Peculiarities of the propagation of the fast magnetosonic mode in a curved magnetic field: a case of the hemicylindrical model of the magnetosphere

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Abstract

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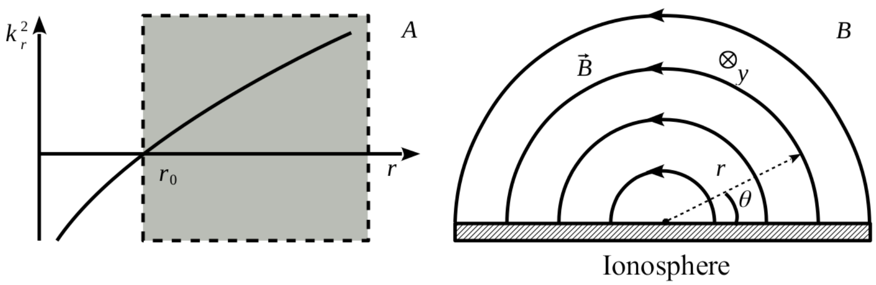

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Key Points:

- A resonance of the fast magnetosonic modes was found in a curved magnetic field
- On this resonance, the value of the wave's magnetic field have a logarithmic singularity
- The resonance surface serves a place of the wave's energy and the plasma density accumulation

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1 Introduction

In the studies of the ultra-low frequency (ULF) waves in planetary magnetospheres, a general framework is provided by the field line resonance (FLR) phenomenon: a compressional fast magnetosonic (FMS) wave propagates into the inhomogeneous magnetosphere generating a shear Alfvén mode on a magnetic surface where the wave frequency equals the local Alfvén frequency (*Glassmeier et al.*, 1999). This concept is usually applied to the dayside magnetosphere, where the role of driver of the FMS mode is provided by the processes related to the solar wind, such as solar wind pressure impulses, Kelvin-Helmholtz instability on the magnetopause, or compressional waves in the solar wind (*Menk and Waters*, 2013). The FLR can take place also on the night side, where the FMS can be driven by the bursty bulk flows (*Lysak et al.*, 2015).

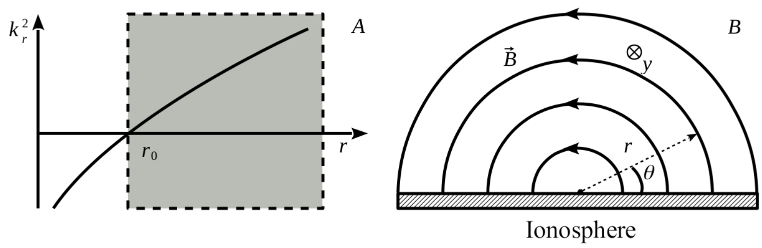

Figure 1. (A) The fast modes wave vector radial component squared k_r^2 as a function of the radial coordinate in the box-model. The shaded rectangle depicts is the FMS transparent region. (B) Hemicylindrical model of the magnetosphere

Originally, the FLR concept was established for the box-model with straight field lines (*Tamao*, 1965; *Southwood*, 1974; *Chen and Hasegawa*, 1974). In this case, the FMS' wave vector radial component is given by the equation $k_r^2 = (\omega^2 - k_y^2 v_A^2 - k_\|^2 v_A^2)/v_A^2$, where ω is the wave's frequency, k_y and $k_\|$ are the azimuthal and parallel wave vector components, and v_A is the Alfvén velocity. The FMS transparent region, that is, the region where $k_r^2 > 0$ is bounded by the magnetic shell where $\omega^2 = (k_y^2 + k_\|^2)v_A^2$ and k_r vanishes (Fig.1, A). Reflecting from this boundary, incident FMS forms the wave standing across the magnetic shells — the cavity mode (*Kivelson and Southwood*, 1985). The only wave field singularity in this model is the surface of the Alfvén (or field line) resonance.

The box-model cannot be considered as a realistic model of the magnetosphere since it does not take into account the field line curvature. The simplest model of the magnetosphere with the curvature is the hemicylindrical model where field lines are concentric semi-circles. While the dynamics of the shear Alfvén waves in this model has received significant attention (*Allan et al.*, 1986, 1987), the behaviour of the FMS mode has not been properly understood. The same model was used also for studies of the MHD oscillations in the solar coronal arcades (*Kaneko et al.*, 2015; *Klimushkin et al.*, 2017). The aim of the present paper is to consider some basic features of the field line resonance in the hemicylindrical model.

