seshasayanan & Gallet (2019))

$$\overline{\mathcal{U}} = \frac{(1-i)D}{2}\overline{\mathcal{U}_\mathrm{S}}\left(1 + \frac{1}{1+iD^2/2}\right)e^{(1+i)z/\delta_\mathrm{E}} - \frac{\overline{\mathcal{U}_\mathrm{S}}e^{2kz}}{1+iD^2/2}, \tag{7}$$

where $D \equiv \delta_\mathrm{E}/\delta_\mathrm{S}$ is the fixed ratio of Ekman to Stokes depths. In the limit $D \to 0^+$, equation (7) tends to $-\overline{\mathcal{U}_\mathrm{S}}\exp(2kz)$: up to an inertial oscillation this is the so-called 'anti-Stokes' Eulerian-mean flow, predicted by Hasselmann (1970) to be induced by periodic waves in a rotating, inviscid ocean. Viscosity acts to reduce the shear in the anti-Stokes flow, so that a nonzero Lagrangian-mean velocity remains.

2.3 Ekman–Stokes kernel

We now use the Laplace convolution theorem to write the unsteady solution for the Ekman–Stokes mean flow as a function of time for arbitrary Stokes drift as

$$\mathcal{U}(\boldsymbol{x}, z, t) = \mathcal{U}_\mathrm{S}|_{z=0} * K(z,t), \tag{8}$$

where $*$ denotes convolution in time and

$$K(z,t) = \mathcal{L}^{-1}\left\{\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} + \frac{if}{s+if-4k^2\nu}\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} - e^{2kz}\right)\right\}. \tag{9}$$

The convolution kernel $K(z,t)$, which we will term the Ekman–Stokes kernel, can be evaluated by deforming the integration contour involved in the inverse Laplace transform to obtain (see supplementary material)

$$K(z,t) = 2k\sqrt{\nu}e^{-ift}\frac{e^{-z^2/(4\nu t)}}{\sqrt{\pi t}} - ife^{\left(4k^2\nu - if\right)t}\sum_\pm \frac{e^{\pm 2kz}}{2}\operatorname{erfc}\left(\sqrt{4k^2\nu t} \pm \frac{z}{\sqrt{4\nu t}}\right), \tag{10}$$

where $\sum_\pm$ denotes the sum of the plus and minus terms and we use the complementary error function $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. An equivalent form emphasising the dependence on wave parameters uses the scaled error function $\operatorname{erfcx}(t) = e^{t^2}\operatorname{erfc}(t)$ and reads

$$K(z,t) = 2k\sqrt{\nu}e^{-ift}\frac{e^{-z^2/(4\nu t)}}{\sqrt{\pi t}} - ife^{-ift}\frac{e^{-z^2/(4\nu t)}}{2}\sum_\pm \operatorname{erfcx}\left(\sqrt{4k^2\nu t} \pm \frac{z}{\sqrt{4\nu t}}\right). \tag{11}$$

Limit	Behaviour	Theory
$t \to \infty$	$2k\sqrt{\nu}e^{-ift}/\sqrt{\pi t}\left[1 - if/(4k^2\nu)\left(1-(1+2k^2z^2)/(4k^2\nu t)\right)\right]$	long-time limit
$t \to 0^+$	$8\nu k^2\delta(z/\delta_{\mathrm{S}}) - ife^{2kz}$	short-time limit
$\nu \to 0^+$	$-ife^{-ift}e^{2kz}$	Hasselmann (1970)
$f \to 0^+$	$2k\sqrt{\nu}e^{-z^2/(4\nu t)}/\sqrt{\pi t}$	Longuet-Higgins (1953)

Table 1. Asymptotic behaviour of the Ekman–Stokes kernel $K(z, t)$.

The Ekman–Stokes kernel K captures the (Eulerian-mean) flow response to the Stokes drift. The $1/\sqrt{t}$ describes the establishment of an Ekman spiral driven by the wave-induced surface stress; the if terms describe the impact of the Coriolis–Stokes forcing. Note that the dimension of $K(z, t)$ is time^{-1}.

Several limits of the kernel are of interest; they are given in dimensional terms in Table 1. The limits $\nu \to 0^+$ and $f \to 0^+$ are best understood by rewriting (11) in terms of the dimensionless parameters $D = \delta_{\mathrm{E}}/\delta_{\mathrm{S}}$, $\zeta = 2kz$ and $\tau = ft$ to obtain

$$K(\zeta,\tau)/f = De^{-i\tau}\frac{e^{-\zeta^2/(2D^2\tau)}}{\sqrt{2\pi\tau}} - \frac{i}{2}\sum_{\pm} e^{-i\tau-\zeta^2/(2D^2\tau)}\,\mathrm{erfcx}\left(D\sqrt{\frac{\tau}{2}} \pm \frac{\zeta}{\sqrt{2D^2\tau}}\right). \quad (12)$$

When $D \gg 1$, e.g. because $f \to 0^+$, the Coriolis–Stokes sum term in (12) is negligible and the flow becomes the Longuet-Higgins (1953) response to the wave stress at the surface. In contrast, for $D \ll 1$, e.g. as $\nu \to 0^+$, the anti-Stokes result of Hasselmann (1970) is approached but non-uniformly in ζ. This singular behaviour arises since for any nonzero D the shear condition at the surface cannot be met by an exact anti-Stokes flow, resulting in a thin layer of depth $\sim \sqrt{\nu/f}$ near the surface where cancellation of the Stokes drift is imperfect (e.g. Seshasayanan & Gallet (2019)). Over long times $\tau \to \infty$, the Coriolis–Stokes terms decay on the viscous rather than the inertial timescale, despite owing their existence to Earth's rotation.

The magnitude and argument of the dimensionless kernel $K(\zeta, \tau)$ are shown in Figure 1 for $D = 1$. The magnitude is largest towards $(\tau, \zeta) = (0, 0)$ due to the singular behaviour discussed above. The kernel has the character of an amplitude-decaying inertial oscillation with period $2\pi/f$ with an orientation in the horizontal plane that oscillates with the inertial period. Equation (11) together with the convolution in time (8) is the key result of this letter. Taking as inputs a time series of Stokes drift and estimates of the peak wavenumber k, Coriolis parameter f and turbulent viscosity ν, these equations produce a time series of the associated (Eulerian-mean) Ekman–Stokes current at any vertical elevation z, which can simply be added to the time series of the Stokes drift to give the Lagrangian-mean current relevant for marine litter transport. An open-source implementation in Python is provided as supplementary material.

