Generalised Langevin Equations and the Climate Response Problem

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Abstract

There can be few greater scientific challenges than predicting the response of the global system to anthropogenic disruption, even with the array of sensing tools available in the "digital Anthropocene". Rather than depend on one approach, climate science thus employs a hierarchy of models, trading off the tractability of Energy Balance Models (EBMs) [1] against the detail of Global Circulation Models. Since the 70s Hasselmann-type stochastic EBMs have allowed treatment of climate fluctuations and noise. They remain topical, e.g. their use by Cox et al to propose an emergent constraint on climate sensitivity [2]. Insight comes from exploiting a mapping between Hasselmann's EBM and the original stochastic model in physics, the Langevin equation of 1908. However, it has recently been claimed that the wide range of time scales in the global system may require a heavy-tailed response [3,4] to perturbation, instead of the familiar exponential. Evidence for this includes long range memory (LRM) in GMT, and the success of a fractional Gaussian model in predicting GMT [5]. Our line of enquiry is complementary to [3-5] and proposes mapping a model well known in statistical mechanics, the Green-Kubo "Generalised Langevin Equation" (GLE) to generalise the Hasselmann EBM [6]. If present LRM then simplifies the GLE to a fractional Langevin equation (FLE). As well as a noise term the FLM has a dissipation term not present in [3,4], generalising Hasselmann's damping constant. We describe the corresponding EBM [7] that maps to the FLE, discuss its solutions, and relate it to existing models. References: [1] Ghil M (2019) Earth and Space Sciences, in press. [2] Cox P et al. (2018) Nature 553: 319-322 [3] Rypdal K. (2012) JGR 117: D06115 [4] Rypdal M and Rypdal K (2014) J Climate 27: 5240-5258. [5] Lovejoy et al (2015) ESDD 6:1–22 [6] Watkins N W (2013) GRL 40:1-9 [7] Watkins et al, to be submitted.

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Simplest energy balance models for the global temperature anomaly are deterministic, c.f. those used in IAMs (DICE/PAGE/FUND).

$$\frac{d\Delta Q}{dt} = C\frac{d\Delta T}{dt} = -\lambda \Delta T + F(t)$$

Hasselmann's approach of adding white (delta-correlated) noise improved realism.

$$\left(\frac{d}{dt} + \frac{\lambda}{C}\right)\Delta T = \frac{F(t)}{C} + \frac{\xi}{C}$$

$$< \xi(t)\xi(t+t') >= \sigma_Q^2 \delta(t-t')$$

However as noted by Leith, 1994 there is a case for making the noise itself in this model red [c.f Padilla et al, 2011], or even long-range dependent ("1/f"), and/or exploring non-exponential long tailed response kernels to replace constant lambda [c.f. ongoing work of Tromso group starting with Rypdal, JGR, 2012].

In this poster we modify the kernel, and propose using the mapping between the Langevin equation of Brownian motion and the Hasselmann equation in climate to suggest other equations. Noise can be left white-we don't assume presence of a fluctuation dissipation theorem relating kernel to noise.

Heat capacity	<- ->	Mass
Temperature anomaly		Velocity
Feedback parameter		Damping constant
Forcing		Drift Force

We show how this approach gives a "generalised Hasselmann model" for arbitrary forcing and noise, and a "fractional Hasselman model" in the special1/f case of long range memory. The LHS of the latter is the same as that of Lovejoy et al's FEBE model.

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CLIMATE

"Generalised Hasselmann Equation"

$$\left(\frac{d}{dt}\Delta T(t) + \int_0^t \gamma(t-t')\Delta T(t')\,dt'\right) = \frac{F}{C} + \frac{\xi}{C}$$

Allows full choice of memory kernel γ long range, short range, hybrid etc.

Specialise to long range memory: power law kernel $\gamma \sim t^{-\alpha}$,

"Fractional Hasselmann Equation"

when integral operator $\int_0^t dt' \frac{v(t')}{(t-t')^{\alpha}}$

written (equivalently) as fractional derivative

$$\left(\frac{d}{dt}\Delta T(t) + \gamma_\alpha \frac{d^{\alpha-1}}{dt^{\alpha-1}}\Delta T(t)\right) = \frac{F(t)}{C} + \frac{\xi}{C}$$

Fractionally integrate w.r.t $t^{\alpha-1}$

(LHS) is Lovejoy et al FEBE (Fractional Energy Balance Equation)

$$\left(\frac{d^{2-\alpha}}{dt^{2-\alpha}}\Delta T(t) + \gamma_\alpha \Delta T(t)\right) =$$

If Lovejoy et al's H $\equiv$ our 2-α

and their τ^{-H} $\equiv$ our γ_α.