2 The model

In the hemicylindrical model, the magnetic field field lines and the magnetic shells are represented by concentric semi-circles and by half cylinder, respectively (Fig. 1, B). The plane cutting the cylinder along its axis represents the Earth's ionosphere. All equilibrium parameters of the are taken to depend only on the radial coordinate r, the field line curvature radius. The coordinate θ is the angle changed along the field line, being equal 0 and π on the ionosphere. The coordinate y is directed along the cylinder. It corresponds to the azimuthal coordinate in the magnetosphere.

The plasma is considered to be cold, $\beta = 0$. The background magnetic field $B_0(r) = \{0, B_{0\theta}(r), 0\}$ satisfies the equilibrium condition

$$\frac{\partial}{\partial r} B_0(r) = -\frac{B_0(r)}{r}.$$

The solution of this equation is $B_0 \propto r^{-1}$.

3 The governing equations

The MHD oscillations in the cold plasma are governed by the equation system

$$\rho_0\omega^2\vec{v} = \frac{1}{4\pi}\vec{B}_0 \times \nabla \times \left\{\nabla \times [\vec{v} \times \vec{B}_0]\right\}, \tag{1}$$

where $\vec{v}$ is the plasma velocity and ω is the wave's frequency. The ionosphere is supposed to be ideally conductive, thus $\vec{E}(\theta = 0) = \vec{E}(\theta = \pi)$ and $\vec{v}(\theta = 0) = \vec{v}(\theta = \pi)$. Then the velocity can be represented as

$$\vec{v}(r, \theta, y) = \vec{v}(r)\sin(N\theta)\, e^{-i(\omega t - k_y y)}$$

where N is an integer and k_y is the azimuthal component of the wave vector. After some algebra, Eq. (1) is reduced to the form

$$\partial_r\left[\frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)}\partial_r v_r\right] - \partial_r\left[\frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)} \cdot \frac{v_r}{r}\right] + \frac{\omega^2 - \omega_A^2(r)}{v_A^2}v_r = 0, \tag{2}$$

where $v_A = \sqrt{B_0/4\pi\rho_0}$ is the Alfvén speed,

$$\omega_A^2(r) = k_\parallel^2 v_A^2, \tag{3}$$

$$\omega_0^2(r) = (k_\parallel^2 + k_y^2)v_A^2, \tag{4}$$

$k_\parallel = N/r$ is the radial component of the wave vector. The azimuthal component of the plasma velocity v_y and the wave's magnetic field can be expressed in terms of the radial component of the plasma velocity v_r as

$$\begin{aligned}
v_y &= -ik_y\frac{v_A^2}{\omega^2 - \omega_0^2(r)}\left[\partial_r v_r - \frac{v_r}{r}\right], &(5)\\
B_r &= -k_\parallel\frac{B_0}{\omega}v_r, &(6)\\
B_y &= i\frac{B_0}{\omega}\frac{k_y k_\parallel v_A^2}{\omega^2 - \omega_0^2(r)}\left[\partial_r v_r - \frac{v_r}{r}\right], &(7)\\
B_\theta &= -i\frac{B_0}{\omega}\frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)}\left[\partial_r v_r - \frac{v_r}{r}\right]. &(8)
\end{aligned}$$

4 Approximate solution for the fast mode

The main subject of our study is the fast mode. Assuming $\omega_A \ll \omega_0$ and $\omega \gg \omega_A$, Eq. (2) is reduced to the form

$$\partial_r^2 v_r + \frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)}\partial_r v_r + \left(-\frac{1}{r}\frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)} + \frac{\omega^2 - \omega_0^2(r) - v_A^2/r^2}{v_A^2}\right)v_r = 0. \tag{9}$$

Here the prime means the derivative over the radial coordinate, $(...)' = \partial(...)/\partial r$.