3 Sample calculations of the Ekman–Stokes flow

3.1 Idealised storm

To demonstrate the use of the Ekman–Stokes kernel, we calculate the Eulerian response to a Gaussian Stokes drift profile lasting approximately 24 hours to represent an idealised storm. Specifically, we set $u_{\mathrm{S}}(z = 0) = u_{\mathrm{S}}^* \exp(-(t - t^*)^2/(\sigma^2))$ (and $v_{\mathrm{S}} = 0$) with $\sigma = 6$ hrs and magnitude $u_{\mathrm{S}}^* = 0.070$ m/s being reached at $t^* = 24$ hrs. Choosing $f = 1.0 \times 10^{-4}$ s^{-1} and $\nu = 1.0 \times 10^{-2}$ m^{2}s^{-1} ($D = 1.1$), we set $\mathcal{U}(z = 0, t = 0) = 0$ and evaluate the response for 1 week.

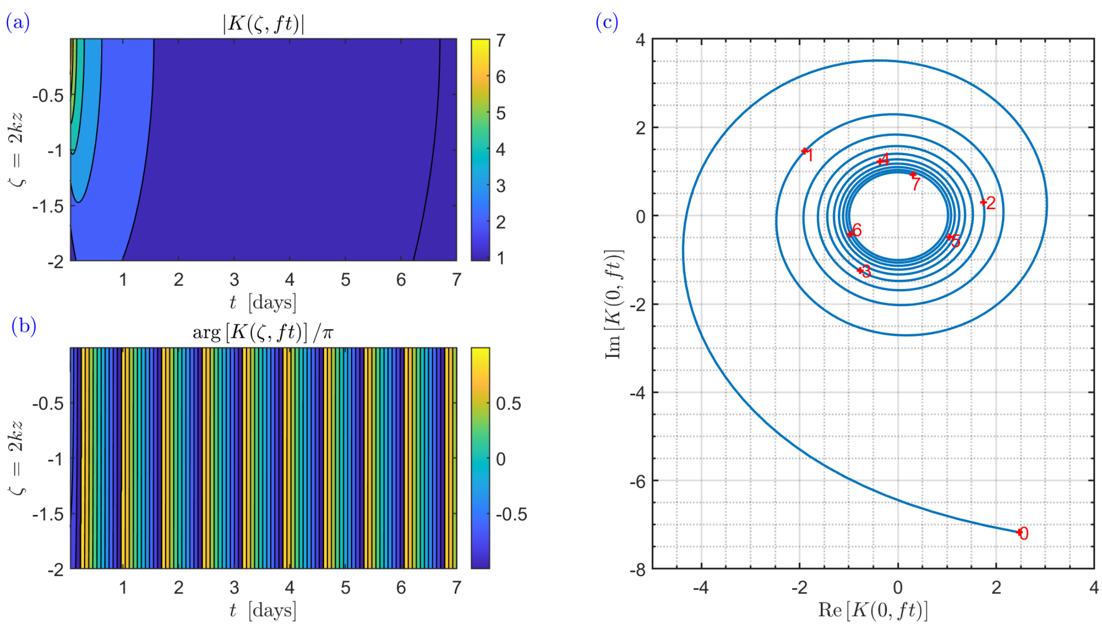

Figure 1. Ekman–Stokes kernel $K(\zeta, \tau)$ for $D = 1$ (with $f = 1 \times 10^{-4}\mathrm{s}^{-1}$): (a) magnitude and (b) argument as a function of depth and time, and (c) hodograph at the surface ($\zeta = 0$) with time (in days) shown in red. In panel (a) we have saturated the colour scale, as the kernel is singular at $(\zeta, \tau) = (0, 0)$.

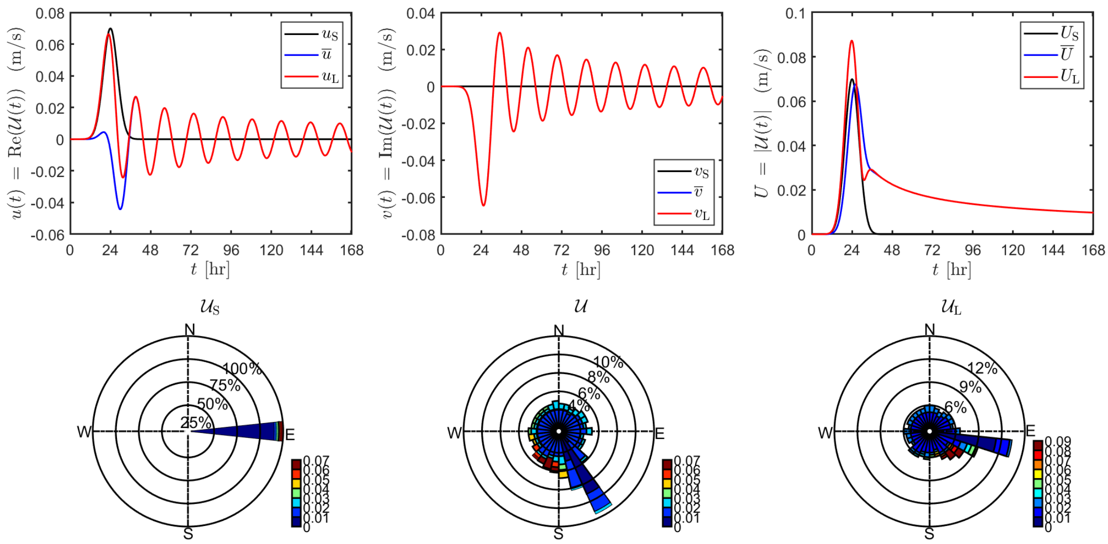

Figure 2. **Top:** Time series of wave-induced velocities formed in response to an idealised 24-hr Gaussian storm in the Northern Hemisphere showing the two components and magnitude of the Stokes drift $\mathcal{U}_\mathrm{S}$ (black), Eulerian-mean velocity $\mathcal{U}$ (blue) and Lagrangian velocity $\mathcal{U}_\mathrm{L}$ (red). **Bottom:** Wave roses for $\mathcal{U}_\mathrm{S}$, $\mathcal{U}$, and $\mathcal{U}_\mathrm{L}$, with radial distance representing the fraction of time during which the velocity has a given direction, and colour indicating magnitude in m/s.