Meaning of RHS is more opaque ...

$$\frac{d^{1-\alpha}}{dt^{1-\alpha}}\frac{1}{C}F(t) + \frac{d^{1-\alpha}}{dt^{1-\alpha}}\frac{1}{C}\xi\ )$$

but can see that even if ξ white, noise will usually become fractional.

Infrared Climate — Leith, NCAR Conference, 1994

Unfortunately, there is evidence that such a model would be unsatisfactory to capture some of the low-frequency phenomena observed in the atmosphere. This is referred to as the *infrared climate problem* and appears to be caused by non-linear interactions of the chaotic internal weather frequencies that potentially induce a "piling up" of extra variance at the low frequencies.

This infrared climate problem further obscures the predictability of climate change induced by slowly changing external influences, anthropogenic or not.

Specialise to absence of memory $\gamma(t) = (\lambda / C)\delta(t)$

[Hasselmann, 1976] Equation

$$\left(\frac{d}{dt} + \frac{\lambda}{C}\right)\Delta T(t) = \frac{F(t)}{C} + \frac{\xi}{C}$$

Hasselmann solution for starting time of 0

$$\Delta T(t) = \Delta T(0)e^{-(\lambda/C)t} + \frac{1}{C}\int_0^t e^{-(\lambda/C)(t-\tau)}[F(\tau)+\xi]d\tau$$

(NB in some cases the anomaly $\Delta T(0) = 0$)

becomes

$$\Delta T(t) = \frac{1}{C}\int_{-\infty}^t e^{-(\lambda/C)(t-\tau)}[F(\tau)+\xi]d\tau$$

in the case when start time is -∞.

In "Fractional Hasselmann" case exponential will be replaced by Mittag-Leffler functions c.f. Lovejoy FEBE model (e.g. EGU, 2019)

[Mori, 1965; Kubo, 1966] Generalised Langevin Equation

$$\left(\frac{d}{dt}v(t) + \int_0^t \gamma(t-t')\,v(t')\,dt'\right) = \frac{F(t)}{M} + \frac{\xi}{M}$$

$$v = \dot{x}$$

Specialise to long range memory: kernel $\gamma \sim t^{-\alpha}$,

Specialise to absence of memory $\gamma(t) = (\eta / M)\delta(t)$

Fractional Langevin Equation [e.g. Lutz, PRE, 2001]

when integral operator $\int_0^t dt' \frac{v(t')}{(t-t')^{\alpha}}$

written (equivalently) as fractional derivative

$$\left(\frac{d}{dt}v(t) + \gamma_\alpha \frac{d^{\alpha-1}}{dt^{\alpha-1}}v(t)\right) = \frac{F(t)}{M} + \frac{\xi}{M}$$

If an FDT holds, ξ becomes fractional Gaussian noise, ξ_H whose Hurst exponent obeys $2H = 2 - \alpha$

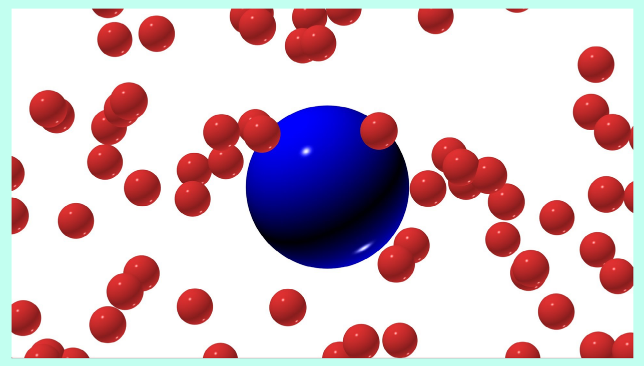
Fractionally integrate w.r.t $t^{\alpha-1}$

$$\left(\frac{d^{2-\alpha}}{dt^{2-\alpha}}v(t) + \gamma_\alpha v(t)\right) =$$

$$\frac{d^{1-\alpha}}{dt^{1-\alpha}}\left(\frac{F(t)}{M} + \frac{\xi}{M}\right)$$

[Langevin, 1908] Equation

$$\left(\frac{d}{dt} + \frac{\eta}{M}\right)v = \frac{F(t)}{M} + \frac{\xi}{M}$$



BROWNIAN MOTION

Standard text [Gardiner,2004] points out (in F(t)=0 case) that solution for starting time 0

$$v(t) = v(0)e^{-(\eta/M)t} + \frac{1}{M}\int_0^t e^{-(\eta/M)(t-\tau)}[F(\tau)+\xi]d\tau$$

becomes

$$v(t) = \frac{1}{M}\int_{-\infty}^t e^{-(\eta/M)(t-\tau)}[F(\tau)+\xi]d\tau$$

in the case when start time is -∞.

In Fractional Langevin case exponential replaced by Mittag-Leffler functions e.g. Lutz, 2001; Kupferman, 2004.