The box-model (straight field lines) corresponds to the case $r \to \infty$. In this case case, Eq. (2) takes the form known from the previous work on the field line resonance (*Southwood*, 1974; *Chen and Hasegawa*, 1974):

$$\partial_r \left[\frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)} \partial_r \upsilon_r \right] + \frac{\omega^2 - \omega_A^2(r)}{\upsilon_A^2} \upsilon_r = 0. \tag{10}$$

Correspondingly, the fast mode equation (9) is reduced to the form

$$\partial_r^2 \upsilon_r + \frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)} \partial_r \upsilon_r + \frac{\omega^2 - \omega_0^2(r)}{\upsilon_A^2} \upsilon_r = 0. \tag{11}$$

This equation has a singularity in the point r_0 where the equality

$$\omega = \omega_0(r) \tag{12}$$

holds. However, the solution of this equation is regular, as had been mentioned in papers (*Southwood*, 1974; *Chen and Hasegawa*, 1974): in the plasma with straight field lines, there is no singularity in the fast mode wave field. The only singularity is the Alfvén resonance where $\omega = \omega_A(r)$.

Now let us proceed to the $r^{-1} \neq 0$ case. In the the proximity of the resonant point r_0, where the inequality $|\omega - \omega_A| \gg |\omega - \omega_0|$ holds, the function $\omega_0^2(r)$ can be represented as

$$\omega^2 - \omega_0^2(r) \approx -[\omega_0^2(r)]'(r - r_0) = \frac{\omega_0^2}{L}(r - r_0).$$

Then Eq. (9) is reduced to the form

$$\partial_r^2 \upsilon_r - \frac{1}{r - r_0} \partial_r \upsilon_r + \frac{1/r_0}{r - r_0} \upsilon_r = 0, \tag{13}$$

where a and b are constants. Its solution is expressed in terms of the Bessel functions of the second order J_2 and Y_2:

$$\upsilon_r = (r - r_0) \left[a J_2 \left(2\sqrt{\frac{r - r_0}{r_0}} \right) + b Y_2 \left(2\sqrt{\frac{r - r_0}{r_0}} \right) \right]. \tag{14}$$

When $r \to r_0$, the asymptotic of this solution is

$$\upsilon_r = \frac{a}{2r_0}(r - r_0)^2 - \frac{b}{\pi} r_0 - \frac{b}{\pi}(r - r_0) + \frac{b}{\pi r_0}(r - r_0)^2 \ln \sqrt{\frac{r - r_0}{r_0}}. \tag{15}$$

Thus, the resonance at r_0 constitutes the branching point in υ_r component, but the wave's amplitude remains finite.

However, as follows from Eq. (5) the azimuthal component of the plasma velocity υ_y has a logarithmic singularity:

$$\upsilon_y = -i \frac{k_y L}{k_y^2 + k_\parallel^2} \left[\frac{2b}{\pi r_0} \ln \sqrt{\frac{r - r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0} \right]. \tag{16}$$

Using Eqs. (6,7,8), the magnetic field components can be obtained:

$$B_r = -k_\parallel \frac{B_0}{\omega}\left[\frac{a}{2r_0}(r-r_0)^2 - \frac{b}{\pi}r_0 - \frac{b}{\pi}(r-r_0) + \frac{b}{\pi r_0}(r-r_0)^2 \ln\sqrt{\frac{r-r_0}{r_0}}\right], \quad (17)$$

$$B_y = i\frac{B_0}{\omega}\frac{k_\parallel k_y L}{k_y^2+k_\parallel^2}\left[\frac{2b}{\pi r_0}\ln\sqrt{\frac{r-r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0}\right], \quad (18)$$

$$B_\theta = -i\frac{B_0}{\omega}\frac{k_y^2 L}{k_y^2+k_\parallel^2}\left[\frac{2b}{\pi r_0}\ln\sqrt{\frac{r-r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0}\right]. \quad (19)$$

Thus, while the radial component B_r is finite near the resonant point, both azimuthal B_y and parallel B_θ components have the logarithmic singularity.

It is instructive to consider also the perturbation of the plasma density determined from the continuity equation $\partial\rho/\partial t = -\nabla\cdot\rho_0\vec{v}$. Near the resonant point, we have

$$\rho = \frac{i}{\omega}\rho_0\left(ik_y v_y + \frac{\partial v_r}{\partial r}\right) \simeq \ln\sqrt{\frac{r-r_0}{r_0}}. \quad (20)$$

Thus, the resonant point serves as a point of both energy and mass accumulation.