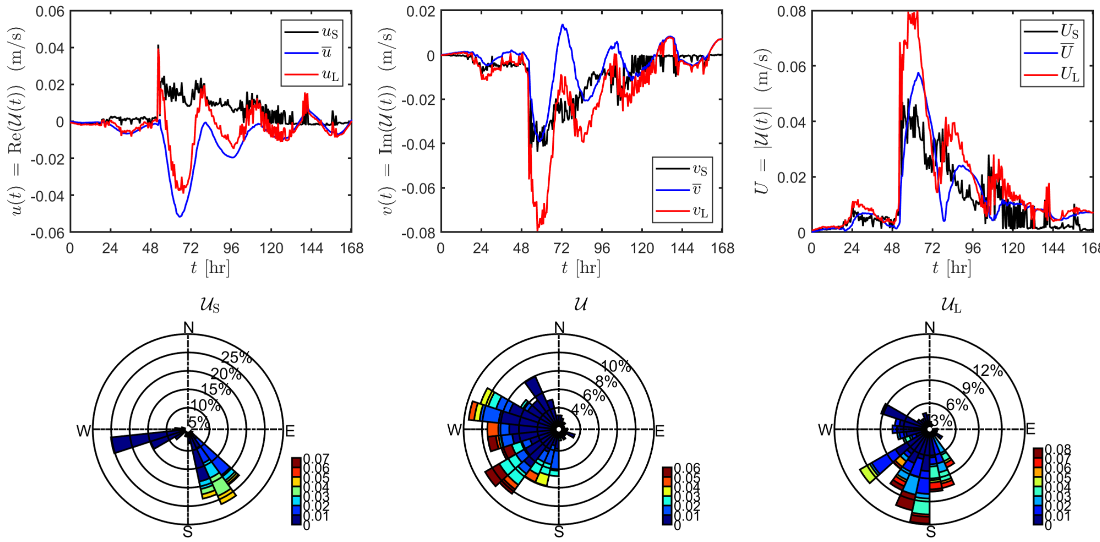

Figure 3. **Top:** Time series (14/05/00 15:41 – 22/05/00 09:41 UTC) of wave-induced velocities computed from buoy data from San Nicolas Island (33.22° N, 119.88° W), with colours as in Fig. 2. **Bottom:** Corresponding wave roses, as in Fig. 2.

In figure 2 we plot the u and v components and magnitudes, respectively, of the second-order currents over a week-long period. The sum of Stokes drift (black) and Ekman–Stokes flow (blue) gives the Lagrangian (transport) velocity (red). Beneath, wave roses are plotted for these second-order currents. The angular direction corresponds to the angle of propagation of the current (separated into 30 bins), the radius of each bar represents the percentage of time during which the velocity has a given direction, and the colour scale divides the data into velocity amplitude ranges. Fig. 2 shows that the Stokes drift is reduced by a (delayed) partial 'anti-Stokes' flow in the opposing direction, a transverse component arises on the same time scale, and damped inertial oscillations are formed which remain after the storm has ceased. The resulting Lagrangian current is deflected by the large transverse component of the Ekman-Stokes flow, to the right in the Northern Hemisphere (and to the left in the Southern Hemisphere).)

3.2 Buoy data

We use half-hourly records for the San Nicolas Island buoy (33.22° N, 119.88° W) obtained from CDIP (the Coastal Data Information Project) and estimate the Stokes drift using the formula

$$\mathcal{U}_\mathrm{S} = g^{-1}\omega_p^3 A_p^2 \exp(2\overline{k}z)\exp(i\theta_p)\,, \text{ where } A_p = H_\mathrm{S}/4. \tag{13}$$

where θ_p is the peak wave direction, H_s is the significant wave height, and ω_p is the peak frequency calculated from the peak period T_p. By making a quasi-monochromatic approximation, we assume the wavenumber spectrum is peaked about $k = \mathrm{mean}(k_p) = \mathrm{mean}(\omega_p^2/g)$, to leading order. We integrate (11) using the Stokes drift (13) by a trapezoidal rule with time-step equal to the buoy sampling time. We define the surface value of the kernel as $\lim_{z\to 0^-} K(z,t)$ instead of directly setting $z = 0$, so that the singular behaviour at $(0,0)$ is avoided.

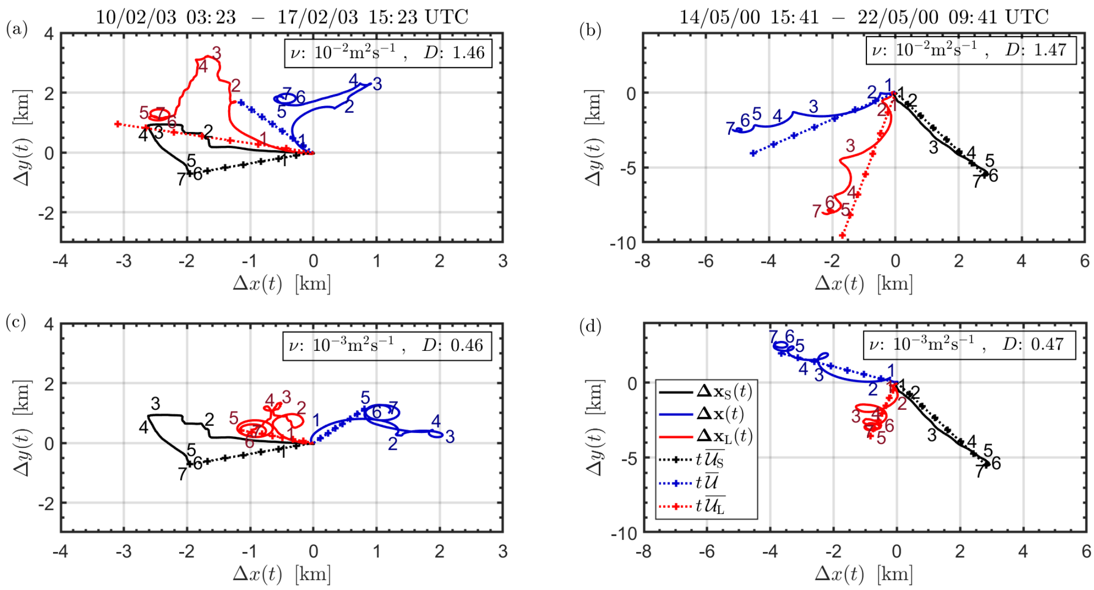

Figure 4. Particle paths at the surface ($z = 0$) computed for the San Nicolas Island buoy using our Ekman–Stokes convolution kernel. **Columns:** two different time samples. **Rows:** different values of turbulent viscosity. The paths shown are obtained using the Stokes drift (black), Eulerian-mean velocity (blue) and Lagrangian-mean velocities (red). The dashed lines ignore time dependence of the Stokes drift and show the response to the average of the Stokes drift over the periods considered. All paths begin at $(\Delta x, \Delta y) = (0, 0)$. Numbers beside each line denote the number of days elapsed.

As in figure 2, the top panels of figure 3 show the u and v components and magnitudes of the second-order currents. The largest Stokes drift at San Nicolas Island over this time period is in a South-Southeasterly direction, though a share of very small values arising from small-amplitude waves are also seen to propagate West-Southwest (cf. bottom-left panel, figure 3). In contrast, the Ekman–Stokes contribution is much more directionally-spread at all velocity amplitudes due to excited inertial oscillations. Superimposing the two flows leads to a directionally-spread Lagrangian drift which veers to the right of the Stokes drift.

