Nonradial Oscillations of Saturn: Forcing of the Slowest Spiral Density Waves in the C Ring by I-modes and R-modes and the Implications for the Internal Structure

Andrew Friedson[1] and Katie Broad[2]

[1]Jet Propulsion Laboratory
[2]Purdue University

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Abstract

It is now known that many spiral density waves in Saturn's C ring are forced by resonant gravitational interactions with the planet's nonradial oscillations. Observations of their periods have the power to reveal new information concerning subtle aspects of the planet's internal structure. The properties of a certain class of nonradial oscillations, consisting of the inertial and rotational normal modes (i-modes and r-modes), are particularly sensitive to small departures of the internal density stratification from adiabatic, and may be especially useful in revealing such departures and their implications for any compositional gradients residing inside Saturn. We calculate the frequencies and eigenfunctions (shapes) of the i-modes and r-modes for a given suite of internal structure models and determine the Lindblad resonance locations and periods of the density waves they excite in the C ring. We find that for reasonable models of Saturn's internal structure, the i-mode and r-mode frequencies overlap the range of frequencies of the slowest density waves observed in the C ring, that is, those having pattern speeds similar to Saturn's rotation rate. In addition, the presence of observable density waves forced by r-modes requires Saturn to possess (at least) two statically stable (non-convective) regions, one in the deep interior and the other in the outer molecular hydrogen envelope. The inner zone is most likely stabilized by a significant compositional gradient, the outer zone by a compositional gradient, by baroclinic processes, or both. Unresolved questions remain. Currently, the model predicts that if energy is equipartitioned among the i-modes, then some slow density waves should be detectable at radii inside ˜84500 km in the C ring. None have been observed inside this radius. In addition, the source of excitation of the normal modes remains to be identified. We are currently exploring whether they can draw their energy from baroclinic instability in the outer stable zone. These questions will be addressed at length at the meeting. This work is supported by the NASA SSW program.

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A. James Friedson (Andrew.Friedson@jpl.Caltech.edu)[1], Katherine Broad[2]

[1]Jet Propulsion Laboratory, California Institute of Technology, [2]Summer Undergraduate Research Fellow, California Institute of Technology/Purdue University

1 OVERVIEW

We calculate the frequencies and eigenfunctions (shapes) of Saturn's inertial and rotational normal modes (i-modes and r-modes). The properties of these oscillations are sensitive to small departures of the internal density stratification from the adiabat. We also explore whether the modes can derive their energy through baroclinic instability associated with a deep zonal jet structure. We find that

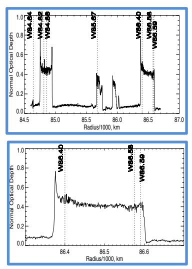
1. Their frequencies fall in the right range to account for density waves in the C ring possessing pattern speeds close to Saturn's rotation rate
2. The presence of density waves in the ring excited by r-modes implies the existence of regions of positive static stability in Saturn, both in the deep interior and in the outer molecular envelope
3. Some of the modes can in principle derive their energy through extraction of potential energy associated with baroclinic structure in Saturn (i.e., non-alignment of density and pressure surfaces due to the presence of shear in the deep zonal jets)

Comparison of the density wave pattern speeds predicted by the model to the observations has the potential to reveal subtle aspects of Saturn's internal stratification.

2 THE SLOW DENSITY WAVES

The table shows the set of slow density waves found by Hedman and Nicholson (2014) with their designation, azimuthal wavenumber *m*, and pattern speed Ω_P. Their periods are too long to be forced by pure f-modes or g-modes; their Ω_P lie close to Saturn's rotation rate. In our model, the waves with $\Omega_P > \Omega_S$ (Saturn's rotation rate, ~816° day^{-1}) must be forced by inertial modes, those with $\Omega_P < \Omega_S$ by r-modes.