5 Discussion

As follows from the previous section, if the wave is propagating radially toward the centre of the cylinder, then when the wave is approaching the resonance point the wave decelerates radially but remains propagating along the cylinder. Thus, plasma can move only along the cylinder at the resonance point. In addition, the wave's energy density grows rapidly up to infinity at the resonance point, since the wave becomes more and more narrowly localized approaching the resonance point. This is due to the fact that the magnetic flux tube volume decreases in toward the centre of the cylinder. This pattern of wave transformation resembles that which occurs during the earthward propagation of the bursty fast flows during substorms (*Shiokawa et al.*, 1997; *Baumjohann*, 2000): a earthward bursty fast flow decelerates and stops due to the strongly earthward decrease of the magnetic flux tube volume in the Earth's dipolar magnetic field. It would be interesting to find out whether that breaking flow surface can be identified with the logarithmic resonance surface found in this paper.

However, it should be noted that the hemicylindrical model is still oversimplified. Thus, it would be interesting to find out whether the FMS singularity found in this paper remains in dipolar geometry, as it happens for the Alfvén resonance (*Chen and Cowley*, 1989; *Leonovich and Mazur*, 1989), or it disappears due to the parallel plasma and magnetic field inhomogeneity. Next, it is worth noting that the similar singularity for the fast mode can occur also in geometry with the straight field line but in the sheared magnetic field (*Mager and Klimushkin*, 2002). Moreover, similar singularity presents also in the slow mode wave field in the cylindrical geometry (*Petrashchuk and Klimushkin*, 2020). Finally, a kind of the singularity presents also for the drift-compressional modes in kinetics, but the formalism of kinetics does not allow to elucidate its precise nature (*Klimushkin and Mager*, 2011; *Mager et al.*, 2013).

6 Conclusions

Let us resume the results of our analysis of the hemicylindrical model of the magnetosphere, where the plasma is assumed to be one-dimensionally inhomogeneous, but which takes into account the field line curvature.

1. The ordinary differential equation for coupled Alfvén and fast modes was derived.
2. The approximate solution of this equation near the surface r_0 determined from the equation $\omega_0^2(r) = (k_\parallel^2 + k_y^2)v_A^2$ was obtained.
3. As in the box model with the straight field lines this surface limits the fast mode propagation region. However, this location became a wave's resonance surface as in the curved field the the azimuthal and compressional components of the wave's magnetic field as well as the plasma density have a logarithmic singularity on this surface. Thus, the r_0 surface can be coined as the surface of the logarithmic resonance. Note that in the straight field lines case, the solution in the vicinity of the r_0 surface is regular (*Southwood*, 1974; *Chen and Hasegawa*, 1974).

Thus, the behavior of the fast mode in the box-model model with straight field lines and in the hemicylindrical models is completely different. This allows one to conclude that the model with straight field lines is too crude to examine the behavior of fast mode in the magnetosphere.

Acknowledgments

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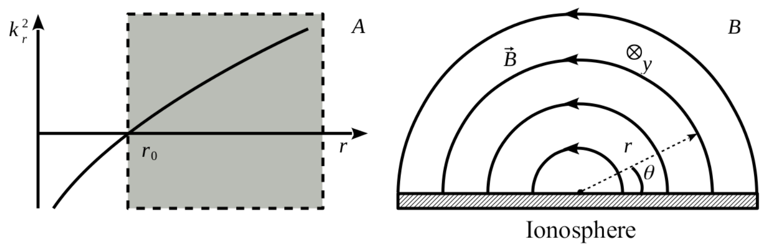

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$$v_y = -i k_y \frac{v_A^2}{\omega^2 - \omega_0^2(r)} \left[\partial_r v_r - \frac{v_r}{r} \right], \tag{5}$$

$$B_r = -k_\parallel \frac{B_0}{\omega} v_r, \tag{6}$$

$$B_y = i \frac{B_0}{\omega} \frac{k_y k_\parallel v_A^2}{\omega^2 - \omega_0^2(r)} \left[\partial_r v_r - \frac{v_r}{r} \right], \tag{7}$$

$$B_\theta = -i \frac{B_0}{\omega} \frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)} \left[\partial_r v_r - \frac{v_r}{r} \right]. \tag{8}$$