To find the displacement associated with the unsteady flows, we simplify the problem by taking the wavenumber and Stokes drift time series to be uniform in space, which is valid for the relatively small accumulated displacements considered. Particle displacements are computed by time integrating the velocities obtained from our Ekman–Stokes kernel and are plotted in figure 4. Panels (a) and (c) show displacements over one week in February 2003 and (b) and (d) over a week in May 2000, with (*b*) corresponding to velocities plotted in figure 3. Line colours are consistent with figures 2 and 3. Straight dotted lines represent steady solutions i.e. (7) multiplied by time elapsed, with $\overline{\mathcal{U}_\mathrm{S}} = \mathrm{mean}\,(\mathcal{U}_\mathrm{S})$. Evidently, the steady approximation causes errors in the prediction of net particle displacement. Instead of simply following the black trajectory being transported by the Stokes drift alone, we predict the particle will follow the red trajectory, being transported by the Lagrangian velocity, the sum of the Stokes drift and the wave-induced Ekman–Stokes flow. For both time samples, the Lagrangian displacement is to the right of the displacement by the Stokes drift, as for the velocities. In the Southern Hemisphere, it will lie to the left of the Stokes drift.

We anticipate that the realistic range for viscosity is $O(10^{-3})$-$O(10^{-2})$ m^{2}s^{-1}, estimated from the vertical mixing coefficient $S_{\mathrm{M}} = 0.30$ in Mellor & Blumberg (2004) by using the law of the wall. Comparing (c) and (d) with (a) and (b), the particle displacement is reduced and inertial oscillations are more pronounced for the smaller viscosity $\nu = 10^{-3}$m^{2}s^{-1} in (c) and (d), since the directly opposing anti-Stokes flow increases in magnitude as viscosity decreases. For both values of ν (and values between them) the displacement is significantly altered in both magnitude and direction when the Ekman–Stokes flow is included.

4 Discussion and conclusions

Our analysis has demonstrated the need to add a so-called Ekman–Stokes flow to the Stokes drift to properly estimate the wave-induced Lagrangian-mean flow which transports floating marine litter. We have derived an Ekman–Stokes convolution kernel which can readily be used to predict the wave-induced Eulerian-mean flow in the turbulent upper-ocean boundary layer on a rotating Earth. It incorporates three important effects: the surface wave stress, the Coriolis–Stokes forcing, and unsteadiness of the forcing and response.

We properly account for the wave stress at the surface. This is often neglected (e.g. Lewis & Belcher (2004); Polton et al. (2005); Onink et al. (2019)), though it may be of the same magnitude as the wind stress (Seshasayanan & Gallet, 2019). Including the wave stress will yield more accurate predictions of the Lagrangian drift, particularly when wind and waves are misaligned. Our model also incorporates the Coriolis–Stokes forcing which induces a partial anti-Stokes flow and alters the response over the Ekman depth $\delta_{\mathrm{E}} = \sqrt{2\nu/f}$ (cf. Polton et al. (2005)). Our results demonstrate that for realistic values of eddy viscosity of 10^{-3}–10^{-2} m^{2}s^{-1} there is only partial cancellation of the Stokes drift by an anti-Stokes flow. Perhaps most importantly, our approach shows that unsteadiness of the Stokes drift and the induced Eulerian response can be readily incorporated into models of Lagrangian drift using a simple convolution. As passage times of storms are typically $O(1/f)$, time variability of the problem is crucial for accurate predictions of drift.

Future work should improve our model in the following ways. For simplicity we have assumed a constant eddy viscosity, although our Ekman–Stokes kernel could be adapted for linearly-increasing eddy viscosity (Madsen (1977), Lewis & Belcher (2004)), which provides a more accurate representation of turbulence in the upper-ocean boundary layer. Additionally, Shrira & Almelah (2020) have presented a solution method accounting for time-dependence of the eddy viscosity due to processes such as mixed-layer restratification or wave breaking (Price & Sundermeyer, 1999). Parametrisations of turbulent viscosity should thus account for both time and depth variation.

In real oceans, wave spectra are broad-banded, leading to a more strongly sheared but depth-persistent Stokes drift than for a monochromatic spectrum (Webb & Fox-Kemper, 2011). When complete information about the wave spectrum is available, the Ekman–Stokes kernel can be used to evaluate the contribution of each wavenumber to the Eulerian-mean velocity, which can then be summed to obtain a complete response. However, shear of the Stokes drift for realistic broad-banded spectra can be approximated using a monochromatic profile with a modified depth dependence (see e.g. Breivik et al. (2014)). Alternative Stokes drift depth-profiles would result in a different wave stress and functional form of the Coriolis–Stokes term in our Ekman–Stokes kernel.

Finally, we note that Seshasayanan & Gallet (2019) have recently shown that the steady Ekman–Stokes spiral is unstable to perturbations. Future work should consider the importance of this instability in the real ocean and how it might interact with unsteadiness of the Stokes drift.

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References

Andrady, A. L. (2011). Microplastics in the marine environment. *Mar. Pollut. Bull.*, *62*(8), 1596-1605. doi: 10.1016/j.marpolbul.2011.05.030

Breivik, Ø., Janssen, P., & Bidlot, J.-R. (2014). Approximate Stokes drift profiles in deep water. *J. Phys. Oceanogr.*, *44*(9), 2433-2445. doi: 10.1175/JPO-D-14-0020.1

Breivik, Ø., Mogensen, K., Bidlot, J.-R., Balmaseda, M. A., & Janssen, P. (2015). Surface wave effects in the NEMO ocean model: Forced and coupled experiments. *J. Geophys. Res.: Oceans*, *120*(4), 2973-2992. doi: 10.1002/2014JC010565

Bühler, O. (2014). *Waves and mean flows* (2nd ed.). Cambridge University Press, Cambridge, UK.

Cózar, A., Echevarría, F., González-Gordillo, J. I., Irigoien, X., Úbeda, M., Hernández-León, S., … Duarte, C. M. (2014). Plastic debris in the open ocean. *Science*, *111*(28), 10239-10244. doi: 10.1073/pnas.1314705111

Delandmeter, P., & van Sebille, E. (2019). The parcels v2.0 Lagrangian framework: new field interpolation schemes. *Geosci. Model Dev.*, *12*(8), 3571-3584. doi: 10.5194/gmd-12-3571-2019

Dobler, D., Huck, T., Maes, C., Grima, N., Blanke, B., Martinez, E., & Ardhuin, F. (2019). Large impact of Stokes drift on the fate of surface floating debris in the South Indian Basin. *Marine Pollution Bulletin*, *148*, 202-209. doi: doi.org/10.1016/j.marpolbul.2019.07.057

Ekman, V. W. (1905). On the influence of the Earth's rotation on ocean-currents. *Ark. Mat. Astron. Fys.*, *2*(11), 1-53.