m	Wave	$\Omega_{Pattern}$, ° day^{-1}
+3	W84.82	833.5
+3	W84.86	833.0
+3	W86.40	810.4
+3	W86.58	807.9
+3	W86.59	807.7

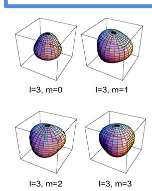


Figures adapted from Hedman and Nicholson (2014)

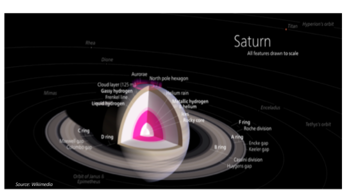
3 NUMERICAL MODEL

A numerical model is used to solve the *barotropic* problem:

1. Solves linearized equations of motion for adiabatic, non-radial oscillations
2. Includes oblateness of planet through Chandrasekhar-Milne expansion
3. Uniform rotation assumed
4. Cowling approximation: effect of self-gravity of normal modes ignored
5. Boundary value problem solved using a modified Magnus Multiple Shooting method to find eigenfrequencies and eigenfunctions

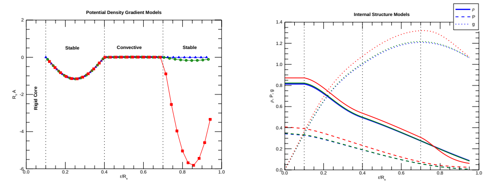


Some generic normal modes of a non-rotating planet. *Images from Tim Bedding, U. Sydney*



4 INTERNAL STRUCTURE MODELS

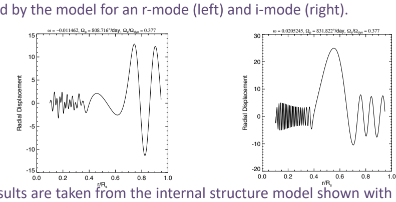
The three trial internal structure models used for the calculations are shown below. The models consist of a rigid inner core, statically-stable outer core, convective mid-layer, and stable outer envelope, as shown in terms of the potential density gradient *A(r)* in the left figure. The right figure shows the density, pressure, and gravity profiles associated with each *A* profile on the left, in units where $G = M_s = R_s = 1$.



The frequencies of the i-modes and r-modes are sensitive to the *A(r)* profile. *A(r)* measures the deviation of the radial density gradient from that of an adiabatic profile. The model produces r-modes capable of creating detectable density waves in the C ring only if statically stable regions exist near the inner core and outer molecular envelope, as indicated in the figure on the left.

5 RADIAL EIGENFUNCTIONS

Below are examples of the (unnormalized) radial displacement eigenfunction computed by the model for an r-mode (left) and i-mode (right).



These results are taken from the internal structure model shown with green circles in Box 4. The shape of the eigenfunction, particularly the number of radial nodes, strongly influences the mode's effectiveness in driving a density wave in the C ring. R-modes are produced in the model only if there is a deep stable (non-convective) region, and will force density waves of significant amplitude only if there is also a statically stable region in the outer molecular envelope.

6 BAROCLINIC INSTABILITY OF NORMAL MODES

The normal modes have been calculated numerically for a barotropic, uniformly rotating state, for which mean pressure and density surfaces coincide. When these conditions are relaxed for a differentially rotating planet (strictly, one for which $\partial\Omega/\partial z \neq 0$, z along rotation axis), the normal modes may become baroclinically unstable, extracting energy from the lateral buoyancy gradient. If we write the equations of motion in terms of the perturbation Lagrangian displacement ξ in the form (e.g. Aerts+2010, *Asteroseismology*, Springer Netherlands)

$$\rho\frac{\partial^2\xi}{\partial t^2} = L(\xi)$$

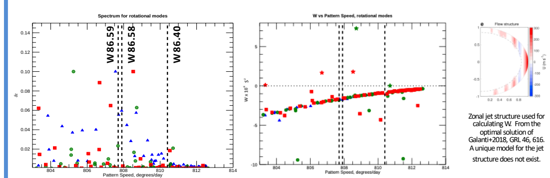
where L is a linear functional of ξ, a necessary and sufficient condition for stability is that $W < 0$, where W is defined as