4 Approximate solution for the fast mode

The main subject of our study is the fast mode. Assuming $\omega_A \ll \omega_0$ and $\omega \gg \omega_A$, Eq. (2) is reduced to the form

$$\partial_r^2 v_r + \frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)} \partial_r v_r + \left(-\frac{1}{r} \frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)} + \frac{\omega^2 - \omega_0^2(r) - v_A^2/r^2}{v_A^2} \right) v_r = 0. \tag{9}$$

Here the prime means the derivative over the radial coordinate, $(...)' = \partial(...)/\partial r$.

The box-model (straight field lines) corresponds to the case $r \to \infty$. In this case case, Eq. (2) takes the form known from the previous work on the field line resonance (*Southwood*, 1974; *Chen and Hasegawa*, 1974):

$$\partial_r \left[\frac{\omega^2 - \omega_A^2(r)}{\omega^2 - \omega_0^2(r)} \partial_r \upsilon_r \right] + \frac{\omega^2 - \omega_A^2(r)}{\upsilon_A^2} \upsilon_r = 0. \tag{10}$$

Correspondingly, the fast mode equation (9) is reduced to the form

$$\partial_r^2 \upsilon_r + \frac{[\omega_0^2(r)]'}{\omega^2 - \omega_0^2(r)} \partial_r \upsilon_r + \frac{\omega^2 - \omega_0^2(r)}{\upsilon_A^2} \upsilon_r = 0. \tag{11}$$

This equation has a singularity in the point r_0 where the equality

$$\omega = \omega_0(r) \tag{12}$$

holds. However, the solution of this equation is regular, as had been mentioned in papers (*Southwood*, 1974; *Chen and Hasegawa*, 1974): in the plasma with straight field lines, there is no singularity in the fast mode wave field. The only singularity is the Alfvén resonance where $\omega = \omega_A(r)$.

Now let us proceed to the $r^{-1} \neq 0$ case. In the the proximity of the resonant point r_0, where the inequality $|\omega - \omega_A| \gg |\omega - \omega_0|$ holds, the function $\omega_0^2(r)$ can be represented as

$$\omega^2 - \omega_0^2(r) \approx -[\omega_0^2(r)]'(r - r_0) = \frac{\omega_0^2}{L}(r - r_0).$$

Then Eq. (9) is reduced to the form

$$\partial_r^2 \upsilon_r - \frac{1}{r - r_0} \partial_r \upsilon_r + \frac{1/r_0}{r - r_0} \upsilon_r = 0, \tag{13}$$

where a and b are constants. Its solution is expressed in terms of the Bessel functions of the second order J_2 and Y_2:

$$\upsilon_r = (r - r_0) \left[a J_2 \left(2\sqrt{\frac{r - r_0}{r_0}} \right) + b Y_2 \left(2\sqrt{\frac{r - r_0}{r_0}} \right) \right]. \tag{14}$$

When $r \to r_0$, the asymptotic of this solution is

$$\upsilon_r = \frac{a}{2r_0}(r - r_0)^2 - \frac{b}{\pi} r_0 - \frac{b}{\pi}(r - r_0) + \frac{b}{\pi r_0}(r - r_0)^2 \ln \sqrt{\frac{r - r_0}{r_0}}. \tag{15}$$

Thus, the resonance at r_0 constitutes the branching point in υ_r component, but the wave's amplitude remains finite.

However, as follows from Eq. (5) the azimuthal component of the plasma velocity υ_y has a logarithmic singularity:

$$\upsilon_y = -i \frac{k_y L}{k_y^2 + k_\parallel^2} \left[\frac{2b}{\pi r_0} \ln \sqrt{\frac{r - r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0} \right]. \tag{16}$$