Eriksen, M., Lebreton, L. C. M., Carson, H. S., Thiel, M., Moore, C. J., Borerro, J. C., … Reisser, J. (2014). Plastic pollution in the world's oceans: More than 5 trillion plastic pieces weighing over 250,000 tons afloat at sea. *PLoS ONE*, *9*(12), e111913. doi: 10.1371/journal.pone.0111913

Fraser, C., Morrison, A., Hogg, A. M., Macaya, E., van Sebille, E., Ryan, P., … Waters, J. (2018). Antarctica's ecological isolation will be broken by storm-driven dispersal and warming. *Nature Climate Change*, *8*(8). doi: 10.1038/s41558-018-0209-7

Geyer, R., Jambeck, J. R., & Lavender Law, K. (2017). Production, use, and fate of all plastics ever made. *Sci. Adv.*, *3*(7), e1700782. doi: 10.1126/sciadv.1700782

Goldstein, M. C., Rosenberg, M., & Cheng, L. (2012). Increased oceanic microplastic debris enhances oviposition in an endemic pelagic insect. *Biol. Lett.*, *8*, 817-20. doi: 10.1098/rsbl.2012.0298

Grue, J., & Kolaas, J. (2017). Experimental particle paths and drift velocity in steep waves at finite water depth. *J. Fluid Mech.*, *810*, R1. doi: 10.1017/jfm.2016.726

Hasselmann, K. (1970). Wave-driven inertial oscillations. *Geophys. Fluid Dyn.*, *1*, 463-502. doi: 10.1080/03091927009365783

Huang, N. E. (1979). On surface drift currents in the ocean. *J. Fluid Mech.*, *91*(1), 191-208. doi: 10.1017/S0022112079000112

Iwasaki, S., Isobe, A., Kako, S., Uchida, K., & Tokai, T. (2017). Fate of microplastics and mesoplastics carried by surface currents and wind waves: A numerical model approach in the Sea of Japan. *Marine Pollution Bulletin*, *121*(1), 85-96.

doi: doi.org/10.1016/j.marpolbul.2017.05.057

Jambeck, J. R., Geyer, R., Wilcox, C., Siegler, T. R., Perryman, M., Andrady, A., ... Lavender Law, K. (2015). Plastic waste inputs from land into the ocean. *Science*, *347*(6223), 768-771. doi: 10.1126/science.1260352

Lavender Law, K. (2017). Plastics in the marine environment. *Annu. Rev. Mar. Sci.*, *9*, 205-229. doi: 10.1146/annurev-marine-010816-060409

Lebreton, L., Egger, M., & Slat, B. (2019). A global mass budget for positively buoyant macroplastic debris in the ocean. *Sci. Rep.*, *9*, 12922. doi: 10.1038/s41598-019-49413-5

Lewis, D., & Belcher, S. (2004). Time-dependent, coupled, Ekman boundary layer solutions incorporating Stokes drift. *Dynam. Atmosph. and Oceans*, *37*(4), 313-351. doi: doi.org/10.1016/j.dynatmoce.2003.11.001

Longuet-Higgins, M. S. (1953). Mass transport in water waves. *Phil. Trans. Roy. Soc. London A*, *245*(903), 535-581. doi: 10.1098/rsta.1953.0006

Madsen, O. S. (1977). A realistic model of the wind-induced ekman boundary layer. *J. Phys. Oceanogr.*, *7*(2), 248-255. doi: 10.1175/1520-0485(1977)007⟨0248:ARMOTW⟩2.0.CO;2

Madsen, O. S. (1978). Mass transport in deep-water waves. *J. Phys. Oceanogr.*, *8*(6), 1009-1015. doi: 10.1175/1520-0485(1978)008⟨1009:MTIDWW⟩2.0.CO;2

Mellor, G., & Blumberg, A. (2004). Wave breaking and ocean surface layer thermal response. *J. Phys. Oceanog.*, *34*(3), 693-698. doi: 10.1175/2517.1

Onink, V., Wichmann, D., Delandmeter, P., & van Sebille, E. (2019). The role of Ekman currents, geostrophy, and Stokes drift in the accumulation of floating microplastic. *J. Geophys. Res.: Oceans*, *124*(3), 1474-1490. doi: 10.1029/2018JC014547

Ostle, C., Thompson, R., Broughton, D., Gregory, L., Wootton, M., & Johns, D. G. (2019). The rise in ocean plastics evidenced from a 60-year time series. *Nat. Commun.*, *10*, 1622. doi: 10.1038/s41467-019-09506-1

Polton, J., Lewis, D., & Belcher, S. (2005). The role of wave-induced Coriolis-Stokes forcing on the wind-driven mixed layer. *J. Phys. Oceanogr.*, *35*(4), 444-457. doi: 10.1175/JPO2701.1

Price, J. F., & Sundermeyer, M. A. (1999). Stratified Ekman layers. *J. Geophys. Res.: Oceans*, *104*(C9), 20467-20494. doi: 10.1029/1999JC900164

Seshasayanan, K., & Gallet, B. (2019). Surface gravity waves propagating in a rotating frame: The Ekman-Stokes instability. *Phys. Rev. Fluids*.

Shrira, V. I., & Almelah, R. B. (2020). Upper-ocean Ekman current dynamics: a new perspective. *J. Fluid Mech.*, *887*, A24. doi: 10.1017/jfm.2019.1059

Stokes, G. G. (1847). On the theory of oscillatory waves. *Trans. Cam. Phil. Soc.*, *8*, 441-455.

Tolman, H. L. (2009). User manual and system documentation of WAVEWATCH III TM version 3.14 Technical Note.

Ünlüata, U., & Mei, C. C. (1970). Mass transport in water waves. *J. Geophy. Res.*, *75*(36), 7611-7618. doi: 10.1029/JC075i036p07611

Ursell, F. (1950). On the theoretical form of ocean swell on a rotating earth. *Mon. Not. Roy. Astron. Soc., Geophys. Suppl.*, *6*(s1), 1-8. doi: 10.1111/j.1365-246X.1950.tb02968.x

van den Bremer, T. S., & Breivik, Ø. (2017). Stokes drift. *Phil. Trans. Roy. Soc. London A*, *376*(2111). doi: 10.1098/rsta.2017.0104

van Sebille, E., Aliani, S., Law, K. L., Maximenko, N., Alsina, J., A. Bagaev, M. B., ... Wichmann., D. (2020). The physical oceanography of the transport of floating marine debris. *Environ. Res. Lett.*, *15*(2), 023003. doi: 10.1088/1748-9326/ab6d7d

van Sebille, E., Wilcox, C., Lebreton, L., Maximenko, N., Hardesty, B. D., van Franeker, J. A., ... Law, K. L. (2015). A global inventory of small floating plastic debris. *Environ. Res. Lett.*, *10*(12), 124006. doi: 10.1088/1748-9326/10/12/124006

Webb, A., & Fox-Kemper, B. (2011). Wave spectral moments and Stokes drift estimation. *Ocean Modelling*, *40*(3), 273-288. doi: doi.org/10.1016/j.ocemod.2011.08.007

Wilcox, C., Hardesty, B. D., & Law, K. L. (2020). Abundance of floating plastic particles is increasing in the western North Atlantic Ocean. *Envir. Sci. Tech.*, *54*(2), 790-796. doi: 10.1021/acs.est.9b04812