$$W = \int dV\xi^* \cdot L(\xi) \Big/ \int dV\rho\xi^* \cdot \xi$$

When $W > 0$ for a particular mode, the barocline is unstable to that mode. We split L into two parts, the barotropic part L_0 and baroclinic part L_1, $L = L_0 + L_1$, and evaluate W for each mode, noting that $L_0(\xi) = -\omega_0^2\xi$ for each eigenmode, with the eigenfrequencies ω_0 already computed by the numerical model. The perturbation due to differential rotation, $L_1(\xi)$, is calculated by including the effects of the latitudinal density gradient and excess centrifugal acceleration associated with the rotational shear. The values of W calculated for each mode, assuming the optimal gravity solution for the zonal jet structure (Galanti+2018) are shown in boxes 7 and 8.

7 R-MODE SPECTRUM

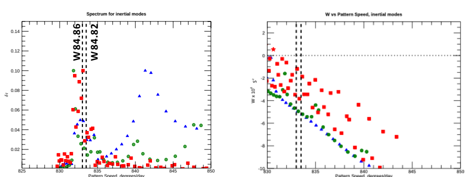
The model produces a rich spectrum of r-modes. Both their amplitudes and shapes determine the modes' effectiveness in producing a sufficiently strong disturbing potential in the C ring. In the **left** panel below, we show the relative perturbation optical depths $\delta\tau$ (normalized to a maximum of 0.1) computed assuming *energy equipartition* among the normal modes of the *barotropic* model. Symbol color indicates the corresponding internal structure model shown in Box 4. The pattern speeds of W86.40, W86.58, and W86.59 are indicated. In the **middle** panel, W is plotted vs. pattern speed, assuming the optimal solution from Galanti+2018 (**right**) for Saturn's jet structure. The modes with $W > 0$ (indicated by stars) are *baroclinically unstable*. These are the modes most likely to develop large enough amplitudes to excite the observed density waves. The baroclinic instability criterion thus may provide a means to determine which of the r modes should register in the C ring.



Zonal jet structure used for calculating W. From the optimal solution of Galanti+2018, GRL 46, 616. A unique model for the jet structure does not exist.

8 I-MODE SPECTRUM

The model also produces a dense spectrum of mixed inertia-gravity modes. Normalized $\delta\tau$ (assuming energy equipartition) is shown on the **left**. W for the jet structure shown above is on the **right**. In the present calculations, the case with the most stably stratified outer envelope (red) shows a baroclinically unstable mode with a pattern speed near 831° day^{-1}. However, the other cases exhibit local maxima in W between 830° and 832° day^{-1}. Modest adjustments to the internal structure model could cause these modes to become unstable as well. Finally, an important caveat: We have not yet accounted for the small frequency shift induced by the differential rotation associated with the baroclinicity. This is work in progress.



9 CONCLUSIONS

1. The frequencies of Saturn's r-modes and i-modes fall in the right range to force the slow density waves in the C ring whose pattern speeds lie close to the planetary rotation rate. Additional work is required before we can unambiguously identify the specific modes responsible for the different density waves observed
2. A few of the computed modes are found to be baroclinically unstable if we assume a zonal jet structure similar to the optimal Cassini gravity solution of Galanti+2018. It seems likely these would have the largest amplitudes and be the best candidates for forcing the slow density waves
3. The presence of observable density waves forced by r-modes requires Saturn to possess two statically stable regions, one in the deep interior and the other at relatively shallow depth
4. Ring seismology applied to the slow density waves has the potential to reveal relatively subtle features of the density stratification, possibly including the effects of compositional gradients in the interior

References: Hedman, M.M. and P.D. Nicholson 2014. MNRAS 444, 1369; Fuller, J. et al. 2014. Icarus 231,34.