Using Eqs. (6,7,8), the magnetic field components can be obtained:

$$B_r = -k_\parallel \frac{B_0}{\omega}\left[\frac{a}{2r_0}(r-r_0)^2 - \frac{b}{\pi}r_0 - \frac{b}{\pi}(r-r_0) + \frac{b}{\pi r_0}(r-r_0)^2 \ln\sqrt{\frac{r-r_0}{r_0}}\right], \quad (17)$$

$$B_y = i\frac{B_0}{\omega}\frac{k_\parallel k_y L}{k_y^2+k_\parallel^2}\left[\frac{2b}{\pi r_0}\ln\sqrt{\frac{r-r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0}\right], \quad (18)$$

$$B_\theta = -i\frac{B_0}{\omega}\frac{k_y^2 L}{k_y^2+k_\parallel^2}\left[\frac{2b}{\pi r_0}\ln\sqrt{\frac{r-r_0}{r_0}} + \frac{a}{r_0} + \frac{2b}{\pi r_0}\right]. \quad (19)$$

Thus, while the radial component B_r is finite near the resonant point, both azimuthal B_y and parallel B_θ components have the logarithmic singularity.

It is instructive to consider also the perturbation of the plasma density determined from the continuity equation $\partial\rho/\partial t = -\nabla\cdot\rho_0\vec{v}$. Near the resonant point, we have

$$\rho = \frac{i}{\omega}\rho_0\left(ik_y v_y + \frac{\partial v_r}{\partial r}\right) \simeq \ln\sqrt{\frac{r-r_0}{r_0}}. \quad (20)$$

Thus, the resonant point serves as a point of both energy and mass accumulation.

5 Discussion

As follows from the previous section, if the wave is propagating radially toward the centre of the cylinder, then when the wave is approaching the resonance point the wave decelerates radially but remains propagating along the cylinder. Thus, plasma can move only along the cylinder at the resonance point. In addition, the wave's energy density grows rapidly up to infinity at the resonance point, since the wave becomes more and more narrowly localized approaching the resonance point. This is due to the fact that the magnetic flux tube volume decreases in toward the centre of the cylinder. This pattern of wave transformation resembles that which occurs during the earthward propagation of the bursty fast flows during substorms (*Shiokawa et al.*, 1997; *Baumjohann*, 2000): a earthward bursty fast flow decelerates and stops due to the strongly earthward decrease of the magnetic flux tube volume in the Earth's dipolar magnetic field. It would be interesting to find out whether that breaking flow surface can be identified with the logarithmic resonance surface found in this paper.

However, it should be noted that the hemicylindrical model is still oversimplified. Thus, it would be interesting to find out whether the FMS singularity found in this paper remains in dipolar geometry, as it happens for the Alfvén resonance (*Chen and Cowley*, 1989; *Leonovich and Mazur*, 1989), or it disappears due to the parallel plasma and magnetic field inhomogeneity. Next, it is worth noting that the similar singularity for the fast mode can occur also in geometry with the straight field line but in the sheared magnetic field (*Mager and Klimushkin*, 2002). Moreover, similar singularity presents also in the slow mode wave field in the cylindrical geometry (*Petrashchuk and Klimushkin*, 2020). Finally, a kind of the singularity presents also for the drift-compressional modes in kinetics, but the formalism of kinetics does not allow to elucidate its precise nature (*Klimushkin and Mager*, 2011; *Mager et al.*, 2013).

6 Conclusions

Let us resume the results of our analysis of the hemicylindrical model of the magnetosphere, where the plasma is assumed to be one-dimensionally inhomogeneous, but which takes into account the field line curvature.

1. The ordinary differential equation for coupled Alfvén and fast modes was derived.
2. The approximate solution of this equation near the surface r_0 determined from the equation $\omega_0^2(r) = (k_\parallel^2 + k_y^2)v_A^2$ was obtained.
3. As in the box model with the straight field lines this surface limits the fast mode propagation region. However, this location became a wave's resonance surface as in the curved field the the azimuthal and compressional components of the wave's magnetic field as well as the plasma density have a logarithmic singularity on this surface. Thus, the r_0 surface can be coined as the surface of the logarithmic resonance. Note that in the straight field lines case, the solution in the vicinity of the r_0 surface is regular (*Southwood*, 1974; *Chen and Hasegawa*, 1974).

Thus, the behavior of the fast mode in the box-model model with straight field lines and in the hemicylindrical models is completely different. This allows one to conclude that the model with straight field lines is too crude to examine the behavior of fast mode in the magnetosphere.

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