Xu, Z., & Bowen, A. (1994). Wave- and wind-driven flow in water of finite depth. *J. Phys. Oceanogr.*, *24*(9), 1850-1866. doi: 10.1175/1520-0485(1994)024⟨1850:WAWDFI⟩2.0.CO;2

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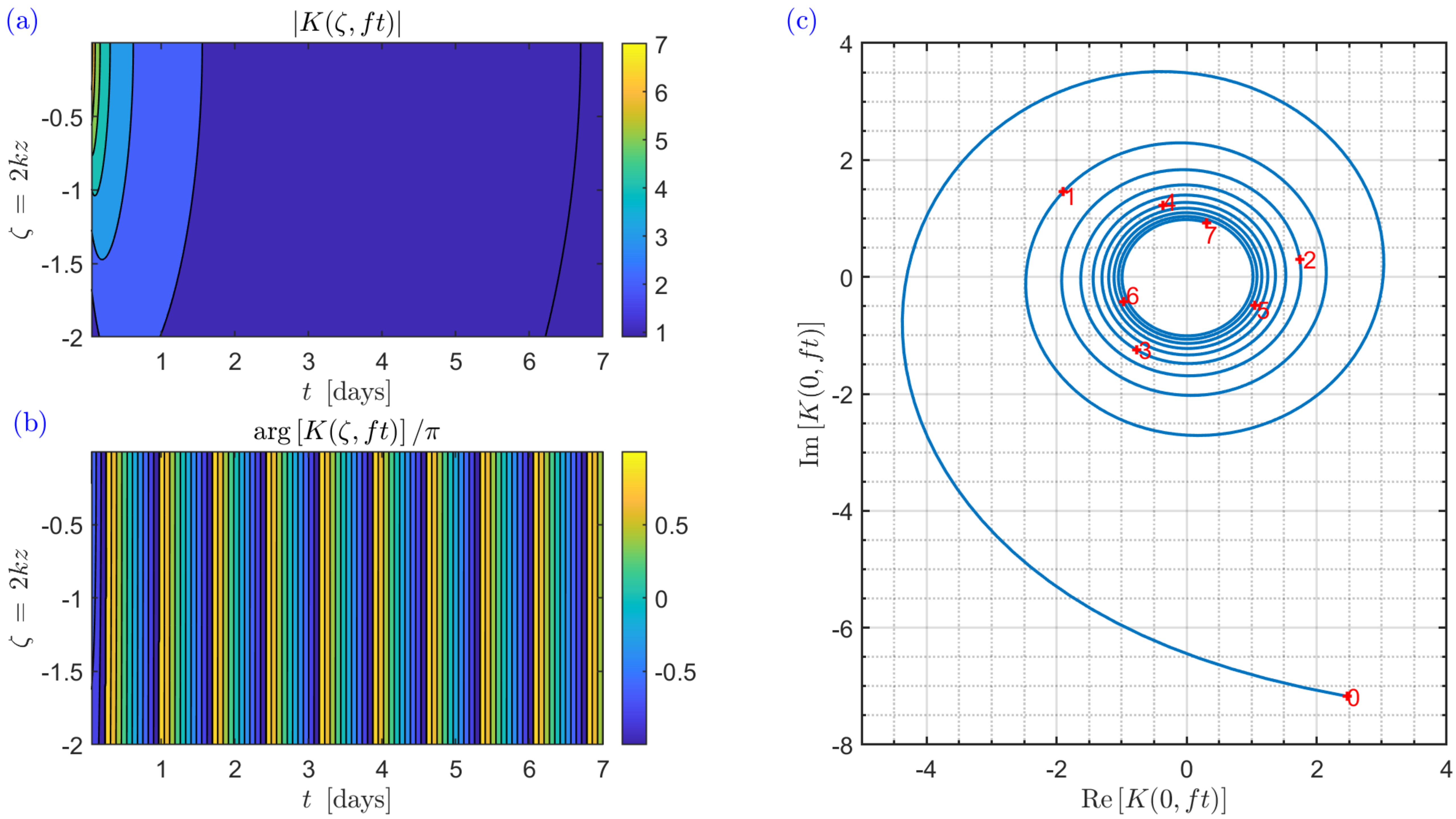

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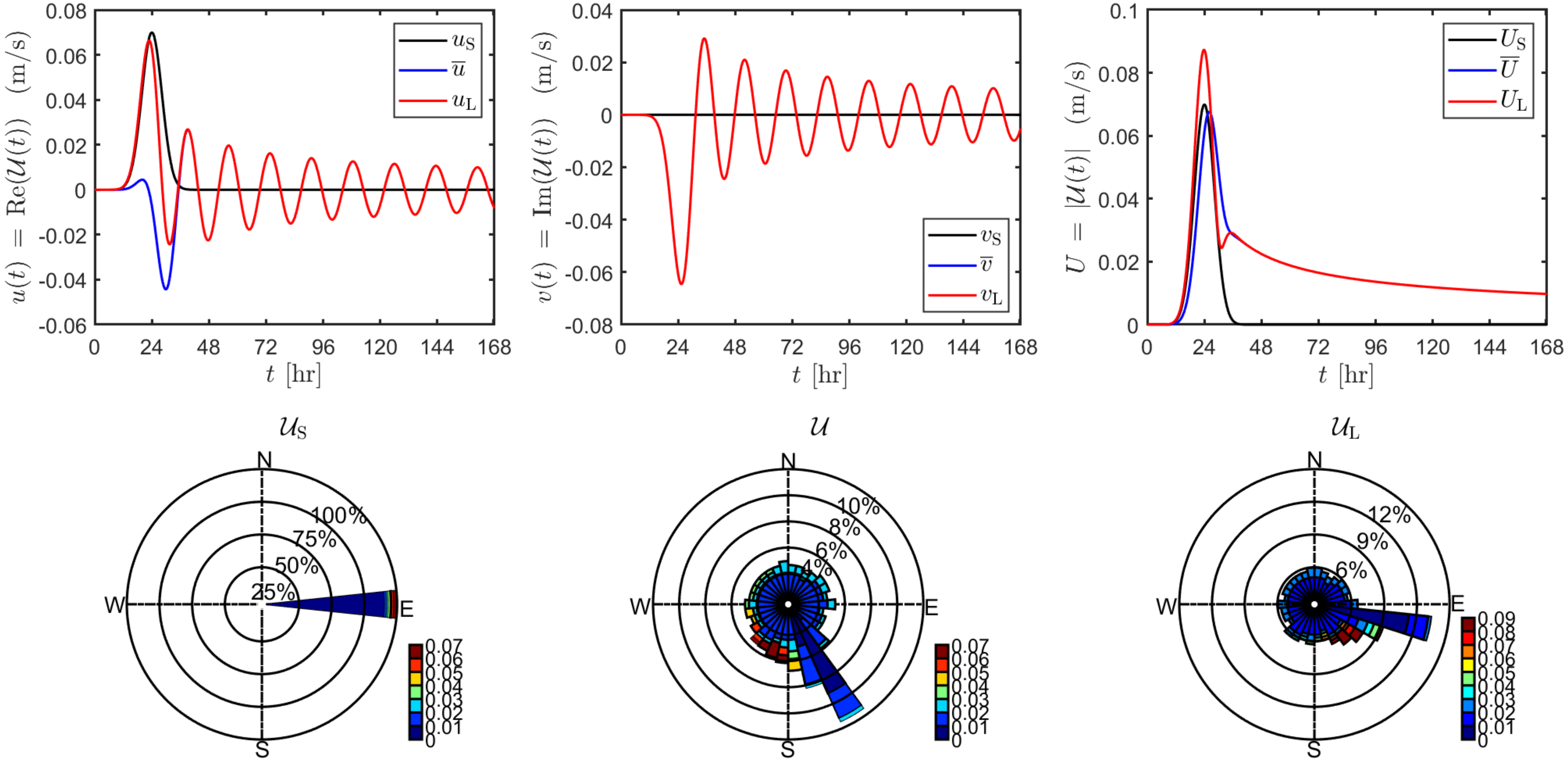

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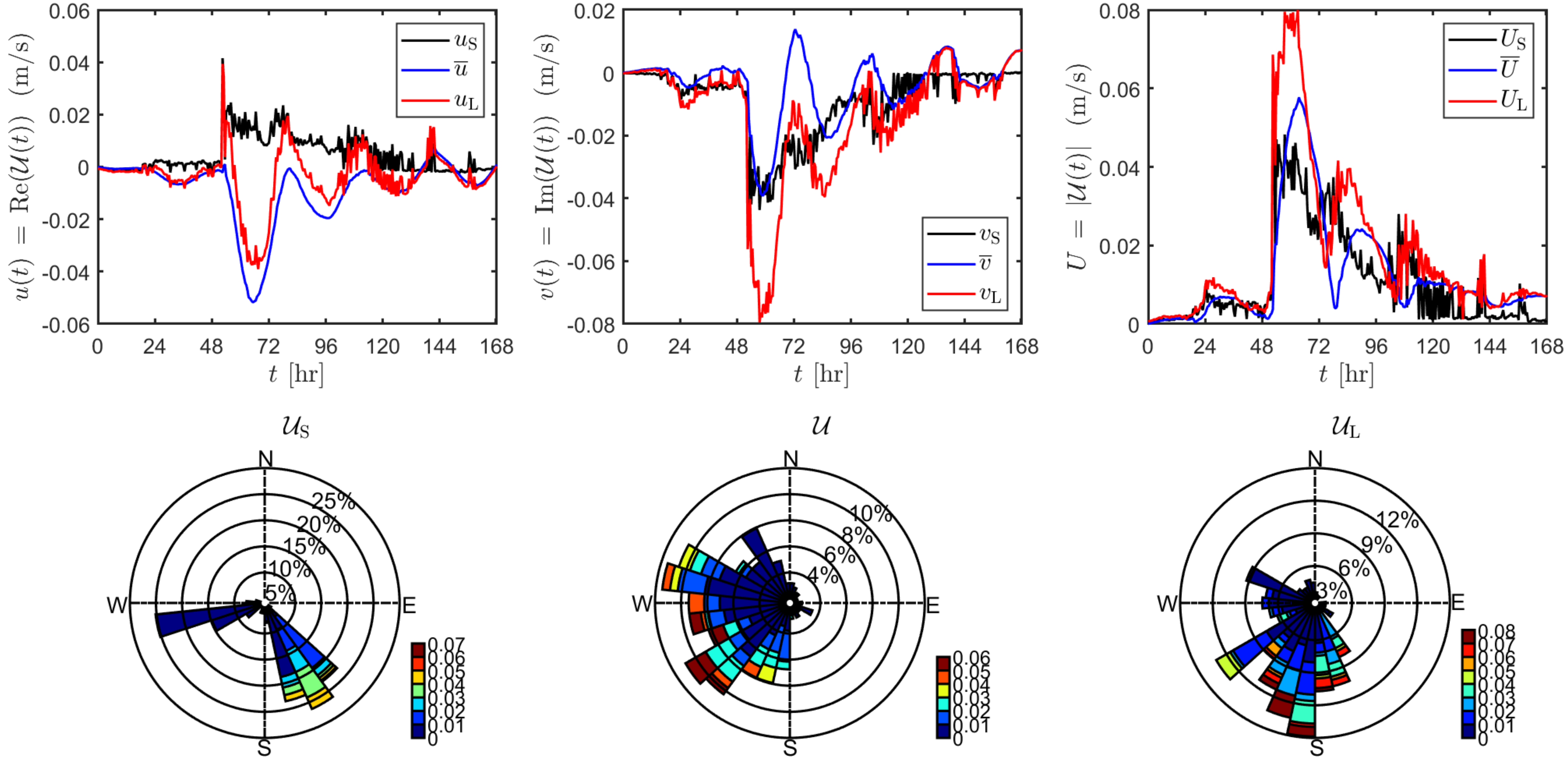

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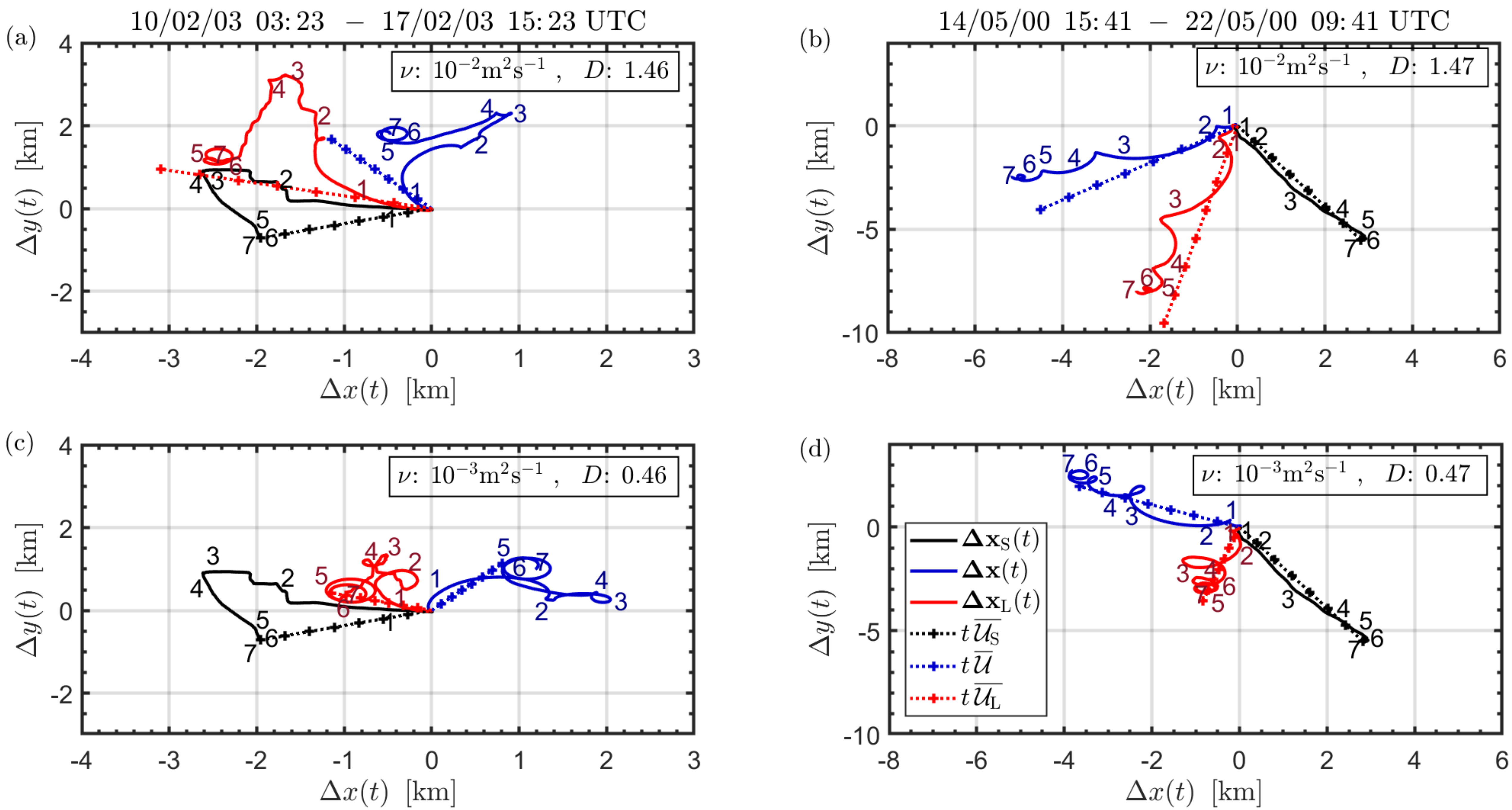

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Supplementary material for: 'Unsteady Ekman–Stokes dynamics: implications for surface-wave induced drift of floating marine litter'

C. Higgins[1,2], J. Vanneste[2], T.S. van den Bremer[1]

[1]Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK
[2]School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, Edinburgh EH9 3FD, UK

Contour integration

To carry out the Laplace inversion in (9) in the main paper, we group the two terms in the round bracket, noting that the second term, which is proportional to $\exp(2kz)$, gives an exponentially-growing solution $\propto \exp(4k^2\nu t)$ if inverted by itself. This may be seen by closing the contour to the left and applying Cauchy's theorem. Using L'Hôpital's rule on the grouped terms shows that $s = 4k^2\nu - if$ is a removable singularity. Defining $S = s + if$ we perform the integration along the contour shown in Figure A1; since the function is analytic within the enclosed region, we have by Cauchy's Theorem

$$\frac{1}{2\pi i}\oint\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} + \frac{if}{s+if-4k^2\nu}\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} - e^{2kz}\right)\right)e^{st}ds = 0. \tag{1}$$

The contribution of the arcs C_1 and C_2 disappears as $R \to \infty$, while the contribution of the small circle C_ε disappears as $\varepsilon \to 0^+$. Applying Cauchy's Theorem, the inverse Laplace transform equals minus the sum of the line integrals either side of the branch cut, L_+ and L_-. Changing variables to $b = \sqrt{|S|/\nu}$ and accounting for the behaviour of the square root when the branch cut is crossed, it is easy to see that

$$\sqrt{S/\nu} = \sqrt{-|S|/\nu} = \begin{cases} +ib & \text{above the branch cut} \\ -ib & \text{below the branch cut} \end{cases}. \tag{2}$$

The inverse transform is equal to the real integral

$$\mathcal{L}^{-1}\left\{\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} + \frac{if}{4k^2\nu - if - s}\left(\frac{2ke^{z\sqrt{(s+if)/\nu}}}{\sqrt{(s+if)/\nu}} - e^{2kz}\right)\right\}$$
$$= \frac{e^{-ift}}{\pi}\int_{-\infty}^{\infty} 2k\sqrt{\nu}e^{-b^2t}\cos\left(bz/\sqrt{\nu}\right)db - \frac{ife^{-ift}}{\pi}\int_{-\infty}^{\infty}\frac{2k\sqrt{\nu}e^{-b^2t}\cos\left(bz/\sqrt{\nu}\right)}{b^2+4k^2\nu}db. \tag{3}$$

The first of these is a Gaussian integral while the second may be evaluated explicitly by using Eq. No. 3.954 in Gradshteyn & Ryzhik (2014, p. 504), resulting in the analytic expression (10) in the main paper.

References

Gradshteyn, I. S., & Ryzhik, I. M. (2014). *Table of integrals, series, and products* (8th ed.). Elsevier.

Corresponding author: Christopher Higgins, `christopher.higgins@keble.ox.ac.uk`

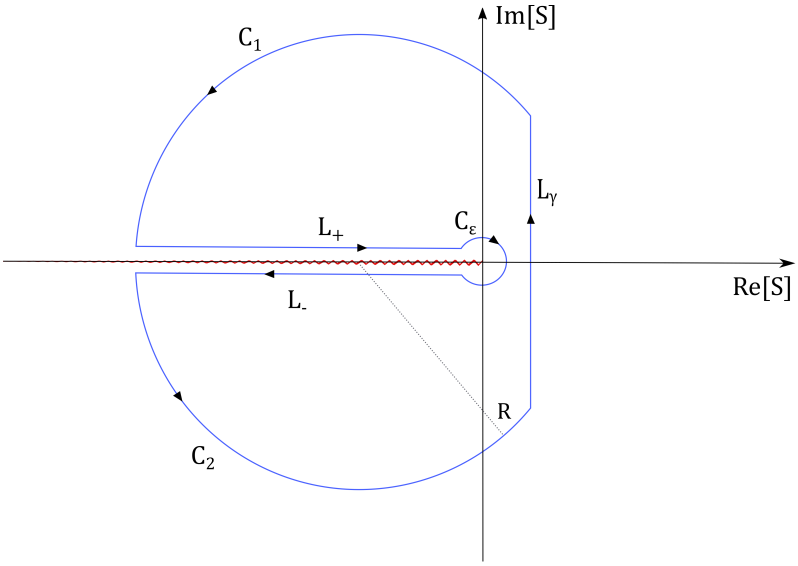

Figure 1. Integration contour for the Laplace inversion of the Ekman–Stokes kernel (**??**). The branch cut of the square root lies along the negative real axis. As $R \to \infty$, the line segment L_γ tends to the Browmwich contour used for the inverse Laplace